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Product of graded submodules

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Abstract

Let Δ be an abelian group. By considering the notion multiplication of Δ-graded modules (see [7]) over a commutative Δ-graded ring with unity, we introduce the notion of product of two Δ-graded submodules which we use to characterize the Δ-graded prime submodules of a multiplication Δ-graded module. Finally we proved a graded version of Nakayama lemma for multiplication Δ-graded modules.

Key Words: Δ-graded Rings, Δ-graded Modules, Δ-graded Submodules

1. Introduction

A grading on a commutative ring with unity and it's modules usually aids computations by allowing one to focus on homogeneous elements, which are simpler or more controllable than random elements. Therefore, the study of graded modules is important.

Graded multiplication modules have been studied by many authors (for example, see [3, 4, 7].) and graded prime submodules have been studied in many papers, (for example, see [1, 2]).

Let Δ be an abelian group, let R be any ring. Then R is called a Δ-graded ring, if $R = \bigoplus_{g\in\Delta} R_g$, such that if $a, b \in \Delta$, then $R_a R_b \subseteq R_{ab}$. Let $R^h = \cup_{g\in\Delta} R_g$. Then R^h is the set of homogeneous elements in R.

Let R be a Δ-graded ring with unity $1 \neq 0 \in R$, then it is easy to see that $1 \in R_e$, where e is the identity element in Δ.

Let M be an R-module. Then M is called a Δ-graded R-module if $M = \bigoplus_{g\in\Delta} M_g$; and for each $g \in \Delta$, M_g is R_e-module and for any $x, y \in \Delta$, we have $R_x M_y \subseteq M_{xy}$. Let $m \in M^h$. We write $\deg(m) = g$ if $m \in M_g$, $g \in \Delta$. Also, we define the annihilator of M to be $\mathrm{ann}(M) = \{r \in R : rM = 0\}$.

We say that M is a torsion free Δ-graded R-module whenever $rm = 0$, then either $r = 0$ or $m = 0$, where $r \in R^h$ and $m \in M^h$.

Throughout this work all rings are commutative Δ-graded rings with identity, and all Δ-graded modules are unitary.

Let N be a proper Δ-graded submodule of M; then N is a prime Δ-graded submodule of M if the condition $rm \in N^h$, where $r \in R^h$, $m \in M^h$, implies that $m \in N$ or $rM \subseteq N$. In this case, if $P = (N : M) = \{t \in R : tM \subseteq N\}$, then we say that N is a P-prime Δ-graded submodule of M; and one

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can easily see that $P = (N : M)$ is a prime Δ-graded ideal in R, that is if $a, b \in R^h$ with $ab \in P$ then $a \in P^h$ or $b \in P^h$.

A Δ-graded R-module M is called a multiplication Δ-graded R-module provided for each Δ-graded submodule N of M there exists a Δ-graded ideal I of R such that $N = IM$. Let M be a Δ-graded R-module; then we say that M is finitely generated, if there exist $m_1, m_2, \cdots, m_n \in M^h$ where $n \in \mathbb{Z}^+$ such that $M = Rm_1 + Rm_2 + \cdots + Rm_n$.

Recently, in [7], we we gave some results about Δ-supergraded prime submodules. These results are generalization of Δ-graded prime submodules and prime submodules (see [6]). In this paper we define the notion of product of Δ-graded submodules of a multiplication Δ-graded R-module and obtain some related results. In particular, we give some equivalent conditions for prime Δ-graded submodules of a multiplication Δ-graded R-modules. The results for the product of submodules of a multiplication module (in the nongraded case) are proved in many papers; see for example [5, 8, 9]. Finally, we state and prove a version of Nakayama lemma for multiplication Δ-graded R-modules.

2. The product of multiplication graded submodules

Let R be a commutative Δ-graded ring with unity and let M be a Δ-graded R-module. Then M is a multiplication Δ-graded R-module if and only if every Δ-graded submodule of M is a multiplication Δ-graded R-module. Let N be a proper Δ-graded submodule of M, then the radical of N, which is denoted by $\sqrt{N}$, is the intersection of all Δ-graded prime submodules of M containing N.

The radical of M, which is denoted by $\sqrt{M}$, is defined to be the intersection of the Δ-graded maximal submodules of M if such exist, and M otherwise.

The proof of the following theorem for the nongraded case is found in [6].

Theorem 2.1 *Let M be a faithful Δ-graded R-module. Then M is a multiplication Δ-graded R-module if and only if*

(1) $\cap_{\lambda\in\Lambda}(I_\lambda M) = (\cap_{\lambda\in\Lambda} I_\lambda)M$ for any nonempty collection of Δ-graded ideals of R; and

(2) For a Δ-graded submodule N of M and a Δ-graded ideal A of R such that $N \subset AM$ there exists a Δ-graded ideal B of R with $B \subset A$ and $N \subseteq BM$.

