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On $\mathcal{F}_s$-supplemented primary subgroups of finite groups

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Abstract

Let G be a finite group and $\mathcal{F}$ a formation of finite groups. A subgroup H of G is called $\mathcal{F}_s$-supplemented in G if there exists a subnormal subgroup T of G such that $G = HT$ and $(H \cap T)H_G/H_G$ is contained in the $\mathcal{F}$-hypercenter $Z_\infty^{\mathcal{F}}(G/H_G)$ of G/H_G. In this paper, we study the structure of finite groups by using $\mathcal{F}_s$-supplemented subgroups.

Key Words: Sylow subgroup, $\mathcal{F}_s$-supplemented subgroup, saturated formation, finite groups

1. Introduction

Throughout this paper, all groups are finite. Recall that a subgroup H of a group G is said to be *supplemented* in G if there exists a subgroup K of G such that $HK = G$. Here, the subgroup K is called a *supplement* of H in G.

The relationship between the properties of subgroups of G and the structure of G has been investigated extensively by many scholars. Particularly, Srinivasan [10] proved that a finite group is supersolvable if every maximal subgroup of every Sylow subgroup is normal. Asaad [1] extended this result using formation theory and proved the following: Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a solvable group. Then $G \in \mathcal{F}$ if there is a normal solvable subgroup H of G such that $G/H \in \mathcal{F}$ and the maximal subgroups of the Fitting subgroup $F(H)$ are π-quasinormal in G. Wang[12] generalized Srinivasan's result as follows: Suppose G is a group with a normal subgroup H such that G/H is supersolvable. If every maximal subgroup of every Sylow subgroup of H is c-supplemented in G, then G is supersolvable.

Recently, Miao and Guo [7] proved that G is supersolvable if and only if every maximal subgroup of a Sylow subgroup of G is supersolvable s-supplemented in G. More recently, Guo in [4] proposed the conception of $\mathcal{F}$-supplemented subgroups and proved the following: Let $\mathcal{F}$ be a S-closed saturated formation containing all supersolvable groups and H be a normal subgroup of G such that $G/H \in \mathcal{F}$. If every maximal subgroup of a non-cyclic Sylow subgroup of H having no supersolvable supplement in G is $\mathcal{F}$-supplemented in G, then

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$G \in \mathcal{F}$. As a continuation of these works, in the present paper, we will analyze the structure of finite groups with $\mathcal{F}_s$-supplemented primary subgroups.

Definition 1.1 *Let G be a finite group and $\mathcal{F}$ a formation of finite groups. A subgroup H of G is called $\mathcal{F}_s$-supplemented in G if there exists a subnormal subgroup T of G such that $G = HT$ and $(H \cap T)H_G/H_G$ is contained in the $\mathcal{F}$-hypercenter $Z_\infty^{\mathcal{F}}(G/H_G)$ of G/H_G.*

Recall that, for a class $\mathcal{F}$ of groups, a chief factor H/K of a group G is called $\mathcal{F}$-central ([3, Definition 2.4.3]) if the semidirect product $[H/K](G/C_G(H/K)) \in \mathcal{F}$. The symbol $Z_\infty^{\mathcal{F}}(G)$ denotes the $\mathcal{F}$-hypercenter of a group G, that is, the product of all such normal subgroups H of G whose G-chief factors are $\mathcal{F}$-central. A subgroup H of G is said to be $\mathcal{F}$-hypercentral in G if $H \leq Z_\infty^{\mathcal{F}}(G)$.

Most of the notation is standard and can be found in [5–6] and [8].We denote by $F(G)$ the Fitting subgroup of G; by $F_p(G)$ the maximal p-nilpotent normal subgroup of G; by $O_p(G)$ the maximal normal p-subgroup of G; by $\Phi(G)$ the intersection of all maximal subgroups of G. $|G|$ denotes the order of a group G; $M < \cdot G$ means M is a maximal subgroup of G.

Let $\mathcal{F}$ be a class of groups. A formation $\mathcal{F}$ is said to be S-closed (S_n-closed) if it contains all subgroups (all normal subgroups, respectively) of all its group. $\mathcal{F}$ is said to be a formation provided that (1) if $G \in \mathcal{F}$ and $H \trianglelefteq G$, then $G/H \in \mathcal{F}$, and (2) if G/M and G/N are in $\mathcal{F}$, then $G/M \cap N$ is in $\mathcal{F}$. A formation $\mathcal{F}$ is said to be saturated if $G \in \mathcal{F}$ whenever $G/\Phi(G) \in \mathcal{F}$. It is well known that the class of all supersolvable groups and the class of all p-nilpotent groups are saturated formations (cf. [3]).

