Weakly normal rings

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Abstract

A ring R is defined to be weakly normal if for all $a, r \in R$ and $e \in E(R)$, $ae = 0$ implies $Rera$ is a nil left ideal of R, where $E(R)$ stands for the set of all idempotent elements of R. It is proved that R is weakly normal if and only if $Rer(1-e)$ is a nil left ideal of R for each $e \in E(R)$ and $r \in R$ if and only if $T_n(R, R)$ is weakly normal for any positive integer n. And it follows that for a weakly normal ring R (1) R is Abelian if and only if R is strongly left idempotent reflexive; (2) R is reduced if and only if R is n-regular; (3) R is strongly regular if and only if R is regular; (4) R is clean if and only if R is exchange. (5) exchange rings have stable range 1.

Key Words: Weakly normal rings, Abelian rings, regular rings, quasi-normal rings, semiabelian rings, exchange rings, clean rings

1. Introduction

Throughout this paper, all rings are associative with identity. Let R be a ring, we use $E(R)$, $N^*(R)$, $J(R)$ and $N(R)$ to denote the set of all idempotents, the nilradical (i.e., the sum of all nil ideals), the Jacobson radical and the set of all nilpotent elements in R, respectively. According to [5], a ring R is called reversible if $ab = 0$ implies $ba = 0$ for $a, b \in R$. In [1], Anderson and Camillo observed the rings whose zero products commute, used the term ZC_2 for what is called reversible; while Krempa and Niewieczerzal [8] took the term C_0 for it. In [17], a generalization of reversible rings is given, that is, a ring R is called weakly reversible if $ab = 0$ implies that $Rbra$ is a nil left ideal of R for all $a, b, r \in R$. Clearly semicommutative rings (e.g., $ab = 0$ implies $aRb = 0$ for all $a, b \in R$) are weakly reversible. A ring R is *Abelian* if every idempotent element of R is contained in the central $C(R)$ of R. Evidently, semicommutative rings are Abelian.

According to [13], an element k of a ring R is called left minimal if Rk is a minimal left ideal of R, and an idempotent e of R is said to be left minimal idempotent if e is a left minimal element of R. We use $ME_l(R)$ to denote the set of all left minimal idempotent elements of R.

According to [13], A ring R is left min-abel if every element of $ME_l(R)$ is left semicentral in R. Clearly, Abelian rings and so semicommutative rings are left min-abel.

A ring R is called quasi-normal if $ae = 0$ implies $eaRe = 0$ for $a \in N(R)$ and $e \in E(R)$ [14], and R is said to be semiabelian [4] if every idempotent of R is either left semicentral or right semicentral. Clearly, Abelian rings are semiabelian and quasi-normal. Following [4], we know that there exists a semiabelian ring which is not Abelian.

A ring R is called weakly normal if for all $a, r \in R$ and $e \in E(R)$, $ae = 0$ implies $Rera$ is a nil left ideal of R. Clearly, weakly reversible rings and Abelian rings are weakly normal.

A ring R is called an exchange ring [9] if for every $x \in R$ there exists $e \in E(R)$ such that $e \in xR$ and $1 - e \in (1 - x)R$, and R is said to be clean if every element of R is a sum of a unit and an idempotent [9]. Clearly, clean rings are always exchange rings. And the converse is true when R is an Abelian ring by [16]. But, as far as we can determine, it appears to be an open question whether exchange rings are exactly clean rings in general.

The present paper is such an attempt in this direction, in other words, we shall give some weaker conditions for exchange rings being clean rings such as weakly normal rings.

Recall that a ring R is said to be have stable range 1 if for any $a, b \in R$ satisfying $Ra + Rb = R$, there exists $y \in R$ such that $a + yb$ is right invertible [11]. In [16, Theorem 6], Yu, H.P. showed that exchange rings with all idempotents central have stable range 1. In this paper, we generalize this result to weakly normal rings.

In section 2, using trivial extensions of rings, we give some characterization of weakly normal rings. In term of these characterizations, we discuss the relations among semiabelian rings, quasi-normal rings, weakly normal rings and left min-abel rings.