Proof. Suppose (1) and (2) hold. Let N be a Δ-graded submodule of M. Let

$$S = \{I \ : \ I \text{ is a } \Delta\text{-graded ideal of R and } N \subseteq IM\}.$$

Clearly $R \in S$. Let I_λ $(\lambda \in \Lambda)$ be a nonempty collection of Δ-graded ideals of R in S. By (1), $\cap_{\lambda\in\Lambda} I_\lambda \in S$ and therefore, by Zorn's Lemma, S has (say) minimal element A. Then $N \subseteq AM$. Suppose $N \neq AM$, then by (2) there exists a Δ-graded ideal B of R with $B \subset A$ and $N \subseteq BM$. In this case $B \in S$, contradicting the choice of A. Thus $N = AM$ and hence M is a multiplication Δ-graded R-module.

Conversely, suppose M is a multiplication Δ-graded R-module. Let I_λ $(\lambda \in \Lambda)$ be a nonempty collection of Δ-graded ideals of R. Let $I = \cap_{\lambda\in\Lambda} I_\lambda$. Clearly $IM \subseteq \cap_{\lambda\in\Lambda}(I_\lambda M)$. Let x be a homogeneous element in $\cap_{\lambda\in\Lambda}(I_\lambda M)$. Let $K = \{r \in R \ : \ rx \in IM\}$. Then K is a Δ-graded ideal of R. Suppose $K \neq R$. Then there exists a Δ-graded maximal ideal P of R such that $K \subseteq P$. By [7, Theorem 3.2], M is P-cyclic. Thus there exists $p \in P^h$ with $\deg(p) = e$, where e is the identity element in Δ, and $m \in M^h$ such that $(1-p)M \subseteq Rm$.

Hence $(1-p)x \in \cap_{\lambda\in\Lambda}(I_\lambda m)$. For each $\lambda \in \Lambda$ there exists $a_\lambda \in I_\lambda^h$ such that $(1-p)x = a_\lambda m$. Choose $\beta \in \Lambda$. For each $\lambda \in \Lambda$, $a_\beta m = a_\lambda m$ so that $(a_\beta - a_\lambda)m = 0$. Now

$$(1-p)(a_\beta - a_\lambda)M = (a_\beta - a_\lambda)(1-p)M \subseteq (a_\beta - a_\lambda)Rm = 0$$

implies that $(1-p)(a_\beta - a_\lambda) = 0$. Therefore, $(1-p)a_\beta = (1-p)a_\lambda \in I_\lambda$ for any $\lambda \in \Lambda$ and hence $(1-p)a_\beta \in I$. Thus $(1-p)^2 x = (1-p)a_\beta m \in IM$. It follows that $(1-p)^2 \in K \subseteq P$, a contradiction. Thus $K = R$ and $x \in IM$. Therefore $\cap_{\lambda\in\Lambda}(I_\lambda M) \subseteq IM$ and hence $IM = \cap_{\lambda\in\Lambda}(I_\lambda M)$. □

Results about faithful multiplication Δ-graded R-modules can easily be extended to non-faithful ones. Therefore we leave it to the reader to show that Theorem 2.1 gives the following immediate corollary. For the nongraded case see [6].

Corollary 2.2 *Let M be a Δ-graded R-module. Then M is a multiplication Δ-graded R-module if and only if*

(1) $\cap_{\lambda\in\Lambda}(I_\lambda M) = (\cap_{\lambda\in\Lambda}[I_\lambda + \operatorname{ann}(M)])M$ for any nonempty collection of Δ-graded ideals of R, and

(2) For a Δ-graded submodule N of M and a Δ-graded ideal A of R such that $N \subset AM$ there exists a Δ-graded ideal B of R with $B \subset A$ and $N \subseteq BM$. □

Let $\mathcal{M}$ denotes the collection of all Δ-graded maximal ideals of R. Define $\mathcal{M}_1 = \{P \in \mathcal{M} \mid M \neq PM\}$ and $\mathcal{M}_2 = \{P \in \mathcal{M} \mid (0 : M) \subseteq P\}$. Define $J_1(M) = \cap_{P\in\mathcal{M}_1} P$ and $J_2(M) = \cap_{P\in\mathcal{M}_2} P$.