2. Preliminaries

For the sake of convenience, we first list here some known results which will be useful in the sequel.

Lemma 2.1 *[4, Lemma 2.1] Let G be a group and $A \leq G$. Let $\mathcal{F}$ be a non-empty saturated formation and $Z = Z_\infty^{\mathcal{F}}(G)$. Then*
(1) If A is normal in G, then $AZ/A \leq Z_\infty^{\mathcal{F}}(G/A)$.
(2) If $\mathcal{F}$ is S-closed, then $Z \cap A \leq Z_\infty^{\mathcal{F}}(A)$.
(3) If $\mathcal{F}$ is S_n-closed and A is normal in G, then $Z \cap A \leq Z_\infty^{\mathcal{F}}(A)$.
(4) If $G \in \mathcal{F}$, then $Z = G$.

Lemma 2.2 *Let G be a group and $H \leq K \leq G$. Then*
(1) H is $\mathcal{F}_s$-supplemented in G if and only if G has a subnormal subgroup T such that $G = HT$, $H_G \leq T$ and $(H/H_G) \cap (T/H_G) \leq Z_\infty^{\mathcal{F}}(G/H_G)$.
(2) Suppose that H is normal in G. Then K/H is $\mathcal{F}_s$-supplemented in G/H if and only if K is $\mathcal{F}_s$-supplemented in G .
(3) Suppose that H is normal in G. Then, for every $\mathcal{F}_s$-supplemented subgroup E in G satisfying $(|H|, |E|) = 1$, HE/H is $\mathcal{F}_s$-supplemented in G/H.
(4) If H is $\mathcal{F}_s$-supplemented in G and $\mathcal{F}$ is S-closed, then H is $\mathcal{F}_s$-supplemented in K.
(5) If H is $\mathcal{F}_s$-supplemented in G, K is normal in G and $\mathcal{F}$ is S_n-closed, then H is $\mathcal{F}_s$-supplemented in K.

(6) If $G \in \mathcal{F}$, then every subgroup of G is $\mathcal{F}_s$-supplemented in G.

Proof. A slight modification of the proof of [4, Lemma 2.2] gives the result. □

Lemma 2.3 *([3, Theorem 1.8.17]) Let N be a nontrivial solvable normal subgroup of a group G. If $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G which is contained in N.*

Lemma 2.4 *[3, Lemma 1.8.19] If G is a p-solvable group where p is a prime divisor of $|G|$, then $C_G(F_p(G)) \leq F_p(G)$.*

Lemma 2.5 *[9,Lemma 1.9] Let $\mathcal{F}$ be a saturated formation containing all supersolvable groups and G be a group with a normal subgroup E such that $G/E \in \mathcal{F}$. If E is cyclic, then $G \in \mathcal{F}$.*

Lemma 2.6 *[3, Lemma 3.6.10] Let K be a normal subgroup of G and P a p-subgroup of G where p is a prime divisor of $|G|$. Then $N_{G/K}(PK/K) = N_G(P_1)K/K$, here P_1 is a Sylow p-subgroup of PK.*

Lemma 2.7 *If L is a subnormal p-subgroup of G where p is a prime divisor of $|G|$, then $L \leq O_p(G)$.*

Proof. Since L is subnormal in G, there exists a subnormal series

$$L \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq \ldots\ldots \trianglelefteq N_t = G.$$

It is easy to know that $L \leq O_p(N_1)$ char $N_1 \trianglelefteq N_2$. This induces that $O_p(N_1) \trianglelefteq N_2$, and hence $O_p(N_1) \leq O_p(N_2)$. So $L \leq O_p(N_2)$. Analogously, we can obtain that $L \leq O_p(G)$. □

3. Main results

Theorem 3.1 *Let G be a finite group and P a Sylow p-subgroup of G where p is the smallest prime divisor of $|G|$. Then G is p-nilpotent if and only if every maximal subgroup of P is $\mathcal{F}_s$-supplemented in G where $\mathcal{F}$ is a class of all p-nilpotent groups.*

Proof. If G is p-nilpotent, then by Lemma 2.2(6) every subgroup of G is $\mathcal{F}_s$-supplemented in G and so is every maximal subgroup of P.

