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On the automorphisms of direct product of monogenic semigroups and monoids

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Abstract

This paper investigates the automorphism group of monogenic [4] semigroups (or monoids) to find its relationship with the automorphism group of cyclic groups. Then, by considering a presentation related to the direct product of monogenic semigroups, verify the relationship between the automorphism group of These and the automorphism group of the group presented by the same presentation. This study gives us some [4]explicit formulas for computing the order of automorphism groups of these algebraic structures.

Key Words: Automorphism groups of algebraic structures, Monogenic [4] semigroups and monoids

1. Introduction

We know that the set of all endomorphisms of an algebraic structure forms a monoid with composition of maps, and the set of all automorphisms of an algebraic structure forms a group with the same operation.

As had the automorphism group of finite groups has been studied widely, in recent years the automorphism group of some well-known classes of semigroups has also been studied by various authors (see [7, 10, 11, 12, 13, 16, 17]).

In this paper we study the structure of automorphism group of monogenic semigroups (or monoids) and the automorphism group of direct product of such semigroups by examining a larger semigroup than the direct product of two monogenic semigroups. This attempt is to get explicit formulas for the orders of automorphism groups.

Let S be a semigroup and M be a monoid with identity element 1 and let $Aut(S)$ and $Aut(M)$ be the group of automorphisms of S and M, respectively. Consider $\pi = \langle A \mid R \rangle$ as a presentation for a semigroup, monoid or group. To avoid confusion we denote a semigroup presentation by $Sg(\pi)$, a monoid presentation by $Mon(\pi)$, and a group presentation by $Gp(\pi)$. For more information on the presentation of semigroups and monoids, one may consult [1, 2, 3, 4, 5, 6, 9, 14, 15]. We also denote the cyclic group of order n by C_n, the group of units of the monoid $(\mathbb{Z}_n, .)$ by $U(n)$, and the Euler's phi-function by ϕ.

Our main result considering the presentation

$$\pi = \langle a, b |\ a^m = a,\ b^n = b,\ ab = ba \rangle,$$

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is the following theoreme.

Theorem 1 *For every positive integers $m, n \geq 2$,*

(i) $Aut(Sg(\pi)) \cong H \leqslant Aut(Gp(\pi))$*, and so* $|Aut(Sg(\pi))|$ *divides* $|Aut(Gp(\pi))|$*,*

(ii)

$$| Aut(Sg(\pi)) | = \begin{cases} \phi(m-1)\phi(n-1) & if \quad m \neq n, \\ 2\phi(m-1)\phi(n-1) & if \quad m = n, \end{cases}$$

(iii) If $gcd(m-1, n-1) = 1$ *then,* $Aut(Sg(\pi)) \cong Aut(Gp(\pi))$.

First we give the following remark concerning certain easy statements on the monogenic semigroups and their automorphisms.

Remark 1.2

(i) Let S be the monogenic semigroup defined by $\langle a \mid a^n = a^m \rangle$, where m and n are positive integers with $n > m$. If $m \neq 1$ then, since a is the unique generator of S, the automorphism group of S is the trivial group. If $m = 1$ then, since S is the cyclic group C_{n-1} of order $n-1$, $Aut(S) = Aut(C_{n-1})$.

(ii) Let $S = Sg(\pi)$ be the semigroup defined by the presentation

$$\pi = \langle a, b | \; a^m = a, \; b^n = b, \; ab = ba \