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Small covers over products of a polygon with a simplex

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Abstract

The equivariant homeomorphism class of an (orientable) small cover over a simple convex polytope P^n bijectively corresponds to the equivalence class of its (orientable) coloring under the action of automorphism group of face poset of P^n. By calculating the number of orbits of group actions we determine the number of equivariant homeomorphism classes of small covers over products of a polygon with a simplex. Moreover, we calculate the number of equivariant homeomorphism classes of all orientable small covers over the product.

Key Words: Small cover, equivariant homeomorphism, polytope, coloring

1. Introduction

A small cover, defined by Davis and Januszkiewicz in [5], is a smooth closed manifold M^n with a locally standard $(\mathbb{Z}_2)^n$-action such that its orbit space is a simple convex polytope. This establishes a direct connection between equivariant topology and combinatorics. From [5], we know that the connected sum of some $RP(2)^{'}s$ is a small cover over the m-gon P_m and that real projective space $RP(n)$ is a small cover over the n-simplex Δ_n. Thus, their product is a small cover over $P_m \times \Delta_n$.

In [6], L$\ddot{u}$ and Masuda showed that the equivariant homeomorphism class of a small cover over a simple convex polytope P^n agrees with the equivalence class of its corresponding $(\mathbb{Z}_2)^n$-coloring under the action of automorphism group of face poset of P^n. This holds for orientable small covers by the orientability condition in [7] (see Theorem 5.3). But there are no general formulas to calculate the number of equivariant homeomorphism classes of (orientable) small covers over an arbitrary simple convex polytope.

In recent years, several studies have attempted to enumerate the number of equivalence classes of all small covers over a specific polytope. Cai, Chen and L$\ddot{u}$ calculated the number of equivariant homeomorphism classes of small covers over 3-dimensional prisms [2]. In 2008, S. Choi determined the number of equivariant homeomorphism classes of small covers over cubes [3]. There are few results about orientable small covers. S. Choi calculated the number of D-J equivalence classes of orientable small covers over cubes [4].

This paper gives a calculation formula of the number of equivariant homeomorphism classes of all small covers over $P_m \times \Delta_n$ (see Theorem 4.1). When n=1, $P_m \times \Delta_n$ is a 3-dimensional m-sided prism and the present result is the same as Theorem 4.1 in [2]. So our result is a generalization of Theorem 4.1 in [2]. Furthermore,

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we determine the number of equivariant homeomorphism classes of orientable small covers over $P_m \times \Delta_n$ (see Theorem 5.5).

This paper is organized as follows. In Section 2, we review the basic theory about small covers and calculate the automorphism group of face poset of $P_m \times \Delta_n$. In Section 3, we determine the number of all colorings on $P_m \times \Delta_n$, so that in Section 4 we obtain a calculation formula of the number of equivariant homeomorphism classes of all small covers over $P_m \times \Delta_n$. In Section 5, similarly we determine the number of equivariant homeomorphism classes of orientable small covers over $P_m \times \Delta_n$.

2. Preliminaries

A convex polytope P^n of dimension n is said to be simple if every vertex of P^n is the intersection of exactly n facets (i.e. faces of dimension $(n-1)$) [8]. An n-dimensional smooth closed manifold M^n is said to be a small cover if it admits a smooth $(\mathbb{Z}_2)^n$-action such that the action is locally isomorphic to a standard action of $(\mathbb{Z}_2)^n$ on R^n and the orbit space $M^n/(\mathbb{Z}_2)^n$ is a simple convex polytope of dimension n.

Let P^n be a simple convex polytope of dimension n and $\mathcal{F}(P^n)=\{F_1,\cdots,F_l\}$ be the set of facets of P^n. Suppose that $\pi: M^n \to P^n$ is a small cover over P^n. Then there are l connected submanifolds $\pi^{-1}(F_1),\cdots,\pi^{-1}(F_l)$. Each submanifold $\pi^{-1}(F_i)$ is fixed pointwise by a $\mathbb{Z}_2$-subgroup $\mathbb{Z}_2(F_i)$ of $(\mathbb{Z}_2)^n$, so that each facet F_i corresponds to the $\mathbb{Z}_2$-subgroup $\mathbb{Z}_2(F_i)$. Obviously, the $\mathbb{Z}_2$-subgroup $\mathbb{Z}_2(F_i)$ actually agrees with an element ν_i in $(\mathbb{Z}_2)^n$ as a vector space. For each face F of codimension u, since P^n is simple, there are u facets $F_{i_1},\cdots,F_{i_u}$ such that $F=F_{i_1}\cap\cdots\cap F_{i_u}$. Then, the corresponding submanifolds $\pi^{-1}(F_{i_1}),\cdots,\pi^{-1}(F_{i_u})$ intersect transversally in the $(n-u)$-dimensional submanifold $\pi^{-1}(F)$, and the isotropy subgroup $\mathbb{Z}_2(F)$ of $\pi^{-1}(F)$ is a subtorus of rank u and is generated by $\mathbb{Z}_2(F_{i_1}),\cdots,\mathbb{Z}_2(F_{i_u})$ (or is determined by $\nu_{i_1},\cdots,\nu_{i_u}$ in $(\mathbb{Z}_2)^n$). Thus, this actually gives a characteristic function [5]

$$\lambda : \mathcal{F}(P^n) \longrightarrow (\mathbb{Z}_2)^n$$

defined by $\lambda(F_i)=\nu_i$ such that, whenever the intersection $F_{i_1}\cap\cdots\cap F_{i_u}$ is non-empty, $\lambda(F_{i_1}),\cdots,\lambda(F_{i_u})$ are linearly independent in $(\mathbb{Z}_2)^n$. If we regard each nonzero vector of $(\mathbb{Z}_2)^n$ as being a color, then the characteristic function λ means that each facet is colored by a color. Here, we also call λ a $(\mathbb{Z}_2)^n$-coloring on P^n.

