Turk J Math
36 (2012) , 59 – 65.

doi:10.3906/mat-1101-76

A homotopy for a complex of free Lie algebras

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Abstract

Using the Guichardet construction, we compute the cohomology groups of a complex of free Lie algebras introduced by Alekseev and Torossian.

1. Introduction

In their study of the relation between the KV-conjecture and Drinfeld's associators, Alekseev and Torossian [1] studied the Eilenberg-MacLane differential $\delta_A : L_n \to L_{n+1}$, where L_n is the free Lie algebra in n variables, and computed the cohomology groups of δ_A in dimensions $1, 2$. Following the construction of Guichardet [2] (see also [3]), we remark that the complex δ_A is acyclic, except in dimensions $1, 2$, where the cohomology is of dimension 1. We also identify the cohomology groups of a similar complex $\delta_A : T_n \to T_{n+1}$, where T_n is the free associative algebra in n variables: the cohomology is of dimension 1 in any degree. The Guichardet construction provides an explicit homotopy.

Alekseev and Torossian used the computations in dimension 2 to deduce the existence of a solution to the KV problem from the existence of an associator. A simple by-product of their computation is the existence and the uniqueness of the Campbell-Hausdorff formula. We do not have any other application of the computations of higher cohomologies.

In this note, we start with a review of the construction of Guichardet. Then we adapt it to free associative algebras and free Lie algebras.

I am thankful to the referee for his careful reading.

2. The Guichardet construction

Let V be a finite dimensional real vector space. Let F^n be the space of polynomial functions f on $V \oplus V \oplus \cdots \oplus V$. An element f of F^n is written as $f(v_1, v_2, \ldots, v_n)$.

Define

$$(\delta_n f)(v_1, \ldots, v_{n+1}) = \sum_{i=1}^{n} (-1)^i f(v_1, v_2, \ldots, v_{i-1}, \hat{v}_i, v_{i+1}, \ldots, v_n).$$

For example:

$$(\delta_1 f)(v_1, v_2) = -f(v_2) + f(v_1)$$

$$(\delta_2 f)(v_1, v_2, v_3) = -f(v_2, v_3) + f(v_1, v_3) - f(v_1, v_2).$$

We define $F^0 = \mathbb{R}$, and embed $F^0 \to F^1$ as the constant functions.

The complex $0 \to F^0 \to F^1 \to \cdots$ is acyclic except in degree 0. Indeed, $s : F^n \to F^{n-1}$ given by

$$(sf)(v_1, v_2, \ldots, v_{n-1}) = f(0, v_1, v_2, \ldots, v_{n-1}) \tag{1}$$

satisfies $\mathrm{Id} := s\delta + \delta s$.

Now the additive group V operates on F^n by translations: if $\alpha \in V$, we write

$$(\tau(\alpha)f)(v_1, \ldots, v_n) = f(v_1 - \alpha, \ldots, v_n - \alpha).$$

The differential δ commutes with translations, so that it induces a differential δ_A on the subspace of translation invariant functions.

It is well known that the cohomology of the complex δ_A is isomorphic with $\Lambda^{n-1}V^*$. Here, we recall Guichardet's explicit construction of the isomorphism as we will adapt it to the "universal case" considered by Alekseev-Torossian.

Let Ω^{n-1} be the space of differential forms of exterior degree $n-1$ on V, with polynomial coefficients, equipped with the de Rham differential.

Consider the simplex $S := S_{v_1, v_2, \ldots, v_n}$ in V with vertices $(v_1, v_2, \ldots, v_n)$. Thus the map $\Omega^{n-1} \to F^n$ defined by $\omega \to \int_S \omega$ induces a map from Ω^{n-1} to F^n. This map commutes with the differentials (as follows from Stokes formula) and with the natural action by translations.