In section 3, we give some applications of weakly normal rings. It is shown that: (1) R is an Abelian ring if and only if R is a weakly normal strongly left idempotent reflexive ring; (2) R is a strongly regular ring if and only if R is a weakly normal von Neumann regular ring; (3) A weakly normal ring R is exchange if and only if R is clean; (4) A weakly normal exchange ring has stable range 1.

2. Some properties of weakly normal rings

We begin with the following characterization of weakly normal rings.

Theorem 2.1 *The following conditions are equivalent for a ring R:*

(1) R is a weakly normal ring.

(2) $Rer(1 - e)$ is a nil left ideal of R for all $e \in E(R)$ and $r \in R$.

(3) $eR(1 - e) \subseteq N^*(R)$ for any $e \in E(R)$.

Proof. (1) $\Longrightarrow$ (2) Let $e \in E(R)$. Then $(1 - e)e = 0$ implies $Rer(1 - e)$ is a nil left ideal of R for any $r \in R$ because R is a weakly normal ring.

(2) $\Longrightarrow$ (3) Let $e \in E(R)$. Then $Rer(1 - e) \subseteq N^*(R)$ for all $r \in R$ by (2). Hence $eR(1 - e) \subseteq N^*(R)$.

(3) $\Longrightarrow$ (1) Assume that $ae = 0$, where $a \in R$ and $e \in E(R)$. Hence $a = a(1 - e)$. By (3), $eRa = eRa(1 - e) \subseteq eR(1 - e) \subseteq N^*(R)$, so, we have $Rera \subseteq N^*(R)$ for any $r \in R$, which shows that R is a weakly normal ring. $\square$

Call an element a of a ring R left projective if $_R Ra$ is a projective module. Clearly, every idempotent element of R is left projective. Call an element e of a ring R a $op - idempotent$ if $e^2 = -e$. Clearly, op-idempotent is not idempotent in general. For example, let $R = Z/3Z$. Then $\bar{2} \in R$ is a op-idempotent, while it is not idempotent. Call an element $e \in R$ $potent$ in case there exists some integer $n \geq 2$ such that $e^n = e$ (We write $p(e)$ for the minimal positive integer n). Clearly, idempotent is potent, while there exists a potent element which is not idempotent. For example, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in M_2(Z)$ is a potent element, while it is not idempotent. We use $P_l(R)$, $E^o(R)$ and $PE(R)$ to denote the set of all left projective elements, the set of all op-idempotent elements and the set of all potent elements of R. By Theorem 2.1, we observe that every weakly-normal rings can be characterized by its op-idempotents and potent elements as follows.

Theorem 2.2 *(1) R is a weakly normal ring if and only if $eR(1 + e) \subseteq N^*(R)$ for any $e \in E^o(R)$.*

(2) R is a weakly normal ring if and only if $eR(1 - e^{p(e)-1}) \subseteq N^*(R)$ for any $e \in PE(R)$.

(3) R is a quasi-normal ring if and only if $eR(1 + e)Re = 0$ for all $e \in E^o(R)$.

(4) R is a quasi-normal ring if and only if $eR(1 - e^{p(e)-1})Re = 0$ for any $e \in PE(R)$.

(5) R is a left min-abel ring if and only if $eR(1 - e)Re = 0$ for all $e \in ME_l(R)$.

(6) The following conditions are equivalent for a ring R:

(a) R is a weakly normal ring.

(b) For any $a \in R$ and $k \in P_l(R)$, $ak = 0$ implies $Rkra$ is a nil left ideal of R.

(c) $Rer(1 - e)$ is a nil left ideal of R for all $e \in E(R)$ and $r \in N(R)$.

(d) $eN(R)(1 - e) \subseteq N^*(R)$ for any $e \in E(R)$.

Proof. (1) and (2) are immediate consequences of Theorem 2.1.

(3) and (4) follow from [14, Theorem 2.1].

(5) Assume that R is a left min-abel ring and $e \in ME_l(R)$. Since e is left semicentral, $(1 - e)Re = 0$. Hence $eR(1 - e)Re = 0$.

