Turk J Math
36 (2012) , 147 – 160.

doi:10.3906/mat-0911-145

Invariant parametrizations and complete systems of global invariants of curves in the pseudo-Euclidean geometry

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Abstract

Let $M(n,p)$ be the group of all transformations of an n-dimensional pseudo-Euclidean space E_p^n of index p generated by all pseudo-orthogonal transformations and parallel translations of E_p^n. Definitions of a pseudo-Euclidean type of a curve, an invariant parametrization of a curve and an $M(n,p)$-equivalence of curves are introduced. All possible invariant parametrizations of a curve with a fixed pseudo-Euclidean type are described. The problem of the $M(n,p)$-equivalence of curves is reduced to that of paths. Global conditions of the $M(n,p)$-equivalence of curves are given in terms of the pseudo-Euclidean type of a curve and the system of polynomial differential $M(n,p)$-invariants of a curve $x(s)$.

Key Words: Curve, pseudo-Euclidean geometry, invariant parametrization

1. Introduction

Let R be the field of real numbers, n and p are integers such that $0 \leq p < n$. The n-dimensional pseudo-Euclidean space of index p (that is the space R^n with the scalar product $< x, y >= -x_1y_1 - \cdots - x_py_p + x_{p+1}y_{p+1} + \cdots + x_ny_n$) will be denoted by E_p^n. E_1^4 is the Minkowski spacetime. The group of all pseudo-orthogonal transformations of E_p^n (that is the set of all linear transformations $g : E_p^n \to E_p^n$ such that $< gx, gy >=< x, y >$ for all $x, y \in E_p^n$) is denoted by $O(n,p)$. Put $M(n,p)=\{F : E_p^n \to E_p^n \mid Fx = gx + b, g \in O(n,p),\ b \in E_p^n\}$ and $SM(n,p) = \{F \in M(n,p) : \det g = 1\}$.

The Frenet-Serret formalism for both time-like and space-like curves in spaces E_1^3 and E_1^4 is studied in papers [13, 21] and in the thesis [14]. In papers [2, 5, 6, 9, 20], the Frenet-Serret curve analysis is extended from non-null curves in E_1^4 to null (lightlike, isotropic) curves. For arbitrary n, this theory is extended to the Lorentz space E_1^n and to the space E_2^n in papers [3, 18] and in the book ([10], pp. 52–76). The Frenet-Serret theory for degenerate curves in spaces E_1^n and E_2^n is investigated in [11–12]. The Frenet-Serret theory of curves in E_p^n for arbitrary n and index p is considered in papers [4, 7, 8]. In [7], the fundamental theorem of a naturally-parametrized curve in E_p^n for arbitrary n and index p is obtained. It is found necessary and sufficient conditions under which given real-valued functions $\varphi_1, \ldots, \varphi_{n-1}$, $n \geq 2$, on an interval I of the real

2000 *AMS Mathematics Subject Classification:* 53A35.
This work was supported by the Research Fund of TUBITAK. Project number:107T049.

axis are the successive curvatures of a naturally-parametrized curve in E_p^n which is defined by them uniquely up to congruence for a given distribution of unit and pseudounit vectors in a Frenet $(n-1)$-frame of the curve.

The Frenet-Serret equations for a curve in an Euclidean space E_0^n provide curvature functions $k_1(s),\ldots,k_{n-1}(s)$ of a curve. The curvatures $k_1(s),\ldots,k_{n-2}(s)$ are $M(n,0)$-invariant. But the curvature $k_{n-1}(s)$ is not $M(n,0)$-invariant, it is $SM(n,0)$-invariant. For example, the torsion of a curve in E_0^3 is $SM(3,0)$-invariant, but it is not $M(3,0)$-invariant. Therefore the system $k_1(s),\ldots,k_{n-1}(s)$ gives a solution of the problem of the G-equivalence of curves only for $G=SM(n,0)$ ([19], p.p. 61–64). Besides, the method of moving frames essentially gives only conditions of a local G-equivalence of curves. A similar situation is valid for an arbitrary index p.

In the present paper we use an invariant-theoretic approach to the theory of curves in the pseudo-Euclidean geometry. We give a solution of the problem of global G-equivalence of curves for groups $G=M(n,p), SM(n,p)$ in terms of invariants of a curve.

This paper is organized as follows. In Section 1, the definitions of the pseudo-Euclidean type and an invariant parametrization of a curve are given. The pseudo-Euclidean type of a curve is $M(n,p)$-invariant and it has the following forms: $(0,l)$, where $0<l\leq\infty$, $(-\infty,0)$ and $(-\infty,+\infty)$. All possible invariant parametrizations of a curve with a fixed pseudo-Euclidean type are described. In Theorem 1, the problems of the $M(n,p)$-equivalence and the $SM(n,p)$-equivalence of curves are reduced to that of paths. In Section 2, the conditions of the global G-equivalence of curves are given in terms of the pseudo-Euclidean type and the system of polynomial differential G- invariant functions.

A description of a complete system of correlations between the elements of the complete system of differential invariants of a curve in E_p^n will be considered in our next paper. The theory of regular curves in E_p^n given in the present paper contains also some class of null curves (look at the Remarks 2–3 and Example 4 below). More detailed theory of invariants of null curves in E_p^n will be considered also in our next paper.

2. Invariant parametrizations of a curve

Let $J=(a,b)$ be an open interval of R.

Definition 1 *(see [16, 17]). A C^∞-mapping $x:J\to E_p^n$ will be called a J-path (shortly, a path) in E_p^n.*

Definition 2 *(see [16, 17]). A J_1-path $x(t)$ and a J_2-path $y(r)$ in E_p^n will be called D-equivalent if a C^∞-diffeomorphism $\varphi:J_2\to J_1$ exists such that $\varphi'(r)>0$ and $y(r)=x(\varphi(r))$ for all $r\in J_2$. A class of D-equivalent paths in E_p^n will be called a curve in E_p^n. A path $x\in\alpha$ will be called a parametrization of a curve α.*

If $x(t)$ is a J-path then $Fx(t)$ is a J-path in E_p^n for any $F\in M(n,p)$. Let G be a subgroup of $M(n,p)$.