Conversely, let G be a counterexample of smallest order. By hypotheses, every maximal subgroup P_1 of P is $\mathcal{F}_s$-supplemented in G. Furthermore, we have

1) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, Lemma 2.2(3) guarantees that $G/O_{p'}(G)$ satisfies the hypotheses of the theorem. Thus $G/O_{p'}(G)$ is p-nilpotent by the choice of G. Then G is p-nilpotent, a contradiction.

2) $O_p(G) \neq 1$.

If $O_p(G) = 1$, then $(P_1)_G = 1$ for any maximal subgroup P_1 of P and there exists a subnormal subgroup B of G such that $G = P_1B$ and $P_1 \cap B \leq Z_\infty^{\mathcal{F}}(G)$. If $Z_\infty^{\mathcal{F}}(G) \neq 1$, then we know that every minimal normal subgroup N of G contained in $Z_{\infty}^{\mathcal{F}}(G)$ is $\mathcal{F}$-central in G. Since $\mathcal{F}$ is the class of all p-nilpotent groups, we have $|N| = p$ or N is a $p^{'}$-group. By 1), we have $|N| = p$. By a similar discussion as in 1), we have G/N is p-nilpotent and hence G is p-nilpotent, a contradiction. So we have $Z_{\infty}^{\mathcal{F}}(G) = 1$ and this is equivalent to every maximal subgroup P_1 of P is complemented in G. By the definition of $\mathcal{F}_s$-supplemented subgroup, there exists a subnormal subgroup of K such that $G = P_1K$ and $P_1 \cap K = 1$. Clearly, $|K|_p = p$ and K is p-nilpotent by Burnside p-nilpotent Theorem. It follows that G is p-nilpotent, a contradiction.

3) $O_p(G)$ is the unique minimal normal subgroup of G and $\Phi(G) = 1$.

In fact, $G/O_p(G)$ satisfies the condition of the theorem by Lemma 2.2(2). Thus the minimality of G implies that $G/O_p(G)$ is p-nilpotent and hence G is p-solvable. It follows that every minimal normal subgroup of G is either an elementary abelian p-group or a $p^{'}$-group. By (1), $O_{p'}(G) = 1$. Then every minimal normal subgroup N of G is an elementary abelian p-group and hence contained in $O_p(G)$. Let N be a minimal normal subgroup of G. Clearly, G/N satisfies the condition of our hypotheses by Lemma 2.2. The minimal choice of G implies that G/N is p-nilpotent. Similarly, if L is another minimal normal subgroup of G, then we may get G/L is p-nilpotent. It follows that $G/N \cap L \cong G$ is p-nilpotent, a contradiction. Therefore G has a unique minimal normal subgroup N. Furthermore, since the class of all p-nilpotent groups is a saturated formation, we have $\Phi(G) = 1$. By Lemma 2.3, we have $O_p(G) = F(G) = N$. Hence $O_p(G)$ is a unique minimal normal subgroup of G.

(4) The final contradiction.

By (3), there exists a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. Since $G/N \cong M$ is p-nilpotent, we know that M has a normal Hall $p^{'}$-subgroup $M_{p'}$. It is clear that $G = NM = NN_G(M_{p'}) = PN_G(M_{p'})$. Now we let P_1 be a maximal subgroup of P containing $P \cap N_G(M_{p'})$. By the hypotheses of the theorem, P_1 is $\mathcal{F}_s$-supplemented in G and there exists a subnormal subgroup T of G such that $G = P_1T$ and $P_1 \cap T \leq Z_{\infty}^{\mathcal{F}}(G)$ since $(P_1)_G = 1$. Based on the discussion of 2), we have the P_1 is complemented in G. Clearly, $|T|_p = p$ and we know T is p-nilpotent by the Burnside p-nilpotent Theorem. Therefore we have G is p-nilpotent since T has a normal p-complement.

Final contradiction completes our proof. □

Theorem 3.2 *Let p be an odd prime divisor of $|G|$ and P be a Sylow p-subgroup of G. Then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent and every maximal subgroup of P is $\mathcal{F}_s$-supplemented in G,where $\mathcal{F}$ is the class of all p-nilpotent groups.*

Proof. Necessity part is obvious. So we only need to prove the sufficiency part.