In fact, Davis and Januszkiewicz gave a reconstruction process of a small cover by using a $(\mathbb{Z}_2)^n$-coloring $\lambda : \mathcal{F}(P^n) \longrightarrow (\mathbb{Z}_2)^n$. Let $\mathbb{Z}_2(F_i)$ be the subgroup of $(\mathbb{Z}_2)^n$ generated by $\lambda(F_i)$. Given a point $p \in P^n$, by $F(p)$ we denote the minimal face containing p in its relative interior. Assume $F(p)=F_{i_1}\cap\cdots\cap F_{i_u}$ and $\mathbb{Z}_2(F(p))=\bigoplus_{j=1}^u \mathbb{Z}_2(F_{i_j})$. Note that $\mathbb{Z}_2(F(p))$ is a u-dimensional subgroup of $(\mathbb{Z}_2)^n$. Let $M(\lambda)$ denote $P^n \times (\mathbb{Z}_2)^n/\sim$, where $(p,g)\sim(q,h)$ if $p=q$ and $g^{-1}h \in \mathbb{Z}_2(F(p))$. The free action of $(\mathbb{Z}_2)^n$ on $P^n \times (\mathbb{Z}_2)^n$ descends to an action on $M(\lambda)$ with quotient P^n. Thus $M(\lambda)$ is a small cover over P^n [5].

Two small covers M_1 and M_2 over P^n are said to be weakly equivariantly homeomorphic if there is an automorphism $\varphi : (\mathbb{Z}_2)^n \to (\mathbb{Z}_2)^n$ and a homeomorphism $f : M_1 \to M_2$ such that $f(t\cdot x)=\varphi(t)\cdot f(x)$ for every $t \in (\mathbb{Z}_2)^n$ and $x \in M_1$. If φ is an identity, then M_1 and M_2 are equivariantly homeomorphic. Following [5], two small covers M_1 and M_2 over P^n are said to be Davis-Januszkiewicz equivalent (or simply, D-J equivalent) if there is a weakly equivariant homeomorphism $f : M_1 \to M_2$ covering the identity on P^n.

By $\Lambda(P^n)$ we denote the set of all $(\mathbb{Z}_2)^n$-colorings on P^n. Then we have the following theorem

Theorem 2.1 *(Davis-Januszkiewicz). All small covers over P^n are given by $\{M(\lambda)|\lambda \in \Lambda(P^n)\}$, i.e. for each small cover M^n over P^n, there is a $(\mathbb{Z}_2)^n$-coloring λ with an equivariant homeomorphism $M(\lambda) \longrightarrow M^n$ covering the identity on P^n.*

Remark 1 Generally speaking, we cannot be sure that there always exist $(\mathbb{Z}_2)^n$-colorings over a simple convex polytope P^n when $n \geq 4$ (see [5, Nonexample 1.22]).

There is a natural action of $GL(n, \mathbb{Z}_2)$ on $\Lambda(P^n)$ defined by the correspondence $\lambda \longmapsto \sigma \circ \lambda$, and the action on $\Lambda(P^n)$ is free. Without loss of generality, we assume that $F_1, \cdots, F_n$ of $\mathcal{F}(P^n)$ meet at one vertex p of P^n. Let $e_1, \cdots, e_n$ be the standard basis of $(\mathbb{Z}_2)^n$. Write $A(P^n) = \{\lambda \in \Lambda(P^n)|\lambda(F_i) = e_i, i = 1, \cdots, n\}$. In fact, $A(P^n)$ is the orbit space of $\Lambda(P^n)$ under the action of $GL(n, \mathbb{Z}_2)$. Then we have this lemma:

Lemma 2.2 $|\Lambda(P^n)| = |A(P^n)| \times |GL(n, \mathbb{Z}_2)|$.

Note that we know from [1] that $|GL(n, \mathbb{Z}_2)| = \prod_{k=1}^{n} \left(2^n - 2^{k-1}\right)$. Two small covers $M(\lambda_1)$ and $M(\lambda_2)$ over P^n are D-J equivalent if and only if there is $\sigma \in GL(n, \mathbb{Z}_2)$ such that $\lambda_1 = \sigma \circ \lambda_2$. So the number of D-J equivalence classes of small covers over P^n is $|A(P^n)|$.

Let P^n be a simple convex polytope of dimension n. All faces of P^n form a poset (i.e., a partially ordered set by inclusion). An automorphism of $\mathcal{F}(P^n)$ is a bijection from $\mathcal{F}(P^n)$ to itself which preserves the poset structure of all faces of P^n, and by $Aut(\mathcal{F}(P^n))$ we denote the group of automorphisms of $\mathcal{F}(P^n)$. One can define the right action of $Aut(\mathcal{F}(P^n))$ on $\Lambda(P^n)$ by $\lambda \times h \longmapsto \lambda \circ h$, where $\lambda \in \Lambda(P^n)$ and $h \in Aut(\mathcal{F}(P^n))$. The following theorem is well known [6].

Theorem 2.3 *Two small covers over an n-dimensional simple convex polytope P^n are equivariantly homeomorphic if and only if there is $h \in Aut(\mathcal{F}(P^n))$ such that $\lambda_1 = \lambda_2 \circ h$, where λ_1 and λ_2 are their corresponding $(\mathbb{Z}_2)^n$-colorings on P^n.*

So the number of orbits of $\Lambda(P^n)$ under the action of $Aut(\mathcal{F}(P^n))$ is just the number of equivariant homeomorphism classes of small covers over P^n. Thus, we are going to count the orbits. Burnside Lemma is very useful in the enumeration of the number of orbits.

Burnside Lemma *Let G be a finite group acting on a set X. Then the number of orbits X under the action of G equals to $\frac{1}{|G|}\sum_{g\in G}|X_g|$, where $X_g = \{x \in X|gx = x\}$.*

Burnside Lemma suggests that, in order to determine the number of the orbits of $\Lambda(P^n)$ under the action of $Aut(\mathcal{F}(P^n))$, we need to understand the structure of $Aut(\mathcal{F}(P^n))$. As stated in Section 1, we shall particularly be concerned with the case in which the simple convex polytope is $P_m \times \Delta_n$.