Theorem 2.3 *Let M be a multiplication Δ-graded R-module. Then*

$$\sqrt{M} = J_1(M)M = J_2(M)M.$$

Proof. By Theorem 2.1.(1) and [7, Corollary 4.5], we have

$$\sqrt{M} = \cap\{PM \mid P \in \mathcal{M}_1\} = J_1(M)M \supseteq J_2(M)M.$$

Moreover Corollary 2.2.(1) also gives us that

$$J_2(M)M = \cap\{PM \mid P \in \mathcal{M}_2\}.$$

Let $Q \in \mathcal{M}_2$. If $M = QM$, then $\sqrt{M} \subseteq QM$. If $M \neq QM$, then $Q \in \mathcal{M}_1$ and hence $\sqrt{M} \subseteq QM$. Thus in any case $\sqrt{M} \subseteq QM$, it follows that $\sqrt{M} \subseteq J_2(M)M$. Therefore, $\sqrt{M} = J_1(M)M = J_2(M)M$. □

Theorem 2.4 *Let M be a multiplication Δ-graded R-module, and let N be a proper Δ-graded submodule of M. If $A = (N : M)$, then $\sqrt{N} = \sqrt{A}M$.*

Proof. Without loss of generality, we may assume that M is faithful. Let ρ be the collection of all Δ-graded prime ideals P of R such that $A \subseteq P$. If $B = \sqrt{A}$ then $B = \cap_{p\in\rho} P$ and, hence, by Theorem 2.1, $BM = \cap_{p\in\rho}(PM)$. Let $P \in \rho$. If $M = PM$ then $\sqrt{N} \subseteq PM$. If $M \neq PM$ then $N = AM \subseteq PM$ implies that $\sqrt{N} \subseteq PM$. It follows that $\sqrt{N} \subseteq BM$.

Conversely, suppose that K is a proper Δ-graded prime submodule of M containing N. By [7, Corollary 3.10] there exists a Δ-graded prime ideal Q of R such that $K = QM$. Since $AM = N \subseteq K = QM \neq M$ it follows that $A \subseteq Q$, by [7, Theorem 3.7], and hence $B \subseteq Q$. Thus $BM \subseteq K$ and so $BM \subseteq \sqrt{N}$. Therefore, $\sqrt{N} = BM$. □

Let M be a Δ-graded R-module and let N be Δ-graded submodule of M such that $N = IM$ for some Δ-graded ideal I of R. Then we say that I is a presentation Δ-graded ideal of N; for short, we say a presentation of N. We denote the set of all presentation Δ-graded ideals of N by $\mathrm{Gpr}(N)$.

Remark 2.5 *It is possible that for a Δ-graded submodule N no such presentation Δ-graded ideal exists. For example, if $V = \bigoplus_{g\in\Delta} V_g$ is a Δ-graded vector space over a field F with a nontrivial proper Δ-graded subspace W of V, then W does not have any presentation.*

It is clear that M is a multiplication Δ-graded R-module if and only if $N = (N : M)M$ for each Δ-graded submodule N of M. Therefore, M is a multiplication Δ-graded R-module if and only if every Δ-graded submodule of M has a presentation Δ-graded ideal. In particular, if N is a Δ-graded submodule of a multiplication Δ-graded R-module M, then $(N : M)$ is a presentation for N.

Let $\mathfrak{L}(R)$ and $\mathfrak{L}(M)$ denote the lattices of Δ-graded ideals of R and Δ-graded submodules of M, respectively.

Define the relation $\sim$ on $\mathfrak{L}(R)$ as follows:

$$I \sim J \Leftrightarrow IM = JM.$$

Then it is easy to verify that this relation is an equivalence relation on $\mathfrak{L}(R)$. We denote the equivalence class of $I \in \mathfrak{L}(R)$ by $[I]$.

Theorem 2.6 *Let M be a faithful multiplication Δ-graded R-module. Then the following statements are equivalent:*

(1) M is finitely generated.

(2) Each equivalence class is singleton.

(3) The map $\phi : \mathfrak{L}(R) \to \mathfrak{L}(M)$ defined by $\phi(I) = IM$ is a lattice isomorphism.

(4) For every proper Δ-graded ideal I of R, $[I] = \{I\}$.

(5) For every Δ-graded maximal ideal P of R, $[P] = \{P\}$.

Proof. (1) $\Rightarrow$ (2) Follows from [7, Corollary 3.10] and [7, Theorem 4.1].

(2) $\Rightarrow$ (3) By [7, Corollary 3.10], we conclude that ϕ is a bijection map, and preserves the order. Moreover by Theorem 2.1, we have $(I + J)M = IM + JM$ and $(I \cap J)M = IM \cap JM$, since M is faithful. Therefore, ϕ is a lattice isomorphism.

(3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (5) are immediate consequences of [7, Corollary 3.10].