Definition 3 *Two J-paths $x(t)$ and $y(t)$ in E_p^n are called G-equivalent if there exists $F\in G$ such that $y(t)=Fx(t)$. This being the case, we write $x(t)\overset{G}{\sim}y(t)$*

Let $\alpha=\{h_\tau,\tau\in Q\}$ be a curve in E_p^n, where h_τ is a parametrization of α. Then $F\alpha=\{Fh_\tau,\tau\in Q\}$ is a curve in E_p^n for any $F\in M(n,p)$.

Definition 4 *(see [16, 17]) Two curves α and β in E_p^n are called G-equivalent if $\beta = F\alpha$ for some $F \in G$. This being the case, we write $\alpha \overset{G}{\sim} \beta$.*

Let $x(t) = (x_1(t), \ldots, x_n(t))$ be a J-path in E_p^n, $x^{'}(t) = (x_1^{'}(t), \ldots, x_n^{'}(t))$ is its first derivative and $x^{(k)}(t)$ is its k-th derivative. Denote the determinant of vectors $x^{'}(t), x^{(2)}(t), \ldots, x^{(n)}(t)$ by $\left[x^{'}(t)x^{(2)}(t)\ldots x^{(n)}(t)\right]$.

Definition 5 *A J-path $x(t)$ in E_p^n will be called pseudo-euclidean regular (regular, for short) if one of the following conditions hold:*
(5_1). $< x^{'}(t), x^{'}(t) > \neq 0$ *for all* $t \in J$;
(5_2). $\left[x^{'}(t)x^{(2)}(t)\ldots x^{(n)}(t)\right] \neq 0$ *for all* $t \in J$;
(5_3). $\left|< x^{'}(t), x^{'}(t) >\right| + \left|\left[x^{'}(t)x^{(2)}(t)\ldots x^{(n)}(t)\right]\right| \neq 0$ *for all* $t \in J$.
A curve α will be called regular if it contains a regular path.

Remark 1 It is obvious that $(5_1) \to (5_3)$ and $(5_2) \to (5_3)$. The following examples 1-3 below show that $(5_1) \not\to (5_2)$, $(5_2) \not\to (5_1)$, $(5_3) \not\to (5_1)$, $(5_3) \not\to (5_2)$ and $(5_3) \not\to (5_1) \cup (5_1)$.

Example 1 Consider the J-path $x(t) = (\frac{1}{2}t^2, \frac{1}{3}t^3)$ in E_1^2, where $J = (0, 2)$. Then $< x^{'}(t), x^{'}(t) >= 0$ for $t = 1$, but $\left[x^{'}(t)x^{(2)}(t)\right] \neq 0$ for all $t \in J$. Hence $(5_2) \not\to (5_1)$. In the case $p = 0$, it is easy to see that $(5_2) \to (5_1)$.

Example 2 Consider the J-path $x(t) = (\frac{1}{3}t^3, \frac{2}{3}t^3)$ in E_1^2, where $J = (0, 2)$. Then $\left[x^{'}(t)x^{(2)}(t)\right] = 0$ for all $t \in J$, but $< x^{'}(t), x^{'}(t) > \neq 0$ for all $t \in J$. Hence $(5_1) \not\to (5_2)$.

Example 3 Consider the J-path $x(t) = (t, \frac{1}{2}t^2, \frac{1}{4}t^4)$ in E_1^3, where $J = (-\frac{1}{2}, 2)$. Then $\left[x^{'}(t)x^{(2)}(t)x^{(3)}(t)\right] = 6t$ and $< x^{'}(t), x^{'}(t) >= 1 + t^2 - t^6$ for all $t \in J$. The equality $\left[x^{'}(t)x^{(2)}(t)x^{(3)}(t)\right] = 6t$ implies that $\left[x^{'}(t)x^{(2)}(t)x^{(3)}(t)\right] = 0$ only for $t = t_1 = 0$. There exists unique $t = t_2 \in J$ such that $< x^{'}(t), x^{'}(t) >= 0$. It is easy to see that $1 < t_2 < 2$. Then $\left[x^{'}(t)x^{(2)}(t)x^{(3)}(t)\right] = 0$ for some $t = t_1 \in J$ and $< x^{'}(t), x^{'}(t) >= 0$ for some $t = t_2 \in J$, where $t_1 \neq t_2$, but $\left|< x^{'}(t), x^{'}(t) >\right| + \left|\left[x^{'}(t)x^{(2)}(t)x^{(3)}(t)\right]\right| \neq 0$ for all $t \in J$. Hence $(5_3) \not\to (5_1) \cup (5_2)$. In particularly, $(5_3) \not\to (5_1)$ and $(5_3) \not\to (5_2)$.

Definition 6 *(see [2]) A J-path $x(t)$ is called null if $< x^{'}(t), x^{'}(t) >= 0$ for all $t \in J$.*

Remark 2 There exists a null J-path such that $\left[x^{'}(t)x^{(2)}(t)\ldots x^{(n)}(t)\right] \neq 0$ for all $t \in J$.

Example 4 Consider the J-path

$$x(t) = (t, \frac{1}{2}t^2, \int_0^1 \sqrt{1 + t^2}dt)$$

in E_1^3, where $J=(0,1)$. Then $<x^{'}(t), x^{'}(t)>=0$ for all $t\in J$ and $\left[x^{'}(t)x^{(2)}(t)x^{(3)}(t)\right]=(1+t^2)^{-\frac{3}{2}}\neq 0$ for all $t\in J$.

Hence there exists a regular null J-path in E_p^n. Therefore the theory of regular curves in E_p^n given below contains also some class of null curves.