Assume that the assertion is false and choose G to be a counterexample of minimal order. We will divide the proof into the following steps.

1) $O_{p'}(G) = 1$.

In fact, if $O_{p'}(G) \neq 1$, then we consider the quotient group $G/O_{p'}(G)$. By Lemma 2.2(3) and Lemma 2.6, $G/O_{p'}(G)$ satisfies the condition of the theorem, and so the minimal choice of G implies that $G/O_{p'}(G)$ is p-nilpotent. Hence G is p-nilpotent, a contradiction.

2) If S is a proper subgroup of G containing P, then S is p-nilpotent.

Clearly, $N_S(P) \leq N_G(P)$ and hence $N_S(P)$ is p-nilpotent. Applying Lemma 2.2(4), we find that S satisfies the hypotheses of our theorem. Now, the minimal choice of G implies that S is p-nilpotent.

3) $G = PQ$, where Q is the Sylow q-subgroup of G with $q \neq p$.

Since G is not p-nilpotent, by Thompson ([11], Corollary), there exists a characteristic subgroup H of P such that $N_G(H)$ is not p-nilpotent. Since $N_G(P)$ is p-nilpotent, we may choose a characteristic subgroup H of P such that $N_G(H)$ is not p-nilpotent, but $N_G(K)$ is p-nilpotent for any characteristic subgroup K of P with $H < K \leq P$. Since $P \leq N_G(H)$ and $N_G(H)$ is not p-nilpotent, we have $N_G(H) = G$ by 2). This leads to $O_p(G) \neq 1$ and $N_G(K)$ is p-nilpotent for any characteristic subgroup K of P such that $O_p(G) < K \leq P$. Now by Lemma 2.6 and Thompson ([11], Corollary), we see that $G/O_p(G)$ is p-nilpotent and therefore, G is p-solvable. Since G is p-solvable, for any $q \in \pi(G)$ with $q \neq p$, there exists a Sylow q-subgroup Q of G such that $PQ = QP$ is a subgroup of G by ([2],Theorem 6.3.5). If $PQ < G$, then PQ is p-nilpotent by 2). This leads to $Q \leq C_G(O_p(G)) \leq O_p(G)$ by Robinson([8], Theorem 9.3.1) since $O_{p'}(G) = 1$, a contradiction. Thus we have proven that $G = PQ$.

4) Conclusion.

Since $O_p(G) \neq 1$, we may choose a minimal normal subgroup L of G with $L \leq O_p(G)$. Clearly, G/L satisfies the condition of the theorem. Now, the minimality of G implies that G/L is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, we may assume L is the unique minimal normal subgroup of G contained in $O_p(G)$ and $L \nleq \Phi(G)$. So $\Phi(G) = 1$. Thus, by Lemma 2.3, we have $F(G) = O_p(G) = L$ is an elementary abelian p-group. Furthermore, there exists a maximal subgroup M of G such that $G = LM$ and $L \cap M = 1$. Hence we have $P = P \cap LM = L(P \cap M)$ and $P \cap M = P^*$ is a Sylow p-subgroup of M. If $P^* = 1$, then $P = L$, and therefore $G = N_G(L) = N_G(P)$ is p-nilpotent, which is a contradiction. So we may assume $P^* \neq 1$. Pick a maximal subgroup P_1 of P with $P^* \leq P_1$. By hypotheses, P_1 is $\mathcal{F}_s$-supplemented in G, that is, there exists a subnormal subgroup K of G such that $G = P_1K$ and $P_1 \cap K \leq Z_\infty^{\mathcal{F}}(G)$ since $(P_1)_G = 1$. If $Z_\infty^{\mathcal{F}}(G) \neq 1$, then we have $Z_\infty^{\mathcal{F}}(G) \leq F(G)$ and hence $Z_\infty^{\mathcal{F}}(G) = O_p(G) = L$. On the other hand, we know that every minimal normal subgroup of G contained in $Z_\infty^{\mathcal{F}}(G)$ is a subgroup of order p or a p'-group. Hence we have $|L| = p$ by 1) and so $P_1 \cap K \leq L$. Furthermore, if $P_1 \cap K = L$, then we have $L \leq P_1$, a contradiction. Thus $Z_\infty^{\mathcal{F}}(G) = 1$, and so $P_1 \cap K = 1$.