(5) $\Rightarrow$ (1) is an immediate consequence of [7, Theorem 4.1]. □

Let M be a Δ-graded R-module and let $N = IM$ and $K = JM$ for some Δ-graded ideals I and J of R. The product of N and K, which is denoted by NK, is defined by IJM. Clearly NK is a Δ-graded submodule of M which is contained in $N \cap K$.

Theorem 2.7 *Let M be a Δ-graded R-module and let $N = IM$ and $K = JM$ for some Δ-graded ideals I and J of R. Then NK is independent of presentations of N and K.*

Proof. Let $I, I_1 \in \mathrm{Gpr}(N)$ and $J, J_1 \in \mathrm{Gpr}(K)$. Let $rsm \in I_1J_1M$ for some $r \in I_1^h$, $s \in J_1^h$ and $m \in M^h$. Since $J_1M = JM$, we conclude that

$$sm = \sum_{i=1}^{n} r_i m_i, \quad r_i \in J^h, \quad m_i \in M^h,$$

where for each i, $\deg(r_i)\deg(m_i) = \deg(s)\deg(m)$. Therefore,

$$rsm = \sum_{i=1}^{n} r(r_i m_i) = \sum_{i=1}^{n} r_i(rm_i),$$

and for each i, $\deg(rsm) = \deg(r_i rm_i)$.

Similarly, $rm_i \in I_1M = IM$ implies that

$$rm_i = \sum_{j=1}^{k} t_{ij} m'_{ij}, \quad t_{ij} \in I^h, \quad m'_{ij} \in M^h,$$

and for all i and j, $\deg(rm_i) = \deg(t_{ij})\deg(m'_{ij})$. Thus,

$$rsm = \sum_{i=1}^{n}\sum_{j=1}^{k} r_i t_{ij} m'_{ij},$$

and $\deg(rsm) = \deg(r_i rm_i) = \deg(r_i)\deg(t_{ij}m'_{ij})$. Therefore, $rsm \in IJM$, so $I_1J_1M \subseteq IJM$. Similarly, we have $IJM \subseteq I_1J_1M$. Thus $IJM = I_1J_1M$. □

If K and L are Δ-graded submodules of a Δ-graded R-module M, then we say that K and L are comaximal if $K + L = M$. By using some properties of the graded ideal Theory and the fact that

$$\sum_{\lambda\in\Lambda} I_\lambda M = (\sum_{\lambda\in\Lambda} I_\lambda)M,$$

we have the following result.

Proposition 2.8 *Let M be a multiplication Δ-graded R-module, and let N, K and L be Δ-graded submodules of M. Then the following statements are satisfied:*

(1) The product is distributive with respect to the sum on $\mathfrak{L}(M)$.

(2) $(K + L)(K \cap L) \subseteq KL$.

(3) If K and L are comaximal then $K \cap L = KL$.

Proof. (1) and (2) follow from the properties of the graded ideal theory.

(3) Since K and L are comaximal, we have $K+L = M$, and hence by (2), $M(K\cap L) = (K+L)(K\cap L) \subseteq KL$. But $M(K \cap L) = K \cap L$, so $K \cap L \subseteq KL$. Clearly, $KL \subseteq K \cap L$. Therefore, $K \cap L = KL$. □

Lemma 2.9 *Let N and K be Δ-graded submodules of a faithful multiplication Δ-graded R-module M. Then,*

(1) The Δ-graded ideals $(N:M).(K:M)$ and $(NK:M)$ are in $\mathrm{Gpr}(NK)$.

(2) If M is finitely generated, then $(N:M).(K:M)=(NK:M)$.

Proof. (1) Since $N=(N:M)M$, $K=(K:M)M$, by using the definition of multiplication Δ-graded submodules, we have

$$NK=(NK:M)M=(N:M)(K:M)M.$$

Therefore, $(N:M).(K:M)$ and $(NK:M)$ are in $\mathrm{Gpr}(NK)$.

(2) Since M is finitely generated, by Theorem 2.6.(4), $(NK:M)=(N:M)(K:M)$. □

Recall that, by [7, Proposition 3.1], for any $m\in M^h$, we have $Rm=IM$ for some Δ-graded ideal I of R. In this case, we say that I is a graded presentation ideal of m, or for short, a presentation of m. We denote the set of all presentation Δ-graded ideals of m by $\mathrm{Gpr}(m)$. In fact, $\mathrm{Gpr}(m)$ equals to $\mathrm{Gpr}(Rm)$.

For $m,n\in M^h$, by mn, we mean the product of Rm and Rn, which equals to IJM for every Δ-graded ideal I in $\mathrm{Gpr}(m)$ and Δ-graded ideal J in $\mathrm{Gpr}(n)$.