To be convenient, we introduce the following notation. By $F_1', \cdots, F_m'$ we denote all edges of the m-gon P_m in their general order, and by $F_{m+1}', \cdots, F_{m+n+1}'$ we denote all facets of the n-simplex Δ_n. Set $\mathcal{F}' = \{F_i = F_i' \times \Delta_n | 1 \leq i \leq m\}$, $\mathcal{F}'' = \{F_i = P_m \times F_i' | m+1 \leq i \leq m+n+1\}$. Then $\mathcal{F}(P_m \times \Delta_n) = \mathcal{F}' \bigcup \mathcal{F}''$.

Next, we determine the automorphism group of $\mathcal{F}(P_m \times \Delta_n)$.

Lemma 2.4 *Let P_m, Δ_n be m-gon and n-simplex respectively. The automorphism group $Aut(\mathcal{F}(P_m \times \Delta_n))$*

is isomorphic to

$$\begin{cases} (\mathbb{Z}_2)^3 \times S_3, & n = 1 \text{ and } m = 4, \\ S_3 \times S_3 \times \mathbb{Z}_2, & n = 2 \text{ and } m = 3, \\ D_m \times S_{n+1}, & n = 1 \text{ and } m \neq 4, n = 2 \text{ and } m \neq 3, \text{ or } n \geq 3, \end{cases}$$

where D_m is the dihedral group of order $2m$ and S_{n+1} is the symmetric group on $n+1$ symbols.

Proof. When n=1 and m=4, $P_m \times \Delta_n$ is a 3-cube I^3. Obviously, the automorphism group $Aut(\mathcal{F}(I^3))$ contains a symmetric group S_3 since there is exactly one automorphism for each permutation of the three pairs of opposite sides of I^3. All elements of $Aut(\mathcal{F}(I^3))$ can be written in a simple form as follows: $\chi_1^{e_1}\chi_2^{e_2}\chi_3^{e_3} \cdot u$, where $e_1, e_2, e_3 \in \mathbb{Z}_2$, with reflections χ_1, χ_2, χ_3 and $u \in S_3$. Thus, the automorphism group $Aut(\mathcal{F}(I^3))$ is isomorphic to $(\mathbb{Z}_2)^3 \times S_3$. In fact, $Aut(\mathcal{F}(I^3))$ has three copies of $D_4 \times \mathbb{Z}_2$ as subgroups.

When n=2 and m=3, $P_m \times \Delta_n$ is $\Delta_2 \times \Delta_2$. In this case, $\mathcal{F}' = \{F_i = F_i' \times \Delta_2 | 1 \leq i \leq 3\}$, $\mathcal{F}'' = \{F_i = \Delta_2 \times F_i' | 4 \leq i \leq 6\}$ and $\mathcal{F}(\Delta_2 \times \Delta_2) = \mathcal{F}' \bigcup \mathcal{F}''$. Let the facets in $\mathcal{F}'$ interchange and the facets in $\mathcal{F}''$ stay unchanged. Then these automorphisms form a group S_3. Let the facets in $\mathcal{F}''$ interchange and the facets in $\mathcal{F}'$ stay unchanged. Then these automorphisms also form a group S_3. We obtain a new group $S_3 \times S_3$, each of which is an automorphism under which the facets in $\mathcal{F}'$ and $\mathcal{F}''$ are mapped to $\mathcal{F}'$ and $\mathcal{F}''$ respectively. We choose an automorphism f such that $f(F_i) = F_{i+3}$ for $1 \leq i \leq 3$ and $f(F_i) = F_{i-3}$ for $4 \leq i \leq 6$. Let $\mathbb{Z}_2 = \{f, 1\}$. Then we again get a new group $S_3 \times S_3 \times \mathbb{Z}_2$, each of which is an automorphism under which the facets in $\mathcal{F}'$ and $\mathcal{F}''$ are mapped to $\mathcal{F}'$ and $\mathcal{F}''$ or to $\mathcal{F}''$ and $\mathcal{F}'$ respectively. In fact, $Aut(\mathcal{F}(\Delta_2 \times \Delta_2))$ is just $S_3 \times S_3 \times \mathbb{Z}_2$ because other bijections from $\mathcal{F}(\Delta_2 \times \Delta_2)$ to itself don't preserve the poset structure of all faces of $\Delta_2 \times \Delta_2$.

When n=1 and $m \neq 4$, n=2 and $m \neq 3$, or $n \geq 3$, the facets in $\mathcal{F}'$ and $\mathcal{F}''$ are mapped to $\mathcal{F}'$ and $\mathcal{F}''$ respectively under automorphisms of $Aut(\mathcal{F}(P_m \times \Delta_n))$. Since the automorphism group $Aut(\mathcal{F}(P_m))$ is isomorphic to D_m and $Aut(\mathcal{F}(\Delta_n))$ is isomorphic to S_{n+1}, $Aut(\mathcal{F}(P_m \times \Delta_n))$ is isomorphic to $D_m \times S_{n+1}$. $\square$

Remark 2 Let x, y, z be automorphisms in $Aut(\mathcal{F}(P_m \times \Delta_n))$ with the following properties, respectively:

(1) $x(F_i) = F_{i+1} (i = 1, 2, \cdots, m-1), x(F_m) = F_1, x(F_j) = F_j, m+1 \leq j \leq m+n+1$;

(2) $y(F_i) = F_{m+1-i} (i = 1, 2, \cdots, m), y(F_j) = F_j, m+1 \leq j \leq m+n+1$;

(3) $z(F_i) = F_i (i = 1, 2, \cdots, m), z(F_j) \in \mathcal{F}'', m+1 \leq j \leq m+n+1$.

Then, when n=1 and $m \neq 4$, n=2 and $m \neq 3$, or $n \geq 3$, all automorphisms in $Aut(\mathcal{F}(P_m \times \Delta_n))$ can be written in the simple form

$$x^u y^v z, \tag{1}$$

with $x^m = y^2 = 1$ and $x^u y = y x^{m-u}$.

3. Colorings on $P_m \times \Delta_n$

This section is devoted to calculating the number of $(\mathbb{Z}_2)^{n+2}$-colorings on $P_m \times \Delta_n$.