If $L \cap K \neq 1$, then $|L \cap K| = p$. If $p < q$, then K is p-nilpotent and therefore Q char K. Moreover, as K is subnormal in G, we have $Q \trianglelefteq G$ and hence G is p-nilpotent, a contradiction. On the other hand, if $q < p$, then since $L \cap K \trianglelefteq K$ and $C_K(L \cap K) = L \cap K$ by Lemma 2.4, we see that $K_q \cong K/L \cap K = N_K(L \cap K)/C_K(L \cap K)$ is isomorphic to a subgroup of $Aut(L \cap K)$ and therefore K_q where K_q is a Sylow q-subgroup of K, and particularly K_q is a cyclic group. Since K_q is also a Sylow q-subgroup of G and $q < p$, we know that G is q-nilpotent and therefore P is normal in G. Hence $N_G(P) = G$ is p-nilpotent, a contradiction.

So we may assume $L \cap K = 1$. Since G is solvable, we have K is solvable. Let T be a minimal normal subgroup of K. We know that T is an elementary abelian p-group or q-group. If T is a p-group, then $T \leq O_p(G) = L$ by Lemma 2.7, a contradiction. So we may assume T is a q-group. By Lemma 2.7, $T \leq O_q(G)$, this is contrary to 1).

The final contradiction completes our proof. □

Theorem 3.3 *Let G be a p-solvable group and P a Sylow p-subgroup of G. Then G is p-supersolvable if and only if every maximal subgroup of P is $\mathcal{F}_s$-supplemented in G, where $\mathcal{F}$ is the class of all p-supersolvable groups.*

Proof. Necessity part is obvious and we only need to prove the sufficiency part.

Assume that the assertion is false and choose G to be a counterexample of minimal order. Furthermore, we have that

1) $O_{p'}(G) = 1$.

If $L = O_{p'}(G) \neq 1$, we consider G/L. Clearly, P_1L/L is a maximal subgroup of Sylow p-subgroup of G/L where P_1 is a maximal subgroup of P. Since P_1 is $\mathcal{F}_s$-supplemented in G, we have P_1L/L is also $\mathcal{F}_s$-supplemented in G/L by Lemma 2.2(3). Therefore G/L satisfies the condition of the theorem. The minimal choice of G implies that G/L is p-supersolvable, and hence G is p-supersolvable, a contradiction

2) $O_p(G) \neq 1$.

Since G is p-solvable and $O_{p'}(G) = 1$, we have that a minimal normal subgroup of G is an abelian p-group and hence $O_p(G) \neq 1$.

3) Final contradiction.

By 2), we may pick a minimal normal subgroup N of G contained in $O_p(G)$. By Lemma 2.2(3), we know that G/N satisfies the condition of the theorem, and so the minimal choice of G implies that G/N is p-supersolvable. On the other hand, since the class of all p-supersolvable groups is a saturated formation, we have N is the unique minimal normal subgroup of G contained in $O_p(G)$ and $O_p(G) = N = F(G) \nleq \Phi(G)$ by Lemma 2.3.

Clearly, there exists a maximal subgroup M of G such that $G = NM$ with $N \cap M = 1$ and $P = NM_p$. We may choose a maximal subgroup P_1 with $M_p \leq P_1$. By hypotheses, P_1 is $\mathcal{F}_s$-supplemented in G. Then there exists a subnormal subgroup K of G such that $G = P_1K$ and $P_1 \cap K \leq Z_\infty^{\mathcal{F}}(G)$ since $(P_1)_G = 1$. If $Z_\infty^{\mathcal{F}}(G) \neq 1$, then every minimal normal subgroup of G contained in $Z_\infty^{\mathcal{F}}(G)$ is either a cyclic group of order p or a p'-group since $\mathcal{F}$ is the class of all p-supersolvable groups, and so $|N| = p$ by 1). It follows from G/N is p-supersolvable that G is p-supersolvable by Lemma 2.5, a contradiction. So we have $P_1 \cap K = 1$ and $|K_p| = p$.

If $N \cap K \neq 1$, we have $|N \cap K| = p$. If p is the smallest prime divisor of $|G|$, by Burnside Theorem, we have K is p-nilpotent and K has a normal p-complement $K_{p'}$. Since K is subnormal, we get $K_{p'}$ is also a normal p-complement of G, a contradiction.