Conversely, associate to $f \in F^n$ a differential form $\omega(f)$ of degree $(n-1)$ by setting for $v_1, v_2, \ldots, v_{n-1}$ vectors in V, identified with tangent vectors at $v \in V$:

$$\langle \omega(f)(v), v_1 \wedge v_2 \wedge \cdots \wedge v_{n-1} \rangle = \sum_{\sigma \in \Sigma_{n-1}} \epsilon(\sigma) \frac{d}{d\epsilon}|_0 f(v, v + \epsilon_1 v_{\sigma(1)}, \ldots, v + \epsilon_{n-1} v_{\sigma(n-1)}).$$

Here if ϕ is a polynomial function of $\epsilon_1, \ldots, \epsilon_{n-1}$, we employ the notation $\frac{d}{d\epsilon}|_0 \phi(\epsilon)$ for the coefficient of $\epsilon_1 \cdots \epsilon_{n-1}$ in ϕ.

The map ω commutes with the differential, and with the action of V by translations. Thus the map $P_n : F^n \to F^n$ defined by

$$P_n(f) = \int_S \omega(f)$$

produces a map from $F^n \to F^n$, commuting with the action of V. This map is the identity on F^1.

Let us give the formulae for P_n so that we see that the map P_n is "universal".

Given $v := (v_1, v_2, \ldots, v_n) \in V$, consider the map $p_v : \mathbb{R}^{n-1} \to V$ given by

$$p_v(t_1, t_2, \ldots, t_{n-1}) = v_1 + t_1(v_2 - v_1) + \cdots + t_{n-1}(v_n - v_1).$$

This map sends the standard simplex Δ_{n-1} defined by

$$t_i \geq 0, \sum_{i=1}^{n-1} t_i \leq 1$$

to the simplex S in V with vertices $v_1, v_2, \ldots, v_n$.

Let us consider the form
$$p_v^*\omega(f) = f(t,v)dt_1 \wedge \cdots \wedge dt_{n-1}.$$

The map P_n is given by
$$(P_n f)(v) = \int_{\Delta_{n-1}} f(t,v)dt,$$
where $f(t,v)$ is the element of F_n depending on t described as follows.

Lemma 2.1 *Let*
$$v(t) = v_1 + t_1(v_2 - v_1) + \cdots + t_{n-1}(v_n - v_1).$$
Define

$$f(t, v_1, v_2, \ldots, v_n) = \frac{d}{d\epsilon}|_0 \sum_{\sigma\in\Sigma[2,\ldots,n]} \epsilon(\sigma) f(v(t), v(t) + \epsilon_1(v_{\sigma(2)} - v_1), \ldots, v(t) + \epsilon_{n-1}(v_{\sigma(n)} - v_1)). \qquad (2)$$

Here $t = (t_1, t_2, \ldots, t_{n-1})$ and $\Sigma([2,\ldots,n])$ is the group of permutations of the set with $(n-1)$ elements $[2,\ldots,n]$.

Then we have the formula
$$(P_n f)(v_1, v_2, \ldots, v_n) = \int_{\Delta_{n-1}} f(t,v)dt_1 dt_2 \cdots dt_{n-1}.$$

Let $H := \mathrm{Id} - P$. Using the injectivity of the vector spaces F^n in the category of V-modules, as it is standard, and we will review the procedure below, to produce a homotopy
$$G : F^n \to F^{n-1}$$
commuting with the action of V by translations such that
$$H = G\delta + \delta G.$$

We first use the following injectivity lemma.

Lemma 2.2 *Let A, B be two real vector spaces provided with a structure of V-modules. Let $u : A \to F^n$ be a V-module map from A to F^n. Let $v : A \to B$ be an injective map of V-modules. Then there exists a map $w : B \to F^n$ of V-modules extending u.*

The formula for a map w (depending on a choice of retraction) is given below in the proof.