Converse, if $e \in ME_l(R)$, then by hypothesis, $eR(1-e)Re = 0$. If $(1-e)Re \neq 0$, then $R(1-e)Re = Re$ because Re is a minimal left ideal of R. Hence $eR(1 - e)Re = eRe \neq 0$, which is a contradiction. Therefore $(1 - e)Re = 0$ and so R is a left min-abel ring.

(6) $(a) \Longrightarrow (b)$ Let $a \in R$ and $k \in P_l(R)$ with $ak = 0$. Since $_R Rk$ is a projective module, there exists $e \in E(R)$ such that $l(k) = l(e)$. Hence $k = ek$ and $ae = 0$ because $a \in l(k)$. Since R is a weakly normal ring, $Rexa$ is a nil left ideal of R for any $x \in R$. Especially, $Rkra = Rekra$ is a nil left ideal of R for any $r \in R$.

$(b) \Longrightarrow (a)$ is clear because $E(R) \subseteq P_l(R)$.

$(a) \Longrightarrow (c)$ and $(a) \Longrightarrow (d)$ are direct results of Theorem 2.1.

$(c) \Longrightarrow (a)$ Since for any $r \in R$ and $e \in E(R)$, $er(1 - e) \in N(R)$. Hence, by (c), $Rer(1 - e) = Re(er(1 - e))(1 - e)$ is a nil left ideal of R.

$(d) \Longrightarrow (a)$ Since for any $e \in E(R)$, $eR(1 - e) \subseteq N(R)$. Hence $eR(1 - e) = e(eR(1 - e))(1 - e) \subseteq eN(R)(1 - e) \subseteq N^*(R)$ by (d). □

By Theorem 2.1 and Theorem 2.2, we have the following corollaries.

Corollary 2.3 *(1) Quasi-normal rings are weakly normal.*

(2) Let R be a left pp ring. Then R is a weakly normal ring if and only if R is a weakly reversible ring.

Proof. (1) Let R be a quasi-normal ring and $e \in E^o(R)$. Then $eR(1+e)Re = 0$ by Theorem 2.2(3), so $ReR(1+e)$ is a nilpotent left ideal of R, which implies that $ReR(1+e) \subseteq N^*(R)$. Hence R is a weakly normal ring by Theorem 2.2(1).

(2) Since R is a left pp ring, $P_l(R) = R$. By Theorem 2.2(6), we know that R is a weakly reversible ring if and only if R is a weakly normal ring. $\square$

Corollary 2.4 *(1) weakly normal rings are left min-abel.*

(2) Weakly normal rings are directly finite.

Proof. (1) Let $e \in ME_l(R)$ and $a \in R$. Write $h = ae - eae$. If $h \neq 0$, then $eh = 0$, $he = h$ and $Rh = Re$. Since R is a weakly normal ring, by Theorem 2.1, $R(1-e)re \subseteq N^*(R)$ for any $r \in R$. Especially, $Re = Rh = R(1-e)h = R(1-e)he \subseteq N^*(R)$, which is a contradiction. Hence $h = 0$, which implies e is left semicentral in R, so R is left min-abel.

(2) Let R be a weakly normal ring and $ab = 1$. Write $e = ba$. Then $eR(1-e) \subseteq N^*(R)$ by Theorem 2.1. Especially, $1-e = ab(1-e) = aeb(1-e) \in N^*(R)$, which implies $1-e = 0$. Hence R is a directly finite ring. $\square$

It is well known that for any positive integer n, the $n \times n$ full matrix rings $M_n(R)$ over real number field R are directly finite. But, by the following Example 2.13, we know that $M_n(R)$ are not weakly normal for $n \geq 2$. Hence, the converse of Corollary 2.4(2) is not true in general.

By Corollary 2.3 and Corollary 2.4, we obtain the following corollary.

Corollary 2.5 *(1) Quasi-normal rings are directly finite and left min-abel.*

(2) Semi-abelian rings are directly finite and left min-abel.

(3) Weakly reversible rings are directly finite and left min-abel.