Now we define invariant parametrizations of regular curves in E_p^n. Let $x(t)$ be a regular J-path in E_p^n. We put

$$l_x(c,d)=\int_c^d |<x'(t),x'(t)>|^{\frac{1}{2}}\,dt.$$

in case (5_1) of Definition 5. If (5_1) doesn't hold and case (5_2) holds, we put

$$l_x(c,d)=\int_c^d \left|\left[x^{'}(t)x^{(2)}(t)\ldots x^{(n)}(t)\right]\right|^{\frac{2}{n(n+1)}}\,dt.$$

If the cases (5_1) and (5_2) don't hold and the case (5_3) holds, we put

$$l_x(c,d)=\int_c^d |<x'(t),x'(t)>|^{\frac{1}{2}}\,dt+\int_c^d \left|\left[x^{'}(t)x^{(2)}(t)\ldots x^{(n)}(t)\right]\right|^{\frac{2}{n(n+1)}}\,dt.$$

The limits $l_x(a,d)=\lim_{c\to a} l_x(c,d)\leq +\infty$ and $l_x(c,b)=\lim_{d\to b} l_x(c,d)\leq +\infty$ exist. There are only four possibilities:

$$(T_1).l_x(a,d)<+\infty, l_x(c,b)<+\infty; \quad (T_2).l_x(a,d)<+\infty, l_x(c,b)=+\infty;$$
$$(T_3).l_x(a,d)=+\infty, l_x(c,b)<+\infty; \quad (T_4).l_x(a,d)=+\infty, l_x(c,b)=+\infty.$$

Suppose that the case (T_1) or (T_2) holds for some $c,d\in J$. Then $l=l_x(a,d)+l_x(c,b)-l_x(c,d)$, where $0\leq l\leq +\infty$, does not depend on $c,d\in J$. In this case we say that x belongs to the pseudo-euclidean type of $(0,l)$. The cases (T_3) and (T_4) do not depend on c,d. In these cases, we say that x belongs to the pseudo-euclidean types of $(-\infty,0)$ and $(-\infty,+\infty)$, respectively. There exist paths of all types $(0,l)$, where $l<+\infty$, $(0,+\infty)$, $(-\infty,0)$ and $(-\infty,+\infty)$. The pseudo-euclidean type of a path x will be denoted by $L(x)$. It is obvious that:

(i) if $x \overset{M(n,p)}{\sim} y$ then $L(x)=L(y)$;

(ii) if x,y is parametrizations of a curve α then $L(x)=L(y)$.

The pseudo-euclidean type of a path $x\in\alpha$, will be called the pseudo-euclidean type of the curve α and denoted by $L(\alpha)$. $L(\alpha)$ is an $M(n,p)$-invariant of a curve α.

Now we define an invariant parametrization of a regular curve in E_p^n. Let $J=(a,b)$ and $x(t)$ be a regular J-path in E_p^n. We define the pseudo-euclidean arc length function $s_x(t)$ for each pseudo-euclidean type as follows. We put $s_x(t)=l_x(a,t)$ for the case $L(x)=(0,l)$, where $l\leq+\infty$, and $s_x(t)=-l_x(t,b)$ for the case $L(x)=(-\infty,0)$. Let $L(x)=(-\infty,+\infty)$. We choose a fixed point in every interval $J=(a,b)$ of R and denote it by a_J. Let $a_J=0$ for $J=(-\infty,+\infty)$. We set $s_x(t)=l_x(a_J,t)$.

Since $s_x^{'}(t)>0$ for all $t\in J$, the inverse function of $s_x(t)$ exists. Let us denote it by $t_x(s)$. The domain of $t_x(s)$ is $L(x)$ and $t_x^{'}(s)>0$ for all $s\in L(x)$.

Proposition 1 *Let* $I=(a,b)$ *and* x *be a regular* I*-path in* E_p^n*. Then*

(*i*) $s_{Fx}(t)=s_x(t)$ *and* $t_{Fx}(s)=t_x(s)$ *for all* $F\in M(n,p)$*;*

(*ii*) *the equalities* $s_{x(\varphi)}(r)=s_x(\varphi(r))+s_0$ *and* $\varphi(t_{x(\varphi)}(s+s_0))=t_x(s)$ *hold for any* C^∞*-diffeomorphism* $\varphi: J=(c,d)\to I$ *such that* $\varphi^{'}(r)>0$ *for all* $r\in J$*, where* $s_0=0$ *for* $L(x)\neq(-\infty,+\infty)$ *and* $s_0=l_x(\varphi(a_J),a_I)$ *for* $L(x)=(-\infty,+\infty)$*.*

Proof. The proof of statement (i) is obvious. We prove statement (ii) for case (5_3) in Definition 5. Let $L(x)=(-\infty,+\infty)$. Then we have $s_{x(\varphi)}(r)=$

$$\int_{a_J}^{r}(\left|<\frac{d}{dr}x(\varphi(r)),\frac{d}{dr}x(\varphi(r))>\right|^{\frac{1}{2}}+\left|\left[\frac{d}{dr}x(\varphi(r))\dots\frac{d^n}{dr^n}x(\varphi(r))\right]\right|^{\frac{2}{n(n+1)}})dr=$$

$$\int_{a_J}^{r}\frac{d\varphi}{dr}(\left|<\frac{d}{d\varphi}x(\varphi(r)),\frac{d}{d\varphi}x(\varphi(r))>\right|^{\frac{1}{2}}+\left|\left[\frac{d}{d\varphi}x(\varphi(r))\dots\frac{d^n}{d\varphi^n}x(\varphi(r))\right]\right|^{\frac{2}{n(n+1)}})dr=$$

$$l_x(\varphi(a_J),\varphi(r))=l_x(a_I,\varphi(r))+l_x(\varphi(a_J),a_I).$$

So $s_{x(\varphi)}(r)=s_x(\varphi(r))+s_0$, where $s_0=l_x(\varphi(a_J),a_I)$. This implies $\varphi(t_{x(\varphi)}(s+s_0))=t_x(s)$. For $L(x)\neq(-\infty,+\infty)$, it is easy to see that $s_0=0$.

Proofs of statement (ii) for cases (5_1) and (5_2) in Definition 5 are similar. □

Let α be a regular curve, $x\in\alpha$. Then $x(t_x(s))$ is a parametrization of α.

Definition 7 *The parametrization* $x(t_x(s))$ *of a regular curve* α *will be called an invariant parametrization of* α*.*

We denote the set of all invariant parametrizations of α by $Ip(\alpha)$. Every $y\in Ip(\alpha)$ is a J-path, where $J=L(\alpha)$.