Next we may assume that p is not the smallest prime divisor of $|G|$. Since $N \cap K \trianglelefteq K$ and K is p-solvable, we have $C_K(N \cap K) = N \cap K$ by Lemma 2.4. Therefore $K/N \cap K = N_K(N \cap K)/C_K(N \cap K)$ is isomorphic to a subgroup of $Aut(N \cap K)$ and $K_{p'}$ is a cyclic group. Clearly, every Sylow subgroup of K is cyclic and hence K is supersolvable. So K has a normal Sylow q-subgroup Q where q is the largest prime divisor of $|K|$. If $p < q$, since K is subnormal in G, then we have $Q \leq O_q(G)$, contrary to 1). So we may assume that p is the largest prime divisor of $|G|$. Since G has a cyclic Hall p'-subgroup, G has a supersolvable type Sylow tower. So we have P is a minimal normal subgroup of G. On the other hand, if P_2 is a maximal subgroup of

P, by hypotheses, P_2 is $\mathcal{F}_s$-supplemented in G. Thus here exists a subnormal subgroup H such that $G = P_2H$ and $P_2 \cap H \leq Z_{\infty}^{\mathcal{F}}(G)$ since $(P_2)_G = 1$. If $P_2 \cap H \neq 1$, then we have $P \cap Z_{\infty}^{\mathcal{F}}(G) \neq 1$. Since every minimal normal subgroup of G contained in $Z_{\infty}^{\mathcal{F}}(G)$ is either a cyclic group of order p or a p'-group, we get $|P| = p$, a contradiction. So we have $P_2 \cap H = 1$. Since P is a minimal normal subgroup of G and G is p-solvable, we have P is an elementary abelian p-group and $P \cap H = 1$. Therefore $P = P \cap P_2H = P_2(P \cap H) = P_2$, a contradiction.

So we may assume $N \cap K = 1$. Since G is p-solvable, we have K is p-solvable. Let T be a minimal normal subgroup of K. We know that T is an elementary abelian p-group or a p'-group. If T is a p-group, then $T \leq O_p(G) = N$ by Lemma 2.7, a contradiction. So we may assume that T is a p'-group. By Lemma 2.7, $T \leq O_{p'}(G)$, contrary to 1).

The final contradiction completes our proof. □

Corollary 3.4 *Let G be a group. Then G is supersolvable if and only if every maximal subgroup of a Sylow subgroup of G is $\mathcal{U}_s$-supplemented in G.*

Theorem 3.5 *Let G be a p-solvable group and p a prime divisor of $|G|$. Then G is p-supersolvable if and only if every maximal subgroup of $F_p(G)$ containing $O_{p'}(G)$ is $\mathcal{F}_s$-supplemented in G, where $\mathcal{F}$ is the class of all p-supersolvable groups.*

Proof. Necessity part is obvious and we only need to prove the sufficiency part.

Assume that the assertion is false and choose G to be a counterexample of minimal order. Furthermore, we have

1) $O_{p'}(G) = 1$.

If $T = O_{p'}(G) \neq 1$, we consider G/T. Firstly, $F_p(G/T) = F_p(G)/T$. Let M/T be a maximal subgroup of $F_p(G/T)$. Then M is a maximal subgroup of $F_p(G)$ containing $O_{p'}(G)$. Since M is $\mathcal{F}_s$-supplemented in G, then M/T is $\mathcal{F}_s$-supplemented in G/T by Lemma 2.2(3). Thus G/T satisfies the hypotheses of the theorem. The minimal choice of G implies that G/T is p-supersolvable and so is G, a contradiction.

2) $\Phi(G) = 1$ and $F_p(G) = F(G) = O_p(G)$.

If not, then $L = \Phi(G) \neq 1$. We consider G/L. Since $O_{p'}(G) = 1$, it is easy to show that $F_p(G) = F(G) = O_p(G)$. This implies that $F_p(G/L) = O_p(G/L) = O_p(G)/L = F_p(G)/L$. If P_1/L is a maximal subgroup of $F_p(G/L)$, then P_1 is a maximal subgroup of $F_p(G)$. Since P_1 is $\mathcal{F}_s$-supplemented in G and hence P_1/L is $\mathcal{F}_s$-supplemented in G/L by Lemma 2.2(3). Thus G/L satisfies the hypotheses of the theorem. The minimal choice of G implies that G/L is p-supersolvable and so is G, since the class of all p-supersolvable groups is a saturated formation, a contradiction.