The following corollary also follows from Theorem 2.1 and Theorem 2.2.

Corollary 2.6 *(1) The subrings and finite direct products of weakly normal rings are weakly normal.*

(2) Let I be an ideal of a weakly normal ring R and idempotents can be lifted modulo I. Then R/I is also a weakly normal ring.

(3) Let R be a weakly normal ring. If $e \in E(R)$ satisfies $ReR = R$, then $e = 1$.

(4) Let R be a weakly normal ring. If $e \in E^o(R)$ satisfies $ReR = R$, then $e = -1$.

(5) Let R be a weakly normal ring. If $e \in PE(R)$ satisfies $ReR = R$, then $e^{p(e)-1} = 1$.

Lemma 2.7 *Let R be a ring and I an ideal of R such that R/I is weakly normal. If $I \subseteq N(R)$, then R is weakly normal.*

Proof. Let $a \in R$ and $e \in E(R)$ with $ae = 0$. Hence, in $\bar{R} = R/I$, $\bar{a}\bar{e} = \bar{0}$. Since $\bar{R}$ is weakly normal, $\bar{R}\bar{e}\bar{r}\bar{a}$ is a nil left ideal of $\bar{R}$ for all $r \in R$. Hence, for any $x \in R$, there exists $n \geq 1$ such that $(xera)^n \in I$. Since

$I \subseteq N(R)$, there exists $m \geq 1$ such that $(xera)^{nm} = 0$, which implies $xera \in N(R)$ for all $x \in R$. Hence $Rera$ is a nil left ideal of R for all $r \in R$. Thus R is a weakly normal ring. □

Theorem 2.8 *Suppose S and T are rings, and M is an (S,T)-bimodule. Let $R = \begin{pmatrix} S & M \\ 0 & T \end{pmatrix}$. Then R is weakly normal if and only if S and T are weakly normal.*

Proof. By Corollary 2.6(1), we know that if R is weakly normal then S and T are weakly normal.

Conversely, suppose S and T are weakly normal. Put $I = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$. Then I is an ideal of R and $R/I \cong S \times T$ is weakly normal by Corollary 2.6(1) and hypothesis. Since $I \subseteq N(R)$, by Lemma 2.7, R is weakly normal. □

The following corollary follows immediately by Theorem 2.8 and induction on n.

Corollary 2.9 *R is a weakly normal ring if and only if, for any $n \geq 1$, the $n \times n$ upper triangular matrix ring $T_n(R)$ is a weakly normal ring.*

Given a ring R and a bimodule $_RM_R$, the trivial extension of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2).$$

This is isomorphic to the ring of all matrix $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

Corollary 2.10 *R is a weakly normal ring if and only if its trivial extension is a weakly normal ring.*

Corollary 2.11 *Let R be a ring. Then R is a weakly normal ring if and only if for any $n \geq 1$, $R[x]/(x^n)$ is a weakly normal ring, where (x^n) is the ideal of $R[x]$ generated by x^n.*

Theorem 2.12 *If R is a subdirect product of a finite family of weakly normal rings $\{R_i : i = 1, 2, \cdots, m\}$, then R is a weakly normal ring.*

Proof. Let $R_i = R/A_i$ where A_i be ideals of R with $\cap_{i=1}^{m} A_i = 0$. Let $e \in E(R)$. Then $e_i = e + A_i \in E(R_i)$, $i = 1, 2, \cdots, m$. Since each R_i is weakly normal, $R_i e_i r_i (1 - e_i) \subseteq N^*(R_i)$ for all $r_i = r + A_i \in R_i$. So, for any $x, r \in R$, there exist a family positive integers $n_i, i = 1, 2, \cdots, m$ such that $(x_i e_i r_i (1 - e_i))^{n_i} = 0$ in R_i, where $x_i = x + A_i$ and $r_i = r + A_i$. This implies $(xer(1 - e))^{n_i} \subseteq A_i$ for all i. Let $n = max\{n_1, n_2, \cdots, n_m\}$. Then we have $(xer(1 - e))^n \subseteq \cap_{i=1}^{m} A_i = 0$ for all $x, r \in R$. Therefore R is a weakly normal ring. □

Example 2.13 *If R is a weakly normal ring and n is any integer greater than 1, then the full matrix ring $M_n(R)$ is not weakly normal.*

For $n = 2$, observe that

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

But $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \notin N(M_2(R))$.