For $m=\Sigma_{g\in\Delta}m_g$, $n=\Sigma_{g\in\Delta}n_g$ in M, by mn, we mean the product of $\Sigma_{g\in\Delta}Rm_g$ and $\Sigma_{g\in\Delta}Rn_g$, which equals to IJM for every presentation graded ideals I and J of $\Sigma_{g\in\Delta}Rm_g$ and $\Sigma_{g\in\Delta}Rn_g$, respectively.

Proposition 2.10 *Let M be a multiplication Δ-graded R-module. Let N,K,N_i, where $i\in I$, be Δ-graded submodules of M, $s\in R^h$, $m\in M^h$ and k any positive integer. Then the following statements are satisfied:*

(1) $Gpr(\Sigma_{i\in I}N_i)=\Sigma_{i\in I}\mathrm{Gpr}(N_i)$;

(2) $Gpr(\cap_{i\in I}N_i)=(\cap_{i\in I}[\mathrm{Gpr}(N_i)+\mathrm{ann}(M)])M$;

(3) $Gpr(sm)=s\mathrm{Gpr}(m)$;

(4) $Gpr(NK)=Gpr(N)\,Gpr(K)$;

(5) $Gpr(N^k)=(\mathrm{Gpr}(N))^k$;

(6) $Gpr(m^k)=(\mathrm{Gpr}(m))^k$;

(7) $Gpr(\sqrt{N})=\sqrt{\mathrm{Gpr}(N)}$.

Proof. (1) Let $J_i\in\mathrm{Gpr}(N_i)$ for every $i\in I$. Then it is easy to verify that

$$\sum_{i\in I}N_i=\sum_{i\in I}(J_iM)=(\sum_{i\in I}J_i)M.$$

Thus, $\mathrm{Gpr}(\Sigma_{i\in I}N_i)=\Sigma_{i\in I}\mathrm{Gpr}(N_i)$.

(2) It is an immediate consequence of Corollary 2.2.(1).

(3), (4), (5) and (6) are immediate consequences of Theorem 2.7.

(7) It follows from Theorem 2.4. □

Proposition 2.11 *Let M be a multiplication Δ-graded R-module, and let $m_1,m_2,...,m_k\in M_g$ for some $g\in\Delta$. Then $Gpr(\Sigma_{i=1}^k m_i)\subseteq\Sigma_{i=1}^k\mathrm{Gpr}(m_i)$.*

Proof. Since $m_1, m_2, ..., m_k$ are homogeneous elements in M_g, where $g \in \Delta$, we have

$$\mathrm{Gpr}(\sum_{i=1}^{k} m_i) = \mathrm{Gpr}(R\sum_{i=1}^{k} m_i) \subseteq \mathrm{Gpr}(\sum_{i=1}^{k} Rm_i) = \sum_{i=1}^{k} \mathrm{Gpr}(m_i).$$

□

Theorem 2.12 *Let P be a proper Δ-graded submodule of a multiplication Δ-graded R-module M. Then P is a prime Δ-graded submodule of M if and only if*

$$UV \subseteq P \Rightarrow U \subseteq P \quad \text{or} \quad V \subseteq P$$

for each Δ-graded submodules U and V of M.

Proof. Let P be a prime Δ-graded submodule of M, and let $UV \subseteq P$ where $U = IM$, $V = JM$. Then $UV = IJM \subseteq P$. Suppose $U \nsubseteq P$ and $V \nsubseteq P$, then there are $ry \in U - P$ and $sx \in V - P$ for some $r \in I^h$, $y \in M^h$ and $s \in J^h$, $x \in M^h$. Thus $rsx \in P$, since $rs \in IJ$ and hence $rM \subseteq P$, that is $ry \in P$, which is a contradiction. Therefore, $U \subseteq P$ or $V \subseteq P$.

Conversely, let $rx \in P$ for some $r \in R^h$ and $x \in M^h - P^h$. We claim that $rM \subseteq P$. Let $m \in M^h$ and let I and J be Δ-graded presentation ideals of rx and m. Then

$$R(rx)(Rm) = (Rx)(Rrm) = IM.JM = IJM \subseteq IM \subseteq P.$$

Now, by hypothesis, we have $Rx \subseteq P$ or $Rrm \subseteq P$ but $x \notin P$ so $rm \in P$, therefore, $rM \subseteq P$ and hence P is a prime. □

A straightforward proof shows that Theorem 2.12 has the following obvious consequence.

Corollary 2.13 *Let P be a proper Δ-graded submodule of a multiplication Δ-graded R-module M. Then P is a prime Δ-graded submodule of M if and only if*

$$mm' \subseteq P \Rightarrow m \in P \quad \text{or} \quad m' \in P$$

for every $m, m' \in M^h$. □

Definition 2.14 *Let M be a multiplication Δ-graded R-module, and let N be a Δ-graded submodule of M. Then*

(1) N is called nilpotent if $N^k = \{0\}$ for some $k \in \mathbb{N}$.