Proof. Denote by τ the action of V on B. Let s be a linear map from B to A such that $sv = \mathrm{Id}$. Let $b \in B$: we define the map w (depending on our choice of linear retraction s) by
$$w(b)(v_1, v_2, \ldots, v_n) = u(s\tau(-v_1)b)(0, v_2 - v_1, \ldots, v_n - v_1).$$

We verify that b satisfy the wanted conditions. The crucial point is that the map w is a map of V-modules, as we now show. Indeed,

$$w(\tau(v_0)b)(v_1, v_2, \ldots, v_n) = u(s(\tau(-v_1)\tau(v_0)b))(0, v_2 - v_1, \ldots, v_n - v_1)$$

$$= u(s(\tau(-v_1 + v_0)b))(0, v_2 - v_1, \ldots, v_n - v_1),$$

while

$$(\tau(v_0)w(b))(v_1, v_2, \ldots, v_n) = w(b)(v_1 - v_0, v_2 - v_0, \ldots, v_n - v_0)$$

$$u(\tau(v_0 - v_1))byu(s(\tau(v_0 - v_1)).$$

□

We now apply this lemma to define G inductively. Consider the injective map deduced from δ from $F^n/\delta(F^{n-1})$ to F^{n+1}.

Recall our linear map $s : F^{n+1} \to F^n$ given by equation (1). We may take as linear inverse (that we still call s) the map $s : F^{n+1} \to F^n$ followed by the projection $F^n \to F^n/\delta(F^{n-1})$.

We define $G^1 = 0$ and inductively G^{n+1} as the map extending

$$H^n - \delta G^n : F^n \to F^n$$

to F^{n+1} constructed in Lemma 2.2. Indeed, $(H^n - \delta G^n)\delta = \delta H^{n-1} - \delta(-\delta G^{n-1} + H^{n-1}) = 0$ so that the map $H^n - \delta G^n$ produces a map from $F^n/\delta(F^{n-1}) \to F^n$ and we use the fact that $F^n/\delta(F^{n-1})$ is embedded in F^{n+1} via δ with inverse s.

More precisely, given v_1 and $f \in F^{n+1}$, we define the function ϕ of n variables given by

$$\phi(w_1, w_2, \ldots, w_n) = f(v_1, v_1 + w_1, \ldots, v_1 + w_n)$$

and define

$$(G^{n+1}f)(v_1, v_2, \ldots, v_n) = ((H^n - \delta G^n)\phi)(0, v_2 - v_1, \ldots, v_n - v_1).$$

For example, this leads to the following formulae for the first elements G^i.

We have $G^1 = 0, G^2 = 0$.

$$(G^3 f)(v_1, v_2) = f(v_1, v_1, v_2) - \int_0^1 \frac{d}{d\epsilon}|_0 f(v_1, v_1 + t(v_2 - v_1), v_1 + t(v_2 - v_1) + \epsilon(v_2 - v_1))dt.$$

$$(G^4 f)(v_1, v_2, v_3) = G_0^4 + G_1^4 + G_2^4$$

with

$$(G_0^4 f)(v_1, v_2, v_3) = f(v_1, v_1, v_2, v_3) - f(v_1, v_2, v_2, v_3) + f(v_1, v_1, v_1, v_3) - f(v_1, v_1, v_1, v_2),$$

$$(G_1^4 f)(v_1, v_2, v_3) = \int_{t=0}^1 \frac{d}{d\epsilon}|_0 f(v_1, v_2, v_2 + t(v_3 - v_2), v_2 + (t + \epsilon)(v_3 - v_2))$$

$$-\int_{t=0}^{1}\frac{d}{d\epsilon}|_0 f(v_1,v_1,v_1+t(v_3-v_1),v_1+(t+\epsilon)(v_3-v_1))$$

$$+\int_{t=0}^{1}\frac{d}{d\epsilon}|_0 f(v_1,v_1,v_1+t(v_2-v_1),v_1+(t+\epsilon)(v_2-v_1)).$$

$$(G_2^4 f)(v_1,v_2,v_3)=-\int_{t\in S_2}\frac{d}{d\epsilon}|_0 f(v_1,V(t),V(t+\epsilon_1),V(t+\epsilon_2))$$

$$-\int_{t\in\Delta_2}\frac{d}{d\epsilon}|_0 f(v_1,V(t),V(t+\epsilon_2),V(t+\epsilon_1)).$$

Here $t=[t_1,t_2]$, $t+\epsilon_1=[t_1+\epsilon_1,t_2]$, $t+\epsilon_2=[t_1,t_2+\epsilon_2]$, $V(t)=v_1+t_1(v_2-v_1)+t_2(v_3-v_1)$, and $\Delta_2:=\{[t_1,t_2],t_1\geq 0,t_2\geq 0;t_1+t_2\leq 1\}$.