So $M_2(R)$ is not weakly normal. One can augment these matrices in a similar way if $n > 2$.

Being weakly normal is not a Morita invariant property by Example 2.13.

The following example is given to show that the converse of Corollary 2.3 is not true in general.

Let F be a division ring and $R = \begin{pmatrix} F & F & F \\ 0 & F & F \\ 0 & 0 & F \end{pmatrix}$. Consider the idempotent $e = e_{11} + e_{33}$, by computing,

we can see that $eR(1-e)Re = \begin{pmatrix} 0 & 0 & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$, so R is not quasi-normal by [14, Theorem 2.1]. But by

Corollary 2.9, we know that R is weakly normal. Hence there exists a weakly normal ring R which is not quasi-normal and so there exists a left min-abel ring R which is not quasi-normal by Corollary 2.4. $\square$

By Example 2.13, we know that there exists a ring S which is not weakly normal. By Corollary 2.6(1), the polynomial ring $S[x]$ is not weakly normal. But $S[x]$ is left min-abel. Hence the converse of Corollary 2.4(1) is not true, that is there exists a left min-abel ring which is not weakly normal.

Example 2.14 *Let R_1 and R_2 be semiabelian rings which are not Abelian. Take $e_1 \in E(R_1)$ to be a right semicentral which is not central and $e_2 \in E(R_2)$ to be a left semicentral which is not central, then the idempotent (e_1, e_2) is neither right nor left semicentral in $R_1 \oplus R_2$. Hence $R_1 \oplus R_2$ is not semiabelian, but $R_1 \oplus R_2$ is quasi-normal.*

By Example 2.14, we know that $\{Abelian\ rings\} \subsetneq \{semiabelian\ rings\} \subsetneq \{quasi-normal\ rings\} \subsetneq \{weakly\ normal\ rings\} \subsetneq \{left\ min-abel\ rings\}$.

It is well known that (1) a ring R is an exchange ring if and only if $R/J(R)$ is an exchange ring and idempotents can be lifted modulo $J(R)$. (2) If R is an exchange ring and $R/J(R)$ is an Abelian ring, then R is a left and right quasi-duo ring. So we have the following theorem.

Theorem 2.15 *Let R be an exchange ring. If R is a weakly normal ring, then $R/J(R)$ is an Abelian ring and R is a left quasi-duo ring.*

Proof. Let $a \in R$ with $a - a^2 \in J(R)$. Since R is an exchange ring, there exists $e \in E(R)$ such that $e - a \in J(R)$. Since R is a weakly normal ring, $(1-e)Re \subseteq N^*(R)$ by Theorem 2.1. Since $N^*(R) \subseteq J(R)$, $(1-e)Re \subseteq J(R)$. Hence $(1-a)Ra \subseteq J(R)$, which implies $R/J(R)$ is an Abelian ring. Consequently, R is a left quasi-duo ring. $\square$

By Theorem 2.15 and Corollary 2.3(1), we have the following corollary.

Corollary 2.16 *Let R be an exchange ring. If R is a quasi-normal ring, then R is a left quasi-duo ring.*

Observing for any division ring F, the ring $R = \begin{pmatrix} F & F & F \\ 0 & F & F \\ 0 & 0 & F \end{pmatrix}$ is an exchange left quasi-duo ring. So

there exists an exchange left quasi-duo ring which is not quasi-normal. Since R is also a weakly normal ring, we naturally ask whether exchange left quasi-duo rings are always weakly normal.

Theorem 2.17 *Let R be a weakly normal ring. Then*

(1) If M is a maximal left ideal of R, then for any $e \in E(R)$, either $e \in M$ or $1 - e \in M$.