Theorem 3.1 *By $\mathbb{N}$ we denote the set of natural numbers. Let a, b, c be the functions from $\mathbb{N}\times\mathbb{N}$ to $\mathbb{N}$ with the following properties:*

(1) $a(j,n) = 2^n a(j-1,n) + 2^{2n+1} a(j-2,n)$ *with* $a(1,n) = 1, a(2,n) = 2^n$*;*

(2) $b(j,n) = b(j-1,n) + 2^{n+1} b(j-2,n)$ *with* $b(1,n) = b(2,n) = 1$*;*

(3) $c(j,n) = 2c(j-1,n) + 2^{n+1}c(j-2,n) - (2^{n+1}+2)c(j-3,n) - (2^{n+1}-1)c(j-4,n) + 2^{n+1}c(j-5,n)$ *with* $c(1,n) = c(2,n) = 1, c(3,n) = 3, c(4,n) = 2^{n+1}+3, c(5,n) = 3\times 2^{n+1}+5$.

Then the number of $(\mathbb{Z}_2)^{n+2}$*-colorings over* $P_m \times \Delta_n$ *is*

$$|\Lambda(P_m \times \Delta_n)| = \prod_{k=1}^{n+2}(2^{n+2}-2^{k-1})[a(m-1,n) + 2b(m-1,n) + c(m-1,n)].$$

Proof. Let $e_1, e_2, \cdots, e_{n+2}$ be the standard basis of $(\mathbb{Z}_2)^{n+2}$, then $(\mathbb{Z}_2)^{n+2}$ contains $2^{n+2}-1$ nonzero elements (or $2^{n+2}-1$ colors). We choose F_1, F_2 from $\mathcal{F}'$ and $F_{m+1}, \cdots, F_{m+n}$ from $\mathcal{F}''$ such that $F_1, F_2, F_{m+1}, \cdots, F_{m+n}$ meet at one vertex of $P_m \times \Delta_n$. Then

$$A(P_m \times \Delta_n) = \{\lambda \in \Lambda(P_m \times \Delta_n) | \lambda(F_1) = e_1, \lambda(F_2) = e_2, \lambda(F_i) = e_{i-m+2}, m+1 \le i \le m+n\}.$$

By Lemma 2.2, we have that

$$|\Lambda(P_m \times \Delta_n)| = |A(P_m \times \Delta_n)| \times |GL(n+2,\mathbb{Z}_2)| = \prod_{k=1}^{n+2}(2^{n+2}-2^{k-1})|A(P_m \times \Delta_n)|.$$

In order to find those facets which have been colored and which meet at one vertex of $P_m \times \Delta_n$ with F_{m+n+1}, we choose F_1, F_2 from $\mathcal{F}'$ and arbitrary $n-1$ facets from $\mathcal{F}''$ which aren't F_{m+n+1}. By the linear independence condition of characteristic functions, the calculation of $|A(P_m \times \Delta_n)|$ is divided into four cases. Write

$$\begin{aligned}
A_0(P_m \times \Delta_n) &= \{\lambda \in A(P_m \times \Delta_n) | \lambda(F_{m+n+1}) = e_3 + \cdots + e_{n+2}\},\\
A_1(P_m \times \Delta_n) &= \{\lambda \in A(P_m \times \Delta_n) | \lambda(F_{m+n+1}) = e_3 + \cdots + e_{n+2} + e_1\},\\
A_2(P_m \times \Delta_n) &= \{\lambda \in A(P_m \times \Delta_n) | \lambda(F_{m+n+1}) = e_3 + \cdots + e_{n+2} + e_2\},\\
A_3(P_m \times \Delta_n) &= \{\lambda \in A(P_m \times \Delta_n) | \lambda(F_{m+n+1}) = e_3 + \cdots + e_{n+2} + e_1 + e_2\}.
\end{aligned}$$

Then we have that $|A(P_m \times \Delta_n)| = \sum_{i=0}^{3} |A_i(P_m \times \Delta_n)|$. Our argument is divided into the following cases.

Case 1. Calculation of $|A_0(P_m \times \Delta_n)|$.

By the linear independence condition of characteristic functions, we see that $\lambda(F_m) = e_2$ or $\lambda(F_m) = e_2 + e_{k_1} + e_{k_2} + \cdots + e_{k_i}, 1 \le k_1 < k_2 < \cdots < k_i \le n+2, k_1 \ne 2, k_2 \ne 2, \cdots, k_i \ne 2$ and $1 \le i \le n+1$. Set $A_0^0(P_m \times \Delta_n) = \{\lambda \in A_0(P_m \times \Delta_n) | \lambda(F_{m-1}) = e_1 + e_{m_1} + \cdots + e_{m_j}, 3 \le m_1 < \cdots < m_j \le n+2, 0 \le j \le n\}$ and $A_0^1(P_m \times \Delta_n) = A_0(P_m \times \Delta_n) - A_0^0(P_m \times \Delta_n)$. Take a coloring λ in $A_0^0(P_m \times \Delta_n)$. Then $\lambda(F_{m-2}), \lambda(F_m) \in \{e_2 + e_{k_1} + e_{k_2} + \cdots + e_{k_i}, 1 \le k_1 < k_2 < \cdots < k_i \le n+2, k_1 \ne 2, k_2 \ne 2, \cdots, k_i \ne 2$ and $0 \le i \le n+1\}$. In this case, we see that the values of λ restricted to F_{m-1} and F_m have 2^{2n+1} possible choices. Thus, $|A_0^0(P_m \times \Delta_n)| = 2^{2n+1}|A_0(P_{m-2} \times \Delta_n)|$. Take a coloring λ in $A_0^1(P_m \times \Delta_n)$. Then $\lambda(F_{m-1}) = e_2$ or $\lambda(F_{m-1}) = e_2 + e_{k_1} + e_{k_2} + \cdots + e_{k_i}, 1 \le k_1 < k_2 < \cdots < k_i \le n+2, k_1 \ne 2, k_2 \ne 2, \cdots, k_i \ne 2$ and $1 \le i \le n+1$. In this case, if we fix any value of $\lambda(F_{m-1})$, then $\lambda(F_m)$ has 2^n possible values. Thus $|A_0^1(P_m\times\Delta_n)| = 2^n|A_0(P_{m-1}\times\Delta_n)|$. Further, we have that

$$|A_0(P_m\times\Delta_n)| = 2^n|A_0(P_{m-1}\times\Delta_n)| + 2^{2n+1}|A_0(P_{m-2}\times\Delta_n)|.$$

A direct observation shows that when m=2, $|A_0(P_m\times\Delta_n)| = 1$, and when m=3, $|A_0(P_m\times\Delta_n)| = 2^n$. Thus, we have that $|A_0(P_m\times\Delta_n)| = a(m-1,n)$.