Proposition 2 *Let* α *be a regular curve,* $x\in\alpha$ *and* x *be a* J*-path, where* $J=L(\alpha)$*. Assume that the condition* (5_1) *in Definition 5 holds for* x*. Then the following conditions are equivalent:*

(*i*) x *is an invariant parametrization of* α*;*

(*ii*) $|<x'(t),x'(t)>|=1$ *for all* $s\in L(\alpha)$*;*

(*iii*) $s_x(s)=s$ *for all* $s\in L(\alpha)$*.*

Proof. $(i)\to(ii)$. Let $x\in Ip(\alpha)$. Then there exists $y\in\alpha$ such that $x(s)=y(t_y(s))$. By Proposition 1, $s_x(s)=s_{y(t_y)}(s)=s_y(t_y(s))+s_0=s+s_0$, where s_0 is as in Proposition 1. Since s_0 does not depend on s, we have $\frac{ds_x(s)}{ds}=|<x'(t),x'(t)>|^{\frac{1}{2}}=1$. Hence $|<x'(t),x'(t)>|=1$ for all $s\in L(\alpha)$.

$(ii)\to(iii)$. Let $|<x'(t),x'(t)>|=1$ for all $s\in L(\alpha)$. Using the definition of $s_x(t)$, we get $\frac{ds_x(s)}{ds}=|<x'(t),x'(t)>|^{\frac{1}{2}}=1$. Therefore $s_x(s)=s+c$ for some $c\in R$. In the case $L(x)\neq(-\infty,+\infty)$, conditions $s_x(s) = s + c$ and $s_x(s) \in L(\alpha)$ for all $s \in L(\alpha)$ implies $c = 0$, that is, $s_x(s) = s$. In the case $L(\alpha) = (-\infty, +\infty)$, equalities $s_x(s) = l_x(a_J, s) = l_x(0, s) = s + c$ implies $0 = l_x(0, 0) = c$, that is , $s_x(s) = s$.

$(iii) \to (i)$. Since $s_x(s) = s$ implies $t_x(s) = s$, we get $x(s) = x(t_x(s)) \in Ip(\alpha)$. □

Similar results are true for conditions (5_2) and (5_3) in Definition 5.

Remark 3 In papers [2–9, 11, 18, 20, 21], in the thesis [14] and in the book [10], essentially the parametrization in the 5_1 of Definition 5 is used and it is used only for curves of the type $(0, l)$, where $0 < l < \infty$. By remark 2 and Examples 1–3, parametrizations in the cases 5_2 and 5_3 are independent of the parametrization in the case 5_1. Hence the class of curves which investigated in the present paper is essentially wider then in the mentioned papers. By Remark 2 and Example 4, parametrizations in the cases 5_2 and 5_3 contain also parametrizations of some class of null curves.

Proposition 3 *Let α be a regular curve and $L(\alpha) \neq (-\infty, +\infty)$. Then there exists the unique invariant parametrization of α.*

Proof. A proof is similar to the proof of Proposition 4 in [16]. □

Let α be a regular curve and $L(\alpha) = (-\infty, +\infty)$. Then it is easy to see that the set $Ip(\alpha)$ is infinite and it is not countable.

Proposition 4 *Let α be a regular curve, $L(\alpha) = (-\infty, +\infty)$ and $x \in Ip(\alpha)$. Then $Ip(\alpha) =$ $\{y : y(s) = x(s + c), c \in (-\infty, +\infty)\}$.*

Proof. A proof is similar to the proof of Proposition 5 in [16]. □

Theorem 1 *Let α, β be regular curves and $x \in Ip(\alpha), y \in Ip(\beta)$. Then:*

(i) for $L(\alpha) = L(\beta) \neq (-\infty, +\infty)$, $\alpha \overset{M(n,p)}{\sim} \beta$ if and only if $x \overset{M(n,p)}{\sim} y$;

(ii) for $L(\alpha) = L(\beta) = (-\infty, +\infty)$, $\alpha \overset{M(n,p)}{\sim} \beta$ if and only if $x \overset{M(n,p)}{\sim} y(\psi_c)$ for some $c \in (-\infty, +\infty)$, where $\psi_c(s) = s + c$.

Proof. (i). Let $\alpha \overset{M(n,p)}{\sim} \beta$ and $h \in \alpha$. Then there exists $F \in M(n, p)$ such that $\beta = F\alpha$. This implies $Fh \in \beta$. Using Propositions 1-3, we get $x(s) = h(t_h(s)), y(s) = (Fh)(t_{Fh}(s))$ and $Fx(s) = F(h(t_h(s))) =$ $(Fh)(t_h(s)) = (Fh)(t_{Fh}(s)) = y(s)$. Thus $x \overset{M(n,p)}{\sim} y$. Conversely, let $x \overset{M(n,p)}{\sim} y$, that is, there exists $F \in M(n, p)$ such that $Fx = y$. Then $\alpha \overset{M(n,p)}{\sim} \beta$.

(ii). Let $\alpha \overset{M(n,p)}{\sim} \beta$. Then there exist J-paths $h \in \alpha, k \in \beta$ and $F \in M(n, p)$ such that $k(t) = Fh(t)$. We have $k(t_k(s)) = k(t_{Fh}(s)) = k(t_h(s)) = (Fh)(t_h(s))$. By Proposition 4, $x(s) = k(t_k(s + s_1)), y(s) =$ $h(t_h(s+s_2))$ for some $s_1, s_2 \in (-\infty, +\infty)$. Therefore $x(s-s_1) = Fy(s-s_2)$. This implies that $x \overset{M(n,p)}{\sim} y(\psi_c)$, where $\psi_c(s) = s+c$ and $c = s_1 - s_2$. Conversely, let $x \overset{M(n,p)}{\sim} y(\psi_c)$ for some $c \in (-\infty, +\infty)$, where $\psi_c = s+c$. Then there exists $F \in M(n, p)$ such that $y(s + c) = Fx(s)$. Since $y(s + c) \in \beta$, then $\alpha \overset{M(n,p)}{\sim} \beta$. □

Theorem 1 reduces the problems of the G-equivalence of regular curves for groups $G = M(n,p), SM(n,p)$ to that of paths only for the case $L(\alpha) = L(\beta) \neq (-\infty, +\infty)$. Let H be a subgroup of $M(n,p)$.