(2) If R is a clean ring, then for any two distinct maximal left ideals M and N of R, there exists $e \in E(R)$ such that $e \in M \setminus N$ and $1 - e \in N \setminus M$.

Proof. (1) If $e \notin M$, then $Re + M = R$. Let $1 = xe + m$ for some $x \in R$ and $m \in M$. Since $(1 - e)xe \in N^*(R) \subseteq J(R) \subseteq M$, $1 - e = (1 - e)xe + (1 - e)m \in M$.

(2) Since $M \neq N$, there exists $a \in M \setminus N$. Hence $Ra + N = R$ and then $1 - ra \in N$ for some $r \in R$. Clearly, $ra \in M \setminus N$. Since R is a clean ring, $ra = e + u$ where $e \in E(R)$ and $u \in R$ is an invertible element of R. Since $u = ra - e \notin M$, $e \notin M$. By (1), $1 - e \in M$. If $e \notin N$, then $1 - e \in N$ by (1). Hence $u = ra - e = (ra - 1) + (1 - e) \in N$, which is a contradiction. Thus $e \in N$, so $1 - e \notin N$. Therefore $e \in N \setminus M$ and $1 - e \in M \setminus N$. $\square$

3. Some applications

A ring R is called (1) left idempotent reflexive if $aRe = 0$ implies $eRa = 0$ for all $a \in R$ and $e \in E(R)$; (2) strongly left idempotent reflexive if $ae = 0$ implies $ea = 0$ for all $a \in J(R)$ and $e \in E(R)$; (3) weakly left idempotent reflexive if $ae = 0$ implies $ea = 0$ for all $a \in R$ and left semi-central idempotent e of R.

A ring R is called (1) von Neumann regular if $a \in aRa$ for all $a \in R$; (2) unit-regular if for any $a \in R$, $a = aua$ for some $u \in U(R)$, where $U(R)$ denotes the group of units of R; (3) strongly regular if $a \in a^2R$ for all $a \in R$; (4) n-regular [12] if $a \in aRa$ for all $a \in N(R)$.

Clearly, strongly regular $\implies$ unit-regular $\implies$ von Neumann regular $\implies$ n-regular; strongly regular $\implies$ reduced $\implies$ n-regular; Abelian rings are strongly left idempotent reflexive.

Lemma 3.1 *(1) Strongly left idempotent reflexive rings are left idempotent reflexive, and left idempotent reflexive rings are weakly left idempotent reflexive.*

(2) Let R be a ring. If $J(R) \subseteq C(R)$, then R is a strongly left idempotent reflexive ring. Hence semiprimitive rings are strongly left idempotent reflexive.

(3) R is an Abelian ring if and only if R is a weakly normal strongly left idempotent reflexive ring.

(4) R is an Abelian ring if and only if R is a quasi-normal left idempotent reflexive ring.

(5) R is an Abelian ring if and only if R is a semiabelian weakly left idempotent reflexive ring.

(6)Let R be a weakly normal ring and $x \in R$. If x is von Neumann regular, then $x \in Rx^2 \cap x^2R$.

Proof. (1) First, we assume that R is a strongly left idempotent reflexive ring and $aRe = 0$, where $a \in R$ and $e \in E(R)$. If $eRa \neq 0$, then there exists $b \in R$ such that $eba \neq 0$. Since $(ebaR)^2 = 0$, $eba \in J(R)$.

Hence $ebae = 0$ implies $e(eba) = 0$ because R is a strongly left idempotent reflexive ring. Hence $eba = 0$ which contradicts $eba \neq 0$. Thus $eRa = 0$ and so R is a left idempotent reflexive ring.

Next, suppose R is a left idempotent reflexive ring and $bg = 0$, where $b \in R$ and g is a left semi-central idempotent of R. Since $bRg = bgRg = 0$ and R is a left idempotent reflexive ring, $gRb = 0$. Thus $gb = 0$ and so R is a weakly left idempotent reflexive ring.

(2) It is evident.