3) Every minimal normal subgroup of G contained in $F(G)$ is cyclic of order p.

By Lemma 2.3 and 2), $F(G)$ is the direct product of minimal normal subgroups of G contained in $F(G)$. Since G is p-solvable and $O_{p'}(G) = 1$, we have $C_G(O_p(G)) \leq O_p(G)$ by Lemma 2.4. Now $\Phi(G) = 1$ implies that $F(G)$ is a nontrivial elementary abelian p-group. Thus $C_G(F(G)) = F(G)$. $P = F(G) = R_1 \times \ldots \times R_t$, where R_i is a minimal normal subgroup of G contained in $F(G)$, $i = 1, 2, \cdots, t$. Fix i, since $\Phi(G) = 1$, there exists a maximal subgroup M of G such that $R_i \nleq M$. Clearly, $P_2 = R_i^* R$ is a maximal subgroup of P where R_i^* is a maximal subgroup of R_i and $R = \prod_{j \neq i} R_j$. By hypotheses, P_2 is $\mathcal{F}_s$-supplemented in G. Evidently, $(P_2)_G = R$. By Lemma 2.2(2), P_2/R is also $\mathcal{F}_s$-supplemented in G/R. There exists a subnormal subgroup T/R such that $G/R = (P_2/R)(T/R)$ and $(P_2 \cap T)/R \leq Z_\infty^{\mathcal{F}}(G/R)$ since $(P_2/R)_{(G/R)} = 1$. If $(P_2 \cap T)/R = 1$, then $P/R = P/R \cap (P_2 T)/R = (P \cap P_2 T)/R = (P_2/R)(P/R \cap T/R) = P_2/R$ since P/R is a minimal normal subgroup of G/R, a contradiction. So we may assume $(P_2 \cap T)/R \neq 1$. Furthermore, since every minimal normal subgroup of G/R contained in $Z_\infty^{\mathcal{F}}(G/R)$ is either a cyclic group of order p or a p'-group, it follows from $(P/R) \cap Z_\infty^{\mathcal{F}}(G/R) \neq 1$ that $|P/R| = p$ and hence $|R_i| = p$.

4) Final contradiction.

Thus $P = F(G) = R_1 \times \ldots \times R_t$, where R_i is a minimal normal subgroup of G of order p. For each i the quotient $G/C_G(R_i)$ is a subgroup of $\mathrm{Aut}(R_i)$ and hence is abelian. Since the class of all p-supersolvable groups is a formation, we have $G/\bigcap_{i=1}^t (C_G(R_i))$ is p-supersolvable, and thus $G/F(G)$ is p-supersolvable because $\bigcap_{i=1}^t (C_G(R_i)) = C_G(F(G)) = F(G)$. But all chief factors of G below $F(G)$ are cyclic groups of order p and hence G is p-supersolvable.

The final contradiction completes our proof. □

Theorem 3.6 *Let G be a p-solvable group and p a prime divisor of $|G|$. Then G is p-supersolvable if and only if every maximal subgroup of a noncyclic Sylow p-subgroup of $F_p(G)$ is $\mathcal{F}_s$-supplemented in G, where $\mathcal{F}$ is the class of all p-supersolvable groups.*

Proof. Necessity part is obvious and we only need to prove the sufficiency part.

Assume that the assertion is false and choose G to be a counterexample of minimal order. Let P be a Sylow p-subgroup of $F_p(G)$. Furthermore, we have that

1) $O_{p'}(G) = 1$.

In fact, if $O_{p'}(G) \neq 1$, we may consider the factor group $G/O_{p'}(G)$. Since $F_p(G/O_{p'}(G)) = F_p(G)/O_{p'}(G)$ and $F_p(G) = O_{p'p}(G)$, we have $F_p(G)/O_{p'}(G) = PO_{p'}(G)/O_{p'}(G)$ and hence $F_p(G)/O_{p'}(G)$ is a p-group. Clearly, there exists a maximal subgroup P_1 of P such that $P_1 O_{p'}(G)/O_{p'}(G) = H/O_{p'}(G)$ for any maximal subgroup $H/O_{p'}(G)$ of $F_p(G)/O_{p'}(G)$. By hypotheses, every maximal subgroup of P is $\mathcal{F}_s$-supplemented in G, $P_1 O_{p'}(G)/O_{p'}(G) = H/O_{p'}(G)$ is also $\mathcal{F}_s$-supplemented in $G/O_{p'}(G)$ by Lemma 2.2(3). Thus $G/O_{p'}(G)$ satisfies the condition of the theorem, the minimal choice of G implies that $G/O_{p'}(G)$ is p-supersolvable. It follows that G is p-supersolvable, a contradiction.