Definition 8 *J-paths $x(t)$ and $y(t)$ will be called $[H, (-\infty, +\infty)]$-equivalent, if there exist $h \in H$ and $d \in (-\infty, +\infty)$ such that $y(t) = hx(t+d)$ for all $t \in J$.*

Theorem 1 reduces the problem of the H-equivalence of curves to $[H, (-\infty, +\infty)]$-equivalence of paths for the case $L(\alpha) = L(\beta) = (-\infty, +\infty)$.

3. Conditions of G-equivalence of paths and curves

Below we use some notations and facts from the differential algebra and the theory of differential invariants of a paths. They may be found in [1, 15, 16, 17].

Definition 9 *A J-path $x(t)$ in E_p^n will be called non-singular if $\left[x^{'}(t)x^{(2)}(t)\ldots x^{(n)}(t)\right] \neq 0$ for all $t \in J$. A curve α will be called non-sigular if it contains a non-singular path.*

Let G be a subgroup of $M(n,p)$.

Definition 10 *(see [1], Definition 8). A differential polynomial function $f\{x\}$ of a path $x(t)$ is called G-invariant if $f\{gx\} = f\{x\}$ for all $g \in G$.*

Let $x(t)$ and $y(t)$ be J-paths in E_p^n such that $x \overset{M(n,p)}{\sim} y$. Then $f\{x\} = f\{y\}$ for any $M(n,p)$-invariant differential polynomial $f\{x\}$. The converse statement (that is conditions of $M(n,p)$-equivalence of J-paths) is true in the following form.

Theorem 2 *Assume that $x(t)$ and $y(t)$ be non-singular J-paths in E_p^n such that*

$$< x^{(i)}(t), x^{(i)}(t) > = < y^{(i)}(t), y^{(i)}(t) > \tag{1}$$

for all $t \in J$ and $1 \leq i \leq n$. Then $x \overset{M(n,p)}{\sim} y$.

Proof. For a proof of this theorem, we use several lemmas. □

Lemma 1 *Assume that $1 \leq i, j, i+j \leq 2n+1$. Then, for each differential polynomial $< x^{(i)}, x^{(j)} >$, a differential polynomial $P_{ij}\{y_1, ..., y_k\}$ exists such that*

$$< x^{(i)}, x^{(j)} > = P_{ij}\left\{< x^{'}, x^{'} >, ..., < x^{(k)}, x^{(k)} >\right\},$$

where $k = \left[\frac{i+j}{2}\right]$.

Proof. A proof is similar to the proof of Proposition 6 in [1]. □

Lemma 2 *The equality*

$$(-1)^p[y_1 \dots y_n][z_1 \dots z_n] = det|| < y_i, z_j > ||_{i,j=1,2,\dots,n}$$

holds for all vectors $y_1, \dots, y_n, z_1, \dots, z_n$ *in* E^n_p.

Proof. Let $Y = \|y_1 \dots y_n\|$ and $Z = \|z_1 \dots z_n\|$ be $n \times n$-matrices of systems $\{y_1, \dots, y_n\}$ and $\{z_1, \dots, z_n\}$ of column vectors $y_1, \dots, y_n, z_1, \dots, z_n \in E^n_p$ and $I_p = \|b_{ij}\|$ be the diagonal $n \times n$-matrix such that $b_{ii} = -1$ for all $i = 1, \dots, p$ and $b_{jj} = 1$ for all $j = p+1, \dots, n$. Then we have $Y^\top I_p Z = \|< y_i, z_j >\|_{i,j=1,2,\dots,n}$, where $Y^\top$ is the transpose matrix of Y. Passing on to determinants, we obtain the desired equality. □

Denote the determinant $det \left\|< x^{(i)}, x^{(j)} >\right\|_{i,j=1,2,\dots,n}$ by $\Delta = \Delta_x$. Equation (1) and Lemma 1 implies that $< x^{(i)}(t), x^{(j)}(t) >=< y^{(i)}(t), y^{(j)}(t) >$ for all $t \in J$ and all $1 \leq i \leq j \leq n$. Using these equalities, we get $\Delta_x(t) = \Delta_y(t)$ for all $t \in J$. Since x, y are non-singular J-paths, we have $\Delta_x(t) \neq 0, \Delta_y(t) \neq 0$ for all $t \in J$. Hence $\Delta_x(t)^{-1} = \Delta_y(t)^{-1}$. Denote the system $\left\{< x^{'}, x^{'} >, ..., < x^{(n)}, x^{(n)} >\right\}$ of differential polynomials by V. Denote the differential R-algebra generated by elements of the system V and the function Δ^{-1} by $R\left\{V, \Delta^{-1}\right\}$. Let $f\left\{x\right\} \in R\left\{V, \Delta^{-1}\right\}$. Then, using Equation (1) and $\Delta_x(t)^{-1} = \Delta_y(t)^{-1}$, we obtain

$$f\left\{x(t)\right\} = f\left\{y(t)\right\} \tag{2}$$

for all $t \in J$.

Denote the matrix $\left\|x^{'}(t)x^{(2)}(t) \dots x^{(n)}(t)\right\|$ by $A(x(t))$, where we consider $x^{(i)}(t)$ as a column-vector. We let $\frac{d}{dt}A(x(t)) = \left\|x^{(2)}(t)x^{(3)}(t) \dots x^{(n+1)}(t)\right\|$. Since $x(t)$ is non-singular, we have $det A(x(t)) = \left[x^{'}(t) \dots x^{(n)}(t)\right] \neq 0$ for all $t \in J$. Hence the matrix $A^{-1}(x(t))$ exists for all $t \in J$. We consider the matrix $A^{-1}(x(t))\frac{d}{dt}A(x(t)) = ||c^x_{ij}(t)||$. It is easy to see that

(a) $c^x_{j+1j}(t) = 1$ for all $t \in J$ and $1 \leq j \leq n-1$;

(b) $c^x_{ij}(t) = 0$ for all $t \in J$ and $j \neq n$, $i \neq j+1$, $1 \leq i \leq n$;

(c) $c^x_{in}(t) = \frac{\left[x^{'}(t) \dots x^{(i-1)}(t)x^{(n+1)}(t)x^{(i+1)}(t) \dots x^{(n)}(t)\right]}{\left[x^{'}(t) \dots x^{(n)}(t)\right]}$

for all $t \in J$ and $1 \leq i \leq n$.