Case 2. Calculation of $|A_1(P_m\times\Delta_n)|$

Similarly to Case 1, set $A_1^0(P_m\times\Delta_n) = \{\lambda\in A_1(P_m\times\Delta_n)|\lambda(F_{m-1}) = e_1\}$ and $A_1^1(P_m\times\Delta_n) = A_1(P_m\times\Delta_n) - A_1^0(P_m\times\Delta_n)$. Take a coloring λ in $A_1^0(P_m\times\Delta_n)$. We have $\lambda(F_{m-2}),\lambda(F_m)\in\{e_2+e_{k_1}+e_{k_2}+\cdots+e_{k_i}, 1\le k_1<k_2<\cdots<k_i\le n+2, k_1\ne 2, k_2\ne 2,\cdots,k_i\ne 2$ and $0\le i\le n+1\}$. Thus, $|A_1^0(P_m\times\Delta_n)| = 2^{n+1}|A_1(P_{m-2}\times\Delta_n)|$. Take a coloring λ in $A_1^1(P_m\times\Delta_n)$. We then have $\lambda(F_{m-1}) = e_2$ or $\lambda(F_{m-1}) = e_2+e_{k_1}+e_{k_2}+\cdots+e_{k_i}, 1\le k_1<k_2<\cdots<k_i\le n+2, k_1\ne 2, k_2\ne 2,\cdots,k_i\ne 2$ and $1\le i\le n+1$. But $\lambda(F_m)$ has only one possible value whichever possible value of $\lambda(F_{m-1})$ is chosen, so $|A_1^1(P_m\times\Delta_n)| = |A_1(P_{m-1}\times\Delta_n)|$. Also, we see that $|A_1(P_2\times\Delta_n)| = |A_1(P_3\times\Delta_n)| = 1$. Thus, $|A_1(P_m\times\Delta_n)| = b(m-1,n)$.

Case 3. Calculation of $|A_2(P_m\times\Delta_n)|$

If we interchange e_1 and e_2, then the problem is reduced to Case 2, so $|A_2(P_m\times\Delta_n)| = b(m-1,n)$.

Case 4. Calculation of $|A_3(P_m\times\Delta_n)|$

In this case, $\lambda(F_m) = e_2$ or e_2+e_1. Set $A_3^0(P_m\times\Delta_n) = \{\lambda\in A_3(P_m\times\Delta_n)|\lambda(F_{m-1}) = e_1\}$, $A_3^1(P_m\times\Delta_n) = \{\lambda\in A_3(P_m\times\Delta_n)|\lambda(F_{m-1}) = e_2$ or $e_2+e_1\}$, and $A_3^2(P_m\times\Delta_n) = \{\lambda\in A_3(P_m\times\Delta_n)|\lambda(F_{m-1}) = e_1+e_{m_1}+\cdots+e_{m_j}$ or $e_2+e_{m_1}+\cdots+e_{m_j}, 3\le m_1<\cdots<m_j\le n+2, 1\le j\le n\}$. Then $|A_3(P_m\times\Delta_n)| = |A_3^0(P_m\times\Delta_n)| + |A_3^1(P_m\times\Delta_n)| + |A_3^2(P_m\times\Delta_n)|$. An easy argument shows that $|A_3^0(P_m\times\Delta_n)| = 2|A_3(P_{m-2}\times\Delta_n)|$ and $|A_3^1(P_m\times\Delta_n)| = |A_3(P_{m-1}\times\Delta_n)|$, so

$$|A_3(P_m\times\Delta_n)| = |A_3(P_{m-1}\times\Delta_n)| + 2|A_3(P_{m-2}\times\Delta_n)| + |A_3^2(P_m\times\Delta_n)|. \tag{2}$$

Set $B(m,n) = \{\lambda\in A_3^2(P_m\times\Delta_n)|\lambda(F_{m-2}) = e_2+e_1\}$. Then it is easy to see that

$$|A_3^2(P_m\times\Delta_n)| = |A_3^2(P_{m-1}\times\Delta_n)| + |B(m,n)| \tag{3}$$

and

$$\begin{aligned}|B(m,n)| \quad = \quad & (2^{n+1}-2)|A_3^2(P_{m-2}\times\Delta_n)| + (2^{n+1}-2)|A_3(P_{m-4}\times\Delta_n)| + (2^{n+1}-2)\\ & |A_3(P_{m-5}\times\Delta_n)| + |B(m-2,n)|\end{aligned} \tag{4}$$

Combining equations (2),(3) and (4), we obtain

$$\begin{aligned}A_3(P_m\times\Delta_n)| \quad = \quad & 2|A_3(P_{m-1}\times\Delta_n)| + 2^{n+1}|A_3(P_{m-2}\times\Delta_n)| - (2^{n+1}+2)|A_3(P_{m-3}\times\\ & \Delta_n)| - (2^{n+1}-1)|A_3(P_{m-4}\times\Delta_n)| + 2^{n+1}|A_3(P_{m-5}\times\Delta_n)|.\end{aligned}$$

A direct observation gives that $|A_3(P_2\times\Delta_n)| = |A_3(P_3\times\Delta_n)| = 1, |A_3(P_4\times\Delta_n)| = 3, |A_3(P_5\times\Delta_n)| = 2^{n+1}+3$, and $|A_3(P_6\times\Delta_n)| = 3\times 2^{n+1}+5$. Thus, we have $|A_3(P_m\times\Delta_n)| = c(m-1,n)$. □

Remark 3 From Theorem 3.1 we know that the number of D-J equivalence classes of small covers over $P_m \times \Delta_n$ is $a(m-1,n) + 2b(m-1,n) + c(m-1,n)$.

4. The number of equivariant homeomorphism classes

In this section, we determine the number of equivariant homeomorphism classes of all small covers over $P_m \times \Delta_n$.