(2) An element m of M^h is called nilpotent if $m^k = 0$ for some $k \in \mathbb{N}$.

We denoted by N_M the set of nilpotent elements of M.

Theorem 2.15 *Let M be a multiplication Δ-graded R-module. A Δ-graded submodule N of M is nilpotent if and only if every presentation Δ-graded ideal I of N, $I^k \subseteq \mathrm{ann}(M)$ for some $k \in \mathbb{N}$.*

Proof. The proof is going straightforward. □

Corollary 2.16 *Let M be a faithful multiplication Δ-graded R-module. A Δ-graded submodule N of M is nilpotent if and only if every presentation Δ-graded ideal I of N is a nilpotent Δ-graded ideal.* □

Theorem 2.17 *Let M be a multiplication Δ-graded R-module. Then*

(1) Let x, y be nonzero homogeneous elements in N_M with $\deg(x) = \deg(y)$ then $x + y \in N_M$.

(2) Let m be a homogeneous element in N_M and let $r \in R^h$ then $rm \in N_M$.

Proof. (1) Let x, y be nonzero homogeneous elements in N_M such that $\deg(x) = \deg(y)$. Say $x^m = 0$ and $y^n = 0$ for some $n, m \in \mathbb{N}$. Suppose $I \in \mathrm{Gpr}(x)$ and $J \in \mathrm{Gpr}(y)$. Then $x^m = I^m M = 0$ and $y^n = J^n M = 0$. Since $Rx = IM$ and $Ry = JM$, we have

$$R(x + y) \subseteq Rx + Ry = IM + JM = (I + J)M,$$

so $I + J \in \mathrm{Gpr}(Rx + Ry)$. Let $l = n + m$. Then,

$$\begin{aligned}(Rx + Ry)^l &= (I + J)^l M \\ &= (\sum_{i=0}^{l} \binom{l}{i} I^i J^{l-i})M \\ &= 0M \\ &= 0,\end{aligned}$$

hence $x + y \in N_M$.

(2) Let m be a homogeneous element in N_M and let $r \in R^h$. Consider the graded presentation ideal I of m. Then $m^k = I^k M = 0$ for some positive integer k. Since $rm \in Rrm = (rI)M \subseteq IM$, we have $(rm)^k \subseteq I^k M = 0$ and hence $rm \in N_M$. □

Corollary 2.18 *Let M be a multiplication Δ-graded R-module. Then N_M is a Δ-graded submodule of M and the Δ-graded R-module M/N_M has no nonzero nilpotent element.*

Proof. In Theorem 2.17 we proved that if $g \in \Delta$ then $(N_M)_g$ is an abelian group under addition and $R_{g'}(N_M)_g \subseteq (N_M)_{g'g}$ for all $g', g \in \Delta$. Therefore to show that N_M is a Δ-graded submodule of M, it is enough to prove that $n + m \in N_M$ for every $n, m \in N_M$.

Let $m = \Sigma_{g\in\Delta} m_g \in N_M$; then $m \in \Sigma_{g\in\Delta} Rm_g = JM$ for some Δ-graded ideal J of R. Let $n = \Sigma_{g\in\Delta} n_g \in N_M$, then $n \in \Sigma_{g\in\Delta} Rn_g = IM$ for some Δ-graded ideal I of R. Since $n, m \in N_M$ we have $n^i = 0$ and $m^j = 0$ for some $i, j \in \mathbb{N}$. Now, $n^i = 0$ implies that $I^i M = 0$ and $m^j = 0$ implies that $J^j M = 0$. Let $k = i + j$, then

$$n + m \in \sum_{g\in\Delta}(Rn_g + Rm_g) \subseteq \sum_{g\in\Delta} Rn_g + \sum_{g\in\Delta} Rm_g \subseteq (I + J)M$$

implies that the graded presentation ideal of $\sum_{g\in\Delta} Rn_g + \sum_{g\in\Delta} Rm_g$ is contained in $I+J$. Since $(I+J)^k M = 0$, we have that $(n + m)^k = 0$ and hence $n + m \in N_M$. Therefore N_M is a Δ-graded submodule of M.

Let $x \in M^h$ and suppose $\overline{x} = x + N_M$. If $(\overline{x})^k = 0$ for some $k \in \mathbb{N}$, then $x^k \in N_M$ and hence $(x^k)^n = 0$ for some $n \in \mathbb{N}$. Therefore, $x \in N_M$ and so $\overline{x} = 0$. Since N_M is a Δ-graded submodule of M, then M/N_M has no nonzero nilpotent element. □

In Corollary 2.18, we proved that N_M is a Δ-graded submodule of M. Therefore, $m = \Sigma_{g\in\Delta} m_g \in N_M$ if and only if each $m_g \in N_M$ for every $g \in \Delta$.