Now let $e \in E(R)$. For any $a \in R$, set $h = ea - eae$. Then $eh = h, he = 0$ and $h^2 = 0$. If R is a weakly normal strongly left idempotent reflexive ring, then $Rerh$ is a nil left ideal of R for any $r \in R$, especially, $Rh = Reh \subseteq N^*(R) \subseteq J(R)$. Hence $he = 0$ implies $h = eh = 0$; If R is a quasi-normal left idempotent reflexive ring, then $ehRe = 0$, that is $hRe = 0$. Hence $eRh = 0$, and so $h = eh = 0$; If R is a semiabelian weakly left idempotent reflexive ring, then e is either left semi-central or right semi-central. If e is left semi-central, then $he = 0$ implies $eh = 0$, which shows that $h = 0$. If e is right semi-central, then $h = 0$ is evident. All these show that every idempotent of R is left semi-central, hence R is Abelian. Consequently, (3), (4) and (5) hold.

(6) Let $x = xyx$ for some $y \in R$. Write $e = yx$. Then $e^2 = e \in R$ and $x = xe$. Since R is a weakly normal ring, $R(1 - e)x$ is a nil left ideal of R. Thus there exists $n \geq 1$ such that $(y(1 - e)x)^n = 0$. Since $y(1 - e)x = yx - yex = e - yex$, there exists $a \in R$ such that $0 = (y(1 - e)x)^n = (e - yex)^n = e - aex$. Hence $x = xe = xaex = xayx^2 \in Rx^2$. Now let $g = xy$. Then $x = gx$ and $g \in E(R)$. By Theorem 2.1, $gR(1 - g) \subseteq N^*(R)$, so $x(1 - g) = gx(1 - g) \in gR(1 - g) \subseteq N^*(R)$. Thus $x(1 - g)y \in N^*(R)$ which implies $x \in x^2R$. $\square$

The following theorem follows from Lemma 3.1(6).

Theorem 3.2 *The following conditions are equivalent for a ring R.*

(1) R is a strongly regular ring.

(2) R is a weakly normal ring and unit-regular ring.

(3) R is a weakly normal ring and von Neumann regular ring.

(4) R is a quasi-normal ring and von Neumann regular ring.

(5) R is a semiabelian ring and von Neumann regular ring.

(6) R is a weakly reversible ring and von Neumann regular ring.

Let R be a ring and $a \in R$. Then a is called Π-regular, if there exists $n \geq 1$ and $b \in R$ such that $a^n = a^nba^n$, and a is said to be strongly Π-regular, if $a^n = ba^{n+1}$. A ring R is called Π-regular and strongly Π-regular, if every element of R is Π-regular and strongly Π-regular, respectively. A ring R is called strongly clean if $a = e + u$ where $e \in E(R)$, $u \in U(R)$ and $eu = ue$ for every $a \in R$.

Corollary 3.3 *Let R be a weakly normal Π-regular ring. Then R is a strongly Π-regular ring, consequently, R is a strongly clean ring.*

Proof. Since R is $\Pi-$ regular, for any $x \in R$, there exists a positive integer n such that x^n is von Neumann regular. Since R is a weakly normal, $x^n \in R(x^n)^2 \subseteq Rx^{n+1}$ by Lemma 3.1(6). Thus R is a strongly $\Pi-$regular. By [10], strongly Π-regular rings are always strongly clean. $\square$

Theorem 3.4 *The following conditions are equivalent for a ring R.*

  (1) R is a reduced ring.

  (2) R is a weakly reversible n-regular ring.

  (3) R is a weakly normal n-regular ring.

  (4) R is a quasi-normal n-regular ring.

  (5) R is a semiabelian n-regular ring.

Proof. $(1) \Longrightarrow (2) \Longrightarrow (3)$ and $(1) \Longrightarrow (5) \Longrightarrow (4) \Longrightarrow (3)$ are trivial.

  $(3) \Longrightarrow (1)$ Suppose $a^2 = 0$, then $a = aba$ for some $b \in R$ because R is n-regular. By Lemma 3.1(6), $a \in Ra^2 = 0$, which implies $a = 0$. Thus R is a reduced ring. $\square$

Following [9], a ring R is called clean if every element of R is a sum of a unit and an idempotent and clean rings are always exchange rings, but the converse is not true unless R is an Abelian ring. Hence, we have the following theorem.