Theorem 4.1 *Let φ denote the Euler's totient function, that is, $\varphi(1) = 1$ and $\varphi(N)$ for a positive integer $N (N \geq 2)$ is the number of positive integers both less than N and coprime to N. Let $E(P_m \times \Delta_n)$ denote the number of equivariant homeomorphism classes of small covers over $P_m \times \Delta_n$. Then $E(P_m \times \Delta_n)$ is equal to*

$$\begin{cases} \frac{1}{2m(n+1)!}\{\sum\limits_{t>1,t|m} \varphi(\frac{m}{t})|\Lambda(P_t \times \Delta_n)| + \frac{m}{2}\prod\limits_{k=1}^{n+2}(2^{n+2} - 2^{k-1})[\rho_1(m,n) + \rho_2(m,n)]\}, & \\ & n = 2 \text{ and } m \neq 3, \text{ or } n \geq 3, \\ \frac{1}{4m}\{\sum\limits_{t>1,t|m} \varphi(\frac{m}{t})[|\Lambda(P_t \times I| + 168a(t-1,1)] + 84m\rho_1(m,1) + 168m\rho_2(m,1)\}, & \\ & n = 1 \text{ and } m \neq 4, \\ 1960, & n = 2 \text{ and } m = 3, \\ 259, & n = 1 \text{ and } m = 4, \end{cases}$$

where $\rho_1(m,n)$ is defined recursively as follows:

$$\rho_1(m,n) = \begin{cases} 0, & m \text{ odd}, \\ 3, & m = 2, \\ 2^{n+1} + 4, & m = 4, \\ \rho_1(m-2,n) + 2^{n+1}\rho_1(m-4,n), & m \geq 6 \text{ and } m \text{ even}; \end{cases}$$

and

$$\rho_2(m,n) = \begin{cases} 0, & m \text{ odd}, \\ (2^{n+1})^{\frac{m}{2}-1}, & m \text{ even}. \end{cases}$$

Proof. From Theorem 2.3, and Burnside Lemma and Lemma 2.4, we have that

$$E(P_m \times \Delta_n) = \begin{cases} \frac{1}{2m(n+1)!}\sum_{g \in Aut(\mathcal{F}(P_m \times \Delta_n))} |\Lambda_g|, & n = 1 \text{ and } m \neq 4, n = 2 \text{ and } m \neq 3, \\ & \text{or } n \geq 3, \\ \frac{1}{72}\sum_{g \in Aut(\mathcal{F}(\Delta_2 \times \Delta_2))} |\Lambda_g|, & n = 2 \text{ and } m = 3, \\ \frac{1}{48}\sum_{g \in Aut(\mathcal{F}(I^3))} |\Lambda_g|, & n = 1 \text{ and } m = 4, \end{cases}$$

where $\Lambda_g = \{\lambda \in \Lambda(P_m \times \Delta_n) | \lambda = \lambda \circ g\}$.

When n=2 and $m \neq 3$, or $n \geq 3$, by (1) each automorphism g of $Aut(\mathcal{F}(P_m \times \Delta_n))$ can be written as $x^u y^v z$, and the argument is divided into the following cases.

Case 1. $g = x^u$.

Let $t = gcd(u, m)$ (the greatest common divisor of u and m). Then all facets in $\mathcal{F}'$ are divided into t orbits under the action of g, and each orbit contains $\frac{m}{t}$ facets. Thus, each $(\mathbb{Z}_2)^{n+2}$-coloring of Λ_g gives the same coloring on all $\frac{m}{t}$ facets of each orbit. This means that if $t \neq 1, |\Lambda_g| = |\Lambda(P_t \times \Delta_n)|$. If $t = 1$, then all facets in $\mathcal{F}'$ have the same coloring, which is impossible by the definition of $(\mathbb{Z}_2)^{n+2}$-colorings. On the other hand, for every $t > 1$, there are exactly $\varphi(\frac{m}{t})$ automorphisms of the form x^u, each of which divides all facets in $\mathcal{F}'$ into t orbits. Thus, when $g = x^u$,

$$\sum_{g=x^u} |\Lambda_g| = \sum_{t>1, t|m} \varphi(\frac{m}{t})|\Lambda(P_t \times \Delta_n)|.$$

Case 2. $g = x^u z (z \neq 1)$.

In this case, there exist j_1, j_1' such that (1) $j_1 \neq j_1', m+1 \leq j_1, j_1' \leq m+n+1$ and (2) $g(F_{j_1}) = F_{j_1'}$. Then F_{j_1} and $F_{j_1'}$ have the same coloring, which contradicts the definition of $(\mathbb{Z}_2)^{n+2}$-colorings. Thus, for each such an automorphism g, Λ_g is empty.

Case 3. $g = x^u yz$ with m odd.

Since m is odd, each automorphism always gives an interchange between two neighborly facets in $\mathcal{F}'$, so the two neighborly facets have the same coloring, which contradicts the definition of $(\mathbb{Z}_2)^{n+2}$-colorings. Thus, Λ_g is empty.

Case 4. $g = x^u yz$ with u even and m even.

Let $l = \frac{m-u-2}{2}$. Then it is easy to see that such an automorphism gives an interchange between two neighborly facets F_l and F_{l+1}, so both facets F_l and F_{l+1} have the same coloring. Thus, Λ_g is empty.

Case 5. $g = x^u y$ with u odd and m even.

Since each automorphism $g = x^u y$ contains y as its factor and u is odd, each coloring λ of Λ_g is equivalent to coloring only $\frac{m}{2}+1$ neighborly facets in $\mathcal{F}'$ and all facets in $\mathcal{F}''$. We shall show that for each $g = x^u y$, the number of all colorings in Λ_g is just

$$|\Lambda_g| = \prod_{k=1}^{n+2}(2^{n+2} - 2^{k-1})[\rho_1(m,n) + \rho_2(m,n)], \tag{5}$$

where $\rho_1(m,n)$ and $\rho_2(m,n)$ are stated as in Theorem 4.1. It is easy to see that there are exactly $\frac{m}{2}$ such automorphisms $g = x^u y$ since m is even and u is odd, so

$$\sum_{g=x^u y} |\Lambda_g| = \frac{m}{2} \prod_{k=1}^{n+2}(2^{n+2} - 2^{k-1})[\rho_1(m,n) + \rho_2(m,n)].$$

Now let us show equality (5) as follows.