Theorem 2.19 *Let M be a multiplication Δ-graded R-module, and let N be a Δ-graded submodule of M. Then*

$$\sqrt{N} = \{m \in M \mid m^k \subseteq N \text{ for some } k \in \mathbb{N}\}.$$

Proof. By the same arguments used in the proof of Theorem 2.17 and Corollary 2.18, we can show that $B = \{m \in M \mid m^k \subseteq N \text{ for some } k \in \mathbb{N}\}$ is a Δ-graded submodule of M.

Suppose $m \in B^h$ and $A \subseteq \mathrm{Gpr}(m)$. Then $m^k = A^k M \subseteq N$ for some $k \in \mathbb{N}$ and hence by Theorem 2.4

$$\sqrt{m^k} = \sqrt{A^k M} = \sqrt{A^k} M = \sqrt{A} M \subseteq \sqrt{N}.$$

Thus $m \in Rm = AM \subseteq \sqrt{A}M \subseteq \sqrt{N}$ and hence $B \subseteq \sqrt{N}$.

Conversely, let m be a homogeneous element in $\sqrt{N} = \sqrt{I}M$, where $I = (N : M)$. Then $m = \Sigma_{i=1}^{n} r_i m_i$ where for each i, r_i and m_i are homogeneous elements in $\sqrt{I}$ and M respectively and $\deg(r_i)\deg(m_i) = \deg(m)$. Now, for each i, $r_i m_i \in \sqrt{I}M = \sqrt{IM}$ which implies that $(r_i m_i)^{k_i} \subseteq IM = N$ for some $k_i \in \mathbb{N}$. Thus for a sufficiently large $k \in \mathbb{N}$, we have $m^k \in N$ so $m \in B$ and hence $\sqrt{N} \subseteq B$. Therefore, $B = \sqrt{N}$. □

Corollary 2.20 *Let M be a multiplication Δ-graded R-module. Then N_M is the intersection of all Δ-graded prime submodules of M.*

Proof. By Theorem 2.4, we have $\sqrt{0} = \sqrt{A}M$, where $A = \mathrm{ann}(M)$ and by Theorem 2.19, $\sqrt{0} = N_M$. Therefore N_M is the intersection of all Δ-graded prime submodules of M. □

Corollary 2.21 *Let M be a faithful multiplication Δ-graded R-module. Then $N_M = \mathcal{A}M$, where $\mathcal{A}$ is the Δ-graded nilradical of R.* □

Definition 2.22 *A homogeneous element u of a Δ-graded R-module M is said to be a unit provided that u is not contained in any Δ-graded maximal submodule of M.*

Remark 2.23 *If M is a multiplication Δ-graded R-module and $u \in M^h$. Then, by [7, Proposition 4.4], u is a unit if and only if $< u >= Ru = M$.*

In the final part of this section we prove the graded version of the Nakayama Lemma, but first we need to prove the following theorems. See [11] for the proof of the next theorem in the nongraded case.

Theorem 2.24 *Let M be a faithful multiplication Δ-graded R-module, and let $u \in M^h$ be a unit. Then $m \in M^h$ is an element in $\sqrt{M}$ if and only if $u - rm$ is a unit for every $r \neq 0 \in R^h$ with $\deg(u) = \deg(rm)$.*

Proof. Let $m \in \sqrt{M}$. Suppose for some $r \in R^h$ with $\deg(u) = \deg(rm)$, $u - rm$ is not a unit, then $u - rm \in N$ for some Δ-graded maximal submodule N of M. Since $m \in \sqrt{M}$, $m \in N$ and hence $u \in N$, a contradiction.

Conversely, suppose $u - rm$ is a unit for all $r \in R^h$ with $\deg(u) = \deg(rm)$. Let $m = au$ for some $a \in R^h$. If $m \notin \sqrt{M}$, then $m \notin PM$ for some Δ-graded maximal ideal P of R, so $au \notin PM$ and hence $a \notin P$. Therefore, $P + aR = R$. Now, $1 = q + ar'$ for some $q \in P^h$ and $r' \in R^h$ with $\deg(q) = \deg(ar')$, but $\deg(q) = \deg(ar') = e$ where e is the identity element of the group Δ, so $\deg(r') = (\deg(a))^{-1}$, and since $\deg(u) = (\deg(a))^{-1}\deg(m)$, we have that $\deg(u) = \deg(r')\deg(m) = \deg(r'm)$. Therefore, $u - rm = u - r(au) = (1 - ar)u = (q + ar' - ar)u = (q + a(r' - r))u$ is a unit for all $r \in R^h$ with $\deg(r) = \deg(r')$. Let $r = r'$, then qu is a unit and hence $M = R(qu) \subseteq RPM \subseteq PM$, a contradiction. Thus $m \in \sqrt{M}$. □

Using the fact that the Δ-graded homomorphic image of a multiplication Δ-graded module is a multiplication Δ-graded module we recall that if M is a multiplication Δ-graded R-module and N is a Δ-graded submodule of M, then M/N is a multiplication Δ-graded R-module.