Lemma 3 $c^x_{ij}(t) = c^y_{ij}(t)$ *for all* $t \in J$ *and* $1 \leq i \leq j \leq n$.

Proof. The above equality (a) implies $c^x_{j+1j}(t) = c^y_{j+1j}(t)$ for all $1 \leq j \leq n-1$ and the equality (b) implies $c^x_{ij}(t) = c^y_{ij}(t)$ for all $j \neq n$, $i \neq j+1$, $1 \leq i \leq n$. Prove $c^x_{in}(t) = c^y_{in}(t)$ for all $1 \leq i \leq n$. Using Lemma 2 to vectors $y_i = x^{(i)}(t), z_j = x^{(j)}(t)$ $(i, j = 1, \dots, n)$, we obtain

$$(-1)^p \left[x^{'}(t) \dots x^{(n)}(t)\right]^2 = det|| < x^{(i)}(t), x^{(j)}(t) > ||. \tag{3}$$

Similarly, using Lemma 2 to vectors $x^{'}, \ldots, x^{(i-1)}, x^{(n+1)}, x^{(i+1)}, \ldots, x^{(n)}, x^{'}, \ldots, x^{(n)}$, we have

$$(-1)^p \left[x^{'} \ldots x^{(i-1)} x^{(n+1)} x^{(i+1)} \ldots x^{(n)}\right] \left[x^{'} \ldots x^{(n)}\right] = det|| < x^{(k)}, x^{(l)} > ||, \tag{4}$$

where $k = 1, \ldots, i-1, n+1, i+1, \ldots, n; l = 1, 2, \ldots, n$. From Equation (3), Equation (4), Equation (1), Lemma 1 and the equality $c^x_{in}(t) =$

$$\frac{\left[x^{'} \ldots x^{(i-1)} x^{(n+1)} x^{(i+1)} \ldots x^{(n)}\right]}{\left[x^{'} \ldots x^{(n)}\right]} = \frac{(-1)^p \left[x^{'} \ldots x^{(i-1)} x^{(n+1)} x^{(i+1)} \ldots x^{(n)}\right] \left[x^{'} \ldots x^{(n)}\right]}{(-1)^p \left[x^{'} \ldots x^{(n)}\right]^2},$$

for $1 \leq i \leq n$, we obtain

$$\frac{\left[x^{'} \ldots x^{(i-1)} x^{(n+1)} x^{(i+1)} \ldots x^{(n)}\right]}{\left[x^{'} \ldots x^{(n)}\right]} = \frac{\left[y^{'} \ldots y^{(i-1)} y^{(n+1)} y^{(i+1)} \ldots y^{(n)}\right]}{\left[y^{'} \ldots y^{(n)}\right]}$$

for all $i = 1, \ldots, n$. The lemma is proved. □

Equation (1) and Lemma 3 implies $A^{-1}(x(t))\frac{d}{dt}A(x(t)) = A^{-1}(y(t))\frac{d}{dt}A(y(t))$ for all $t \in J$. The last equality implies

$$\frac{\partial}{\partial t}(A(y)A(x)^{-1}) = (\frac{\partial}{\partial t}A(y))A(x)^{-1} + A(y)\frac{\partial}{\partial t}(A(x)^{-1}) = (\frac{\partial}{\partial t}A(y))A(x)^{-1} -$$

$$A(y)A(x)^{-1}(\frac{\partial}{\partial t}A(x))A(x)^{-1} = A(y)(A(y)^{-1}\frac{\partial}{\partial t}A(y) - A(x)^{-1}\frac{\partial}{\partial t}A(x))A(x)^{-1} = 0.$$

for all $t \in J$. Using this equality and connectedness of J, we obtain that $A(y(t))A(x(t))^{-1}$ does not depend on $t \in J$. Put $F = A(y)A(x)^{-1}$. According to $detA(x(t)) \neq 0$ and $detA(y(t)) \neq 0$ for all $t \in J$, we have $detF \neq 0$ and $A(y(t)) = FA(x(t))$ for all $t \in J$. We prove that $F \in O(n,p)$.

Let $A(x)^\top$ be the transpose matrix of $A(x)$. Let $I_p = \|b_{ij}\|$ be the diagonal $n \times n$-matrix such that $b_{ii} = -1$ for all $i = 1, \ldots, p$ and $b_{jj} = 1$ for all $j = p+1, \ldots, n$. Using the equality $A(x)^\top I_p A(x) = \|< x^{(i)}, x^{(j)} >\|_{i,j=1,2,\ldots,n}$, Lemma 1 and Equation (1), we obtain that $A(x)^\top I_p A(x) = A(y)^\top I_p A(y)$. This equality and the equality $A(y) = FA(x)$ imply that $F^\top I_p F = I_p$. Hence $F \in O(n,p)$.

The equality $Ay(t) = FAx(t)$ implies $\frac{\partial}{\partial t}y(t) = F\frac{\partial}{\partial t}x(t)$ for all $t \in J$. Then there exists a constant vector $b \in E^n_p$ such that $y(t) = Fx(t) + b$ for all $t \in J$. The theorem is completed. □

Corollary 1 *Let α, β be non-singular curves in E^n_p and $x \in Ip(\alpha), y \in Ip(\beta)$. Assume that x, y satisfy the condition (5_1) in Definition 5. Then*

(i) in the case $L(\alpha) = L(\beta) \neq (-\infty, +\infty)$, $\alpha \overset{M(n,p)}{\sim} \beta$ if and only if

$$sgn < x^{'}(s), x^{'}(s) >= sgn < y^{'}(s), y^{'}(s) >, \tag{5}$$

$$< x^{(i)}(s), x^{(i)}(s) >=< y^{(i)}(s), y^{(i)}(s) > \tag{6}$$

for all $s \in L(\alpha)$ and $i = 2, \ldots, n$;