Theorem 3.5 *Let R be a weakly normal ring. Then R is clean if and only if R is exchange.*

Proof. One direction is trivial.

  For the other direction, let R be an exchange ring, then $R/J(R)$ is exchange and idempotents can be lifted modulo $J(R)$. By Corollary 2.6(2), $R/J(R)$ is weakly normal because R is weakly normal. Since $R/J(R)$ is semiprimitive, $R/J(R)$ is strongly left idempotent reflexive by Lemma 3.1(2). Hence $R/J(R)$ is Abelian by Lemma 3.1(3). Therefore $R/J(R)$ is clean by [9], so, by [3, Proposition 7], R is a clean ring. $\square$

The following corollary is an immediate result of Theorem 3.5.

Corollary 3.6 *(1) Let R be a semiabelian ring. Then R is clean if and only if R is exchange [4].*

  (2) Let R be a quasi-normal ring. Then R is clean if and only if R is exchange [14].

  (3) Let R be a weakly reversible ring. Then R is clean if and only if R is exchange.

Recall that a ring R is said to have stable range 1 [11] if for any $a, b \in R$ satisfying $aR + bR = R$, there exists $y \in R$ such that $a + by$ is right invertible. Clearly, R has stable range 1 if and only if $R/J(R)$ has stable range 1. [16 Theorem 6] showed that exchange rings with all idempotents central have stable range 1. We now generalize this result as follows.

Theorem 3.7 *Weakly normal exchange rings have stable range 1.*

Proof. Let R be a weakly normal exchange ring. Then $R/J(R)$ is exchange with all idempotents central by the proof of Theorem 3.5, so, by [16, Theorem 6], $R/J(R)$ has stable range 1. Therefore R has stable range 1. $\square$

Naturally, we have the following corollary.

Corollary 3.8 *(1) Semiabelian exchange rings have stable range 1 [4].*

  (2) Quasi-normal clean rings have stable range 1 [14].

(3) Quasi-normal Π-regular rings have stable range 1.

(4) Weakly normal clean rings have stable range 1.

(5) Weakly normal Π-regular rings have stable range 1.

(6) Quasi-normal exchange rings have stable range 1.

(7) Semiabelian Π-regular rings have stable range 1.

(8) Semiabelian clean rings have stable range 1.

(9) Weakly reversible Π-regular rings have stable range 1.

(10) Weakly reversible clean rings have stable range 1.

(11) Weakly reversible exchange rings have stable range 1.

[6] showed that if R is a unit regular ring, then every element in R is a sum of two units. A ring R is called an $(S, 2)$-ring [7] if every element in R is a sum of two units of R. In [2, Theorem 6] it is proved that if R is an abelian Π-regular ring, then R is an $(S, 2)$-ring if and only if $\mathbb{Z}/2\mathbb{Z}$ is not a homomorphic image of R. Clearly, R is an $(S, 2)$-ring if and only if $R/J(R)$ is an $(S, 2)$-ring.

In [15], a ring R is said to satisfy the unit 1-stable condition if for any $a, b, c \in R$ with $ab + c = 1$, there exists $u \in U(R)$ such that $au + c \in U(R)$. It is easy to prove that R satisfies the unit 1-stable condition if and only if $R/J(R)$ satisfies the unit 1-stable condition. [4, Proposition 3.18] showed that for a semiabelian exchange ring R, R is an $(S, 2)$-ring if and only if R satisfies the unit 1-stable condition.

Combining [4, Proposition 3.18] with Lemma 3.1, Corollary 2.6, we have the following proposition.

Proposition 3.9 *Let R be a weakly-normal exchange ring. Then the following conditions are equivalent:*

(1) R is an $(S, 2)$-ring.

(2) R satisfies the unit 1-stable condition.

(3) Every factor ring R_1 of R is an $(S, 2)$-ring.

(4) $\mathbb{Z}_2$ is not a homomorphic image of R.

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