Theorem 2.25 *Let M be a finitely generated faithful multiplication Δ-graded R-module with $\sqrt{M} = M$. Then $M = 0$.*

Proof. Since M is finitely generated, there must be a minimal generating set $X = \{m_1, m_2, \cdots, m_n\}$ of M^h. If $M \neq 0$, then $m_1 \neq 0$ by minimality. Since M is faithful and $M = \sqrt{M}$, we have $M = J_2(M)M$. Thus, $m_1 = j_1m_1 + j_2m_2 + \cdots + j_nm_n$ ($j_i \in (J_2(M))^h$) whence $j_1m_1 = m_1$, so that $(1 - j_1)m_1 = 0$ if $n = 1$, and

$$(1 - j_1)m_1 = j_2m_2 + \cdots + j_nm_n, \quad \text{if} \quad n > 1.$$

Since $j_1 \in J_2(M)$ with $\deg(j_1) = \deg(1)$, we have $1 - j_1$ is a unit in R, and hence

$$m_1 = (1 - j_1)^{-1}j_2m_2 + \cdots + (1 - j_1)^{-1}j_nm_n.$$

Thus, if $n = 1$, then $m_1 = 0$, which is a contradiction. If $n > 1$, then m_1 is a linear combination of $m_2, m_3, ..., m_n$; consequently, $\{m_2, \cdots, m_n\}$ generates M, which contradicts the choice of X. Therefore, $M = 0$. □

Theorem 2.26 *Let M be a faithful multiplication Δ-graded R-module such that $u \in M^h$ is a unit. Then for every Δ-graded submodule N of M, the following conditions are equivalent:*

(1) N is contained in every Δ-graded maximal submodule of M.

(2) $u - rx$ is a unit for all $r \in R^h$ and $x \in N^h$ with $\deg(rx) = \deg(u)$.

(3) If $NM = M$, then $M = \sqrt{M}$.

Proof. (1) ⇒ (2) is an immediate consequence of Theorem 2.24.

(2) ⇒ (3) Let $I \in \mathrm{Gpr}(N)$. Then $NM = M$ implies that

$$M = NM = IM.M = IM.RM = IM = N,$$

and since M is faithful, then by Theorem 2.3, $\sqrt{M} = J_1(M)M = J_2(M)M$, but by Theorem 2.24, $M = N \subseteq \sqrt{M}$. Therefore, $M = \sqrt{M}$.

(3) $\Rightarrow$ (1) Suppose $NM = M$, then $M = \sqrt{M}$ implies that $N \subseteq \sqrt{M}$, so that N is contained in every Δ-graded maximal submodule of M. □

Now, we use the results proved in Theorem 2.24, Theorem 2.25 and Theorem 2.26 to prove the graded version of the Nakayama lemma for multiplication Δ-graded modules.

Theorem 2.27 (a graded version of Nakayama lemma) *Let M be a finitely generated faithful multiplication Δ-graded R-module. Then for every Δ-graded submodule N of M, the following conditions are equivalent:*

(1) N is contained in every Δ-graded maximal submodule of M.

(2) If $NM = M$, then $M = 0$.

(3) If K is a Δ-graded submodule of M such that $M = NM + K$, then $M = K$.

Proof. (1) $\Rightarrow$ (2) Suppose N is contained in every Δ-graded maximal submodule of M. If $NM = M$, then by Theorem 2.26.(3), $M = \sqrt{M}$, and since M is finitely generated, then by Theorem 2.25, $M = 0$.

(2) $\Rightarrow$ (3) By the recall before Theorem 2.25, M/N is a finitely generated faithful multiplication Δ-graded R-module and $M/N = (NM+K)/N = K/N$. Therefore, $K/N = (K/N)(M/N) = M/N$, and by part (2), $K/N = 0$, so $N = K$ and hence $M = KM + K = K$.

(3) $\Rightarrow$ (1) Let K be a Δ-graded maximal submodule of M, then $K \subseteq NM + K \subseteq M$. If $NM + K = M$ then $M = K$, a contradiction. Therefore, by maximality of K, $K = NM + K$ and hence $N \subseteq NM \subseteq K$. □

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