Let us now consider the action of V by translations on the complex F^n. The differential δ induces a differential $\delta_A : F_A^n \to F_A^n$ on the subspaces of invariants. We identify the space F_A^n with F^{n-1} by the map

$$R : F^{n-1}\to F_A^n$$

given by

$$(Rf)(v_1,v_2,\ldots,v_n)=f(v_2-v_1,v_3-v_2,\ldots,v_n-v_{n-1}).$$

Then the differential δ_A induced by δ becomes the Eilenberg-MacLane differential

$$(\delta_A f)(v_1,v_2,\ldots,v_{n-1})$$

$$=f(v_2,v_3,\ldots,v_{n-1})-f(v_1+v_2,v_3,v_4,\ldots,v_{n-1})+f(v_1,v_2+v_3,\ldots,v_{n-1})+\cdots$$

$$+(-1)^{n-2}f(v_1,v_2,\ldots,v_{n-2}+v_{n-1})+(-1)^{n-1}f(v_1,v_2,\ldots,v_{n-1}).$$

The map $P : F^n\to F^n$ also commutes with translations.

Lemma 2.3 *We have $PR=R\mathrm{Ant}$ where* Ant *is the anti-symmetrization operator of F^{n-1} on the space of $\Lambda^{n-1}V^*$ of antisymmetric functions $f(v_1,v_2,\ldots,v_{n-1})$.*

Proof.

To compute P, we have to compute

$$v(t)=v_1+t_1(v_2-v_1)+\cdots+t_{n-1}(v_{n-1}-v_1)$$

and

$$f(t,v_1,v_2,\ldots,v_{n-1})$$

$$=\frac{d}{d\epsilon}|_0\sum_{\sigma\in\Sigma[2,\ldots,n-1]}\epsilon(\sigma)f(v(t),v(t)+\epsilon_1(v_{\sigma(2)}-v_1),\ldots,v(t)+\epsilon_{n-2}(v_{\sigma(n-1)}-v_1)).$$

Now, if f is invariant by translation, we see that

$$f(t,v_1,v_2,\ldots,v_{n-1})=\frac{d}{d\epsilon}|_0\sum_{\sigma\in\Sigma[2,\ldots,n-1]}\epsilon(\sigma)f(0,\epsilon_1(v_{\sigma(2)}-v_1),\ldots,\epsilon_{n-2}(v_{\sigma(n-1)}-v_1)).$$

We obtain the lemma.

□

The homotopy G commutes with translations and gives an operator G_A on the complex of invariants. It follows that we obtain on the complex δ_A the relation

$$G_A\delta_A + \delta_A G_A = \mathrm{Id} - \mathrm{Ant}.$$

We thus obtain that the cohomology of the operator δ_A is isomorphic in degree n to $\Lambda^{n-1}V^*$.

3. Free variables

Let T_n be the free associative algebra in n variables. We consider $L_n \subset T_n$ as the free Lie algebra in n variables. An element f of T_n is written as $f(x_1, x_2, \ldots, x_n)$.

Define

$$(\delta_n f)(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n}(-1)^i f(x_1, x_2, \ldots, x_{i-1}, \hat{x}_i, x_{i+1}, \ldots, x_n).$$

Consider $T_n(y)$ the free associative algebra generated by $(x_1, x_2, \ldots, x_n, y)$. An operator h on T_n is extended by an operator still denoted by h on $T_n(y)$ where we do not operate on y.