(ii) *in the case* $L(\alpha) = L(\beta) = (-\infty, +\infty)$, $\alpha \overset{M(n,p)}{\sim} \beta$ *if and only if*

$$sgn < x'(s), x'(s) >= sgn < y'(s), y'(s) >,$$
$$< x^{(i)}(s), x^{(i)}(s) >=< y^{(i)}(s+s_1), y^{(i)}(s+s_1) >$$

for some $s_1 \in (-\infty, +\infty)$, *all* $s \in L(\alpha)$ *and* $i = 2, \ldots, n$;

Proof. Let $\alpha \overset{M(n,p)}{\sim} \beta$. Then it is obvious that Equation (5) and Equation (6) hold. Conversely, assume that Equation (5) and Equation (6) hold. By Proposition 2, $\left|< x'(s), x'(s) >\right| = \left|< y'(s), y'(s) >\right| = 1$ for all $s \in L(\alpha)$. This equality and Equation (5) imply that $< x'(s), x'(s) >=< y'(s), y'(s) >$ for all $s \in L(\alpha)$. The last equality and Equation (6), by Theorem 2, imply $x \overset{M(n,p)}{\sim} y$. Applying Theorem 1, we obtain $\alpha \overset{M(n,p)}{\sim} \beta$. Similarly, the proof of statement (ii) follows from statement (ii) of Theorem 1. □

Remark 4 Similar results are true if x, y satisfy conditions (5_2) or (5_3)) in Definition 5.

Let α be a curve and $x \in Ip(\alpha)$.

Remark 5 According to Corollary 1 the system

$$\left\{L(\alpha), sgn < x', x' >, < x^{(2)}, x^{(2)} >, \ldots, < x^{(n)}, x^{(n)} >\right\}$$

is a complete system of $M(n,p)$-invariants of a curve α for the case $L(\alpha) \neq (-\infty, +\infty)$. But they are not invariants of a curve α for the case $L(\alpha) = (-\infty, +\infty)$. They depend on reparametrizations $s \to s + a$ of a curve α.

Let $\delta = \delta_x$ be the determinant of the matrix $|| < y_i, z_j > ||_{i,j=1,2,\ldots,n-1}$, where $y_1 = z_1 = x', y_2 = z_2 = x^{(2)}, \cdots, y_{n-1} = z_{n-1} = x^{(n-1)}$. Denote the system

$$\left\{< x', x' >, ..., < x^{(n-1)}, x^{(n-1)} >, \left[x'(t)x^{(2)}(t) \ldots x^{(n)}(t)\right]\right\}$$

of differential polynomials by Z. Denote the differential R-algebra generated by elements of Z by $R\{Z\}$.

Lemma 4 $< y_i, z_j >\in R\{Z\}$ *for all* $1 \leq i, j, i+j \leq 2n-1$ *and* $\delta \in R\{Z\}$.

Proof. Using Lemma 1, we get $< x^{(i)}, x^{(j)} >\in R\{Z\}$ for all $1 \leq i, j, i+j \leq 2n-1$. Since the element $< y_i, z_j >$ of the determinant δ is the function $< x^{(i)}, x^{(j)} >$, where $1 \leq i, j \leq n-1$, we obtain that $\delta \in R\{Z\}$. □

Theorem 3 *Assume that* $x(t)$ *and* $y(t)$ *be non-singular* J*-paths in* E_p^n *such that* $\delta_x(t) \neq 0$ *and* $\delta_y(t) \neq 0$ *for all* $t \in J$. *Then equalities*

$$< x^{(i)}(t), x^{(i)}(t) >=< y^{(i)}(t), y^{(i)}(t) >, \left[x'(t)x^{(2)}(t) \ldots x^{(n)}(t)\right] = \left[y'(t)y^{(2)}(t) \ldots y^{(n)}(t)\right] \quad (7)$$

for all $t \in J$ *and* $1 \leq i \leq j \leq n, i+j \leq 2n-1$ *implies* $x \overset{SM(n)}{\sim} y$.

Proof. Let $f\{x\} \in R\{Z\}$. Then Equation (7) implies

$$f\{x(t)\} = f\{y(t)\} \tag{8}$$

for all $t \in J$. By Lemma 4, $\delta_x \in R\{Z\}$. Hence Equation (8) implies $\delta_x = \delta_y$ for all $t \in J$. By the assumption of our theorem, we have $\delta_x \neq 0$ and $\delta_y \neq 0$ for all $t \in J$. Hence the equality $\delta_x = \delta_y$ for all $t \in J$ implies $\delta_x^{-1} = \delta_y^{-1}$ for all $t \in J$. Denote the differential R-algebra generated by elements of the system Z, the functions Δ^{-1} and δ^{-1} by $R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$. Let $f\{x\} \in R\left\{Z, \delta^{-1}, \Delta^{-1}\right\})$. Then the equality $\delta_x^{-1} = \delta_y^{-1}$, Equation (7) and Equation (8) imply

$$f\{x(u)\} = f\{y(u)\} \tag{9}$$

for all $t \in J$.

Lemma 5 $\Delta \in R\{Z\}$.

Proof. Using Lemma 2 to vectors $y_1 = z_1 = x^{'}, y_2 = z_2 = x^{(2)}, \cdots, y_n = z_n = x^{(n)}$, we obtain

$$(-1)^p \left[x^{'} x^{(2)} \ldots x^{(n)}\right]^2 = det \left\| < y_i, z_j > \right\|_{i,j=1,2,\ldots n} = \Delta. \tag{10}$$

Since $\left[x^{'} x^{(2)} \ldots x^{(n)}\right] \in Z$, we have $\Delta \in R\{Z\}$. □

Lemma 6 $< x^{(n)}, x^{(n)} > \in R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ *and* $R\left\{V, \Delta^{-1}\right\} \subset R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$.