2) $\Phi(G) = 1$.

Assume that $\Phi(G) \neq 1$. The p-solvability of $G/\Phi(G)$ implies that $F_p(G/\Phi(G)) \neq 1$. By 1), $F_p(G) = P = F(G)$. Since $F_p(G/\Phi(G)) = F_p(G)/\Phi(G)$, we see that $P_1/\Phi(G)$ is $\mathcal{F}_s$-supplemented in $G/\Phi(G)$ for any maximal subgroup $P_1/\Phi(G)$ of $P/\Phi(G)$. The minimal choice of G implies that $G/\Phi(G)$ is p-supersolvable and hence G is p-supersolvable since the class of all p-supersolvable groups is a saturated formation, a contradiction.

3) Final contradiction.

By Lemma 2.3 and 2), $F(G)$ is the direct product of minimal normal subgroups of G contained in $F(G)$. Since G is p-solvable and $O_{p'}(G) = 1$, we have $C_G(O_p(G)) \leq O_p(G)$. Now $\Phi(G) = 1$ implies that $F(G)$ is a nontrivial elementary abelian p-group. Thus $C_G(F(G)) = F(G)$. So we may assume that $P = F(G) = R_1 \times \ldots \times R_t$, where R_i is a minimal normal subgroup of G contained in $F(G)$, $i = 1, 2, \cdots, t$. Since $\Phi(G) = 1$, for each R_i, there exists a maximal subgroup M of G such that $R_i \nleq M$. Thus $G = PM$. Clearly, $P_2 = R_i^* R$ is a maximal subgroup of P where R_i^* is the maximal subgroup of R_i and $R = \prod_{j \neq i} R_j$. By hypotheses, P_2 is $\mathcal{F}_s$-supplemented in G. Evidently, $(P_2)_G = R$. By Lemma 2.2(3), P_2/R is also $\mathcal{F}_s$-supplemented in G/R. There exists a subnormal subgroup T/R such that $G/R = (P_2/R)(T/R)$ and $(P_2 \cap T)/R \leq Z_\infty^{\mathcal{F}}(G/R)$ since $(P_2/R)_{(G/R)} = 1$. If $(P_2 \cap T)/R = 1$, then $P/R = P/R \cap (P_2 T)/R = (P \cap P_2 T)/R = (P_2/R)(P/R \cap T/R) = P_2/R$ since P/R is a minimal normal subgroup of G/R, a contradiction. So we may assume $(P_2 \cap T)/R \neq 1$. Furthermore, since every minimal normal subgroup of G/R contained in $Z_\infty^{\mathcal{F}}(G/R)$ is either a cyclic group of order p or a p'-group, it follows from $(P/R) \cap Z_\infty^{\mathcal{F}}(G/R) \neq 1$ that $|P/R| = p$ and hence $|R_i| = p$.

Thus $P = F(G) = R_1 \times \ldots \times R_t$, where R_i is a minimal normal subgroup of G of order p. For each i the quotient $G/C_G(R_i)$ is a subgroup of $\mathrm{Aut}(R_i)$ and hence is abelian. Since the class of all p-supersolvable groups is a formation, we have $G/\bigcap_{i=1}^t (C_G(R_i))$ is p-supersolvable, and thus $G/F(G)$ is p-supersolvable because $\bigcap_{i=1}^t (C_G(R_i)) = C_G(F(G)) = F(G)$. But all chief factors of G below $F(G)$ are cyclic group of order p and hence G is p-supersolvable, a contradiction.

The final contradiction completes our proof. □

Corollary 3.7 *Let G be a solvable group. Then G is supersolvable if and only if every maximal subgroup of a noncyclic Sylow subgroup of $F(G)$ is $\mathcal{U}_s$-supplemented in G.*

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