We may consider the application $\tau : T_n \to T_n(y)$ defined by

$$(\tau_n f)(x_1, \ldots, x_n) = f(x_1 + y, x_2 + y, \ldots, x_i + y, \ldots, x_n + y).$$

The application τ commutes with δ. Thus the kernel of τ is a subcomplex of T_n. We may identify it with T_{n-1} by $(Rf)(x_1, x_2, \ldots, x_n) = f(x_2 - x_1, x_3 - x_2, \ldots, x_n - x_{n-1})$ and we obtain on T_n the complex δ_A considered by Alekseev-Torossian. Here,

$$(\delta_A f)(x_1, x_2, \ldots, x_{n-1})$$

$$= f(x_2, x_3, \ldots, x_{n-1}) - f(x_1 + x_2, x_3, x_4, \ldots, x_{n-1}) + f(x_1, x_2 + x_3, \ldots, x_{n-1}) + \cdots$$

$$+(-1)^{n-2} f(x_1, x_2, \ldots, x_{n-2} + x_{n-1}) + (-1)^{n-1} f(x_1, x_2, \ldots, x_{n-1}).$$

It is clear that the complex $\delta : 0 \to T_0 \to T_1 \to T_2 \cdots$ is acyclic. Indeed we can define

$$(sf)(x_1, x_2, \ldots, x_n) = f(0, x_1, x_2, \ldots, x_n)$$

and it is immediate to verify that

$$s\delta + \delta s = \mathrm{Id}.$$

If $f \in T_n$, we define a function $f(t, x) \in \mathbb{R}[t] \otimes T_k$ by the same formula as Formula (2):

Definition 3.1 *Let*

$$x(t) = x_1 + t_1(x_2 - x_1) + \cdots + t_{n-1}(x_n - x_1).$$

Define

$$f(t,x_1,x_2,\ldots,x_n)=\frac{d}{d\epsilon}|_0\sum_{\sigma\in\Sigma([2,\ldots,n])}\epsilon(\sigma)f(x(t),x(t)+\epsilon_1(x_{\sigma(2)}-x_1),\ldots,x(t)+\epsilon_{n-1}(x_{\sigma(n)}-x_1)).$$

Define

$$(P_nf)(x_1,x_2,\ldots,x_n)=\int_{\Delta_{n-1}}f(t,x_1,x_2,\ldots,x_n)dt_1dt_2\cdots dt_{n-1}.$$

The following lemma is immediate.

Lemma 3.2 *We have* $\delta P_n=P_n\delta$.

We define $G^1=0$ and inductively G^{n+1} by the same formula as before. More precisely, given $f\in T^{n+1}$, we define the function ϕ of $T^n(x_1)$ given by

$$\phi(w_1,w_2,\ldots,w_n)=f(x_1,x_1+w_1,\ldots,x_1+w_n)$$

and define

$$(G^{n+1}f)(x_1,x_2,\ldots,x_n)=((H^n-\delta G^n)\phi)(0,x_2-x_1,\ldots,x_n-x_1).$$

Then we conclude as before that $G\delta+\delta G=\mathrm{Id}-P$. Restricting to the invariants, we obtain a map G_A such that $\mathrm{Id}-\mathrm{Ant}=G_A\delta_A+G_A\delta_A$. Here Ant is the anti-symmetrization operator $\sum_\sigma\epsilon(\sigma)x_{\sigma(1)}\cdots x_{\sigma(n)}$.

The subspace L_n of T_n is stable under the differential. The operator Ant is equal to 0 on L_n, except in degree $1,2$, as there are no totally antisymmetric elements in L_n for $n\geq 3$. Thus we obtain

Theorem 3.3 • *The cohomology groups* $H^n(T_n,\delta_A)$ *of the complex* $\delta_A:T_n\to T_n$ *are of dimension* 1 *and are generated by* $\sum_\sigma\epsilon(\sigma)x_{\sigma(1)}\cdots x_{\sigma(n)}$.

• *The cohomology groups* $H^n(L_n,\delta_A)$ *of the complex* $\delta_A:L_n\to L_n$ *are of dimension* 0 *if* $n>2$. *For* $n=1,2$,

$$H^1(L_1,\delta_A)=\mathbb{R}x_1,\qquad H^2(L_2,\delta_A)=\mathbb{R}[x_1,x_2].$$

Remark: The Guichardet construction also provides an explicit homotopy.

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Received: 20.01.2011