Proof. For $i = 1, 2, \ldots, n$, denote the cofactor of the element $< y_n, z_j >$ of the matrix $A = \| < y_i, z_j > \|_{i,j=1,2,\ldots n}$ in Equation (10) by D_{ni}. Then we obtain the equality

$$\Delta = < y_n, z_1 > D_{n1} + < y_n, z_2 > D_{n2} + \cdots + < y_n, z_{n-1} > D_{nn-1} + < y_n, z_n > D_{nn}.$$

Since $\delta = D_{nn} \neq 0$, this equality implies

$$\begin{aligned} < y_n, z_n > = < x^{(n)}, x^{(n)} > = \Delta\delta^{-1} - < y_n, z_1 > D_{n1}\delta^{-1} - < y_n, z_2 > D_{n2}\delta^{-1} - \\ \cdots - < y_n, z_{n-1} > D_{nn-1}\delta^{-1}. \end{aligned} \tag{11}$$

By Lemma 1, we have $< y_n, z_j > = < x^{(n)}, x^{(j)} > \in R\{Z\}$ for each $1 \leq j \leq n-1$. We prove that $D_{ns} \in R\{Z\}$ for every $1 \leq s \leq n-1$. We have

$$D_{ns} = (-1)^{n+s} det \| < y_i, z_j \|_{i=1,2,\ldots,n-1; j=1,2,\ldots,s-1,s+1,\ldots,n}.$$

Elements of D_{ns} have forms $< y_i, z_j >$, $< y_i, z_n >$, where $i, j < n$. By $< y_i, z_j > \in R\{Z\}, < y_i, z_n > = < y_n, z_i > \in R\{Z\}$, we obtain $D_{ns} \in R\{Z\}$. Hence Equation (11) implies $< y_n, z_n > \in R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$. Using $V \subset Z \cup \{(y_n, z_n)\}$, we get $R\left\{V, \Delta^{-1}\right\} \subset R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$. □

Using Equations (7), (9)–(11) and $R\left\{V,\Delta^{-1}\right\}\subset R\left\{Z,\delta^{-1},\Delta^{-1}\right\}$ in Lemma 6, we obtain Equation (1). Hence, by Theorem 2, $F\in O(n,p)$ and $b\in E_p^n$ exist such that $y(u)=Fx(u)+b$. Using this equality and $\left[x^{'}(t)x^{(2)}(t)\dots x^{(n)}(t)\right]=\left[y^{'}(t)y^{(2)}(t)\dots y^{(n)}(t)\right]$ in Equation (7), we get $\left[x^{'}(t)x^{(2)}(t)\dots x^{(n)}(t)\right]=detF\left[x^{'}(t)x^{(2)}(t)\dots x^{(n)}(t)\right]$. Since $\left[x^{'}(t)x^{(2)}(t)\dots x^{(n)}(t)\right]\neq 0$ for all $t\in J$, we obtain $detF=1$. Hence $x\overset{SM(n)}{\sim}y$. The theorem is completed. □

Corollary 2 *Let α,β be non-singular curves in E_p^n and $x\in Ip(\alpha),y\in Ip(\beta)$. Assume that x,y satisfy the condition $(5_1))$ in Definition 5 and conditions $\delta_x(t)\neq 0$, $\delta_y(t)\neq 0$ for all $t\in J$. Then*

(i) in the case $L(\alpha)=L(\beta)\neq(-\infty,+\infty)$, $\alpha\overset{SM(n,p)}{\sim}\beta$ if and only if

$$\left[x^{'}(s)\dots x^{(n)}(s)\right]=\left[y^{'}(s)\dots y^{(n)}(s)\right], \tag{12}$$

$$sgn<x^{'}(s),x^{'}(s)>=sgn<y^{'}(s),y^{'}(s)>, \tag{13}$$

$$<x^{(i)}(s),x^{(i)}(s)>=<y^{(i)}(s),y^{(i)}(s)> \tag{14}$$

for all $s\in L(\alpha)$ and all $i=2,\dots,n-1$;

(ii) in the case $L(\alpha)=L(\beta)=(-\infty,+\infty)$, $\alpha\overset{SM(n,p)}{\sim}\beta$ if and only if

$$\left[x^{'}(s)\dots x^{(n)}(s)\right]=\left[y^{'}(s+s_1)\dots y^{(n)}(s+s_1)\right],$$

$$sgn<x^{'}(s),x^{'}(s)>=sgn<y^{'}(s),y^{'}(s)>,$$

$$<x^{(i)}(s),x^{(i)}(s)>=<y^{(i)}(s+s_1),y^{(i)}(s+s_1)>$$

for some $s_1\in(-\infty,+\infty)$, all $s\in L(\alpha)$ and $i=2,\dots,n-1$;

Proof. (i). Let $\alpha\overset{SM(n,p)}{\sim}\beta$. Since elements of Z and the function $sgn<x^{'}(s),x^{'}(s)>$ are $SM(n,p)$-invariant, we obtain that Equation (12)–(14) hold.

Conversely, assume that Equation (12)–(14) hold. According to Proposition 2, we get $\left|<x^{'}(s),x^{'}(s)>\right|=\left|<y^{'}(s),y^{'}(s)>\right|=1$ for all $s\in L(\alpha)$. Then, using Equation (13), we obtain $<x^{'}(s),x^{'}(s)>=<y^{'}(s),y^{'}(s)>$ for all $s\in L(\alpha)$. The latest equality, Equation (12) and Equation (14), by Lemmas 4 and 5, imply $\delta_x=\delta_y,\Delta_x=\Delta_y$. Then, by Lemma 6, we obtain $<x^{(n)},x^{(n)}>=<y^{(n)},y^{(n)}>$. By this equality, Equation (12), Equation (14) and Theorem 3, there exists $F\in SM(n,p)$ such that $y(s)=Fx(s)=gx(s)+b$. The proof of statement (i) is completed. Similarly, the proof of (ii) follows from statement (ii) of Theorem 1. □

Remark 6 Similar results are true for conditions $(5_2))$ or $(5_3))$ in Definition 5.

Let α be a curve and $x\in Ip(\alpha)$.

Remark 7 According to Corollary 2, the system

$$\left\{L(\alpha), sgn < x^{'}, x^{'} >, < x^{(2)}, x^{(2)} >, \ldots, < x^{(n-1)}, x^{(n-1)} >, \left[x^{'} x^{(2)} \ldots x^{(n)}\right]\right\}$$

is a complete system of $SM(n,p)$-invariants of a curve α for the case $L(\alpha) \neq (-\infty, +\infty)$. But they are not invariants of a curve α for the case $L(\alpha) = (-\infty, +\infty)$. They depend on reparametrizations $s \to s + a$ of the curve α.

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Received: 12.11.2009