Actually, the method of Case 1 of Theorem 3.1 can still be carried out here. Also, it suffices to consider the case $g = x^{m-1}y$ (i.e. $g = yx$) since there is no essential difference between this case and other cases. Set $X_1(m,n) = \{\lambda \in \Lambda_g | \lambda(F_{m+n+1}) \neq \lambda(F_{m+1}) + \cdots + \lambda(F_{m+n})\}$ and $X_2(m,n) = \Lambda_g - X_1(m,n)$. Then, by $X_1^0(m,n)$ we denote the set $\{\lambda \in X_1(m,n) | \lambda(F_m), \lambda(F_2), \lambda(F_{k_1}), \cdots, \lambda(F_{k_n})$ are linearly independent, $m+1 \leq k_1 < \cdots < k_n \leq m+n+1\}$, and by $X_1^1(m,n)$ we denote $X_1(m,n) - X_1^0(m,n)$. Similarly to the argument of Case 1 of Theorem 3.1, we have that $|X_1^0(m,n)| = |X_1(m-2,n)|$ and $|X_1^1(m,n)| = 2^{n+1}|X_1(m-4,n)|$ with initial values $|X_1(2,n)| = 3\prod_{k=1}^{n+2}(2^{n+2}-2^{k-1})$ and $|X_1(4,n)| = (2^{n+1}+4)\prod_{k=1}^{n+2}(2^{n+2}-2^{k-1})$. Thus, $|X_1(m,n)| = \prod_{k=1}^{n+2}(2^{n+2}-2^{k-1})\rho_1(m,n)$. For $X_2(m,n)$, in a similar way we may obtain $|X_2(m,n)| = 2^{n+1}|X_2(m-2,n)|$ with $|X_2(2,n)| = \prod_{k=1}^{n+2}(2^{n+2}-2^{k-1})$, which is exactly $\prod_{k=1}^{n+2}(2^{n+2}-2^{k-1})\rho_2(m,n)$.

Case 6. $g = x^u yz (z \neq 1)$ with u odd and m even.

Just as Case 2, Λ_g is empty.

Combing Cases 1-6, we complete the proof for n=2 and $m \neq 3$ or $n \geq 3$.

When n=1 and $m \neq 4$, using the method above, we easily give the proof. This result is the same as Theorem 4.1 of [2].

When n=2 and m=3, the automorphism group $Aut(\mathcal{F}(\Delta_2 \times \Delta_2))$ is isomorphic to $S_3 \times S_3 \times \mathbb{Z}_2$. By the linear independence condition of characteristic functions, we know that Λ_g is empty when g isn't unit element of the automorphism group $Aut(\mathcal{F}(\Delta_2 \times \Delta_2))$. Thus, from Theorem 3.1, we have

$$E(\Delta_2 \times \Delta_2) = \frac{1}{72}|\Lambda(\Delta_2 \times \Delta_2)| = 1960.$$

When n=1 and m=4, then $P_m \times \Delta_n$ is a 3-cube I^3. The automorphism group $Aut(\mathcal{F}(I^3))$ is isomorphic to $(\mathbb{Z}_2)^3 \times S_3$, and it has three copies of $D_4 \times \mathbb{Z}_2$ as subgroups. Similarly we can determine the case of the action of a subgroup $D_4 \times \mathbb{Z}_2$ of $Aut(\mathcal{F}(I^3))$ on I^3. However, each of other 32 automorphisms in $Aut(\mathcal{F}(I^3))$ has no fixed coloring in $\Lambda(I^3))$ since it maps top facet(or bottom facet) to a sided facet. Thus

$$E(I^3) = \frac{1}{48}\{\sum_{t=2,4}\varphi(\frac{4}{t})[|\Lambda(P_t \times I)| + 168a(t-1,1)] + 84 \times 4 \times \rho_1(4,1) + 168 \times 4 \times \rho_2(4,1)\} = 259.$$

This number is the same as that of Theorem 4.1 in [2]. The proof is completed. □

5. Orientable small covers over $P_m \times \Delta_n$

Nakayama and Nishimura found an orientability condition for a small cover [7].

Theorem 5.1 *For a basis $\{e_1, \cdots, e_n\}$ of $(\mathbb{Z}_2)^n$, a homomorphism $\varepsilon : (\mathbb{Z}_2)^n \longrightarrow \mathbb{Z}_2 = \{0,1\}$ is defined by $\varepsilon(e_i) = 1 (i = 1, \cdots, n)$. A small cover $M(\lambda)$ over a simple convex polytope P^n is orientable if and only if there exists a basis $\{e_1, \cdots, e_n\}$ of $(\mathbb{Z}_2)^n$ such that the image of $\varepsilon\lambda$ is $\{1\}$.*

We call a $(\mathbb{Z}_2)^n$-coloring which satisfies the orientability condition in Theorem 5.1 an orientable coloring of P^n. We know that there exists an orientable small cover over every simple convex 3-polytope [7]. Similarly we know the existence of orientable small cover over $P_m\times\Delta_n$ by existence of orientable colorings and determine the number of equivariant homeomorphism classes.

By $O(P^n)$ we denote the set of all orientable colorings on P^n. There is a natural action of $GL(n,\mathbb{Z}_2)$ on $O(P^n)$ defined by the correspondence $\lambda\longmapsto\sigma\circ\lambda$, and the action on $O(P^n)$ is free. Assume that $F_1,\cdots,F_n$ of $\mathcal{F}(P^n)$ meet at one vertex p of P^n. Let $e_1,\cdots,e_n$ be the standard basis of $(\mathbb{Z}_2)^n$. Write $B(P^n)=\{\lambda\in O(P^n)|\lambda(F_i)=e_i, i=1,\cdots,n\}$. It is easy to check that $B(P^n)$ is the orbit space of $O(P^n)$ under the action of $GL(n,\mathbb{Z}_2)$.

Remark 4 In fact, we have $B(P^n)=\{\lambda\in O(P^n)|\lambda(F_i)=e_i, i=1,\cdots,n$ and for $n+1\le j\le\ell, \lambda(F_j)=e_{j_1}+e_{j_2}+\cdots+e_{j_{2h_j+1}}, 1\le j_1<j_2<\cdots<j_{2h_j+1}\le n\}$. Below we show that $\lambda(F_j)=e_{j_1}+e_{j_2}+\cdots+e_{j_{2h_j+1}}$ for $n+1\le j\le\ell$. If $\lambda\in O(P^n)$, there exists a basis $\{e'_1,\cdots,e'_n\}$ of $(\mathbb{Z}_2)^n$ such that for $1\le i\le\ell$, $\lambda(F_i)=e'_{i_1}+\cdots+e'_{i_{2f_i+1}}, 1\le i_1<\cdots<i_{2f_i+1}\le n$. Since $\lambda(F_i)=e_i, i=1,\cdots,n$, then $e_i=e'_{i_1}+\cdots+e'_{i_{2f_i+1}}$. So we obtain that for $n+1\le j\le\ell$, there are not $j_1,\cdots,j_{2k}$ such that $\lambda(F_j)=e_{j_1}+\cdots+e_{j_{2k}}, 1\le j_1<\cdots<j_{2k}\le n$.

Since $B(P^n)$ is the orbit space of $O(P^n)$, then we have

Lemma 5.2 $|O(P^n)|=|B(P^n)|\times|GL(n,\mathbb{Z}_2)|$.

Two orientable small covers $M(\lambda_1)$ and $M(\lambda_2)$ over P^n are D-J equivalent if and only if there is $\sigma\in GL(n,\mathbb{Z}_2)$ such that $\lambda_1=\sigma\circ\lambda_2$. Thus, the number of D-J equivalence classes of orientable small covers over P^n is $|B(P^n)|$.

One can define the right action of $Aut(\mathcal{F}(P^n))$ on $O(P^n)$ by $\lambda\times h\longmapsto\lambda\circ h$, where $\lambda\in O(P^n)$ and $h\in Aut(\mathcal{F}(P^n))$. By improving the classifying result on small covers in [6], we have the following theorem.

Theorem 5.3 *Two orientable small covers over an n-dimensional simple convex polytope P^n are equivariantly homeomorphic if and only if there is $h\in Aut(\mathcal{F}(P^n))$ such that $\lambda_1=\lambda_2\circ h$, where λ_1 and λ_2 are their corresponding orientable colorings on P^n.*

Proof. We know Theorem 5.3 is true by combining Lemma 5.4 in [6] with Theorem 5.1. □

By Theorem 5.3, the number of orbits of $O(P^n)$ under the action of $Aut(\mathcal{F}(P^n))$ is just the number of equivariant homeomorphism classes of orientable small covers over P^n. So we also are going to count the orbits.

In the similar way, we calculate the number of all orientable colorings on $P_m\times\Delta_n$ by Theorem 5.1, Remark 4 and Lemma 5.2.

Theorem 5.4 *Let a', b' be the functions from $\mathbb{N}\times\mathbb{N}$ to $\mathbb{N}$ with the following properties:*

(1) $a'(j,n)=2^{n-1}a'(j-1,n)+2^{2n-1}a'(j-2,n)$ with $a'(1,n)=1, a'(2,n)=2^{n-1}$;

(2) $$b'(j,n)=\begin{cases}0 & j \text{ even},\\ (2^n)^{\frac{j-1}{2}} & j \text{ odd}.\end{cases}$$

Then the number of all orientable colorings on $P_m\times\Delta_n$ is

$$|O(P_m \times \Delta_n)| = \begin{cases} \prod_{k=1}^{n+2}(2^{n+2}-2^{k-1})[a'(m-1,n)+\frac{1+(-1)^m}{2}], & n \ odd, \\ 2\prod_{k=1}^{n+2}(2^{n+2}-2^{k-1})b'(m-1,n), & n \ even. \end{cases}$$

Similarly, we determine the number of equivariant homeomorphism classes of all orientable small covers over $P_m \times \Delta_n$ by Lemma 2.4, the Burnside Lemma and Theorems 5.3, 5.4.

Theorem 5.5 *Let φ denote the Euler's totient function. Let $E_o(P_m \times \Delta_n)$ denote the number of equivariant homeomorphism classes of orientable small covers over $P_m \times \Delta_n$. Then $E_o(P_m \times \Delta_n)$ is equal to*

$$\begin{cases} \frac{1}{2m(n+1)!}\{\sum_{t>1,t|m}\varphi(\frac{m}{t})|O(P_t\times\Delta_n)| + \frac{m}{2}\prod_{k=1}^{n+2}(2^{n+2}-2^{k-1})[\frac{1+(-1)^{n+1}}{2}(\rho_1'(m,n)+\frac{1+(-1)^m}{2}) \\ \quad +\frac{1+(-1)^n}{2}(\rho_2'(m,n)+\rho_3'(m,n))]\}, & n=2 \ and \ m\neq 3, or \ n\geq 3, \\ \frac{1}{4m}\{\sum_{t>1,t|m}\varphi(\frac{m}{t})[|O(P_t\times I)|+168a'(t-1,1)]+168m\rho_1'(m,1)+42m[1+(-1)^m]\}, & \\ & n=1 \ and \ m\neq 4, \\ 0, & n=2 \ and \ m=3, \\ 70, & n=1 \ and \ m=4, \end{cases}$$

where $\rho_1'(m,n), \rho_2'(m,n)$, and $\rho_3'(m,n)$ are defined as

$$\rho_1'(m,n) = \begin{cases} 0, & m \ odd, \\ (2^n)^{\frac{m}{2}-1}, & m \ even \end{cases}$$

$$\rho_2'(m,n) = \begin{cases} 0, & m \ odd, \\ (2^n)^{h-1}, & m=4h \ or \ m=4h-2, h\geq 1 \end{cases}$$

and

$$\rho_3'(m,n) = \begin{cases} 0, & m \ odd, \\ (2^n)^h, & m=4h, h\geq 1 \\ (2^n)^{h-1}, & m=4h-2, h\geq 1. \end{cases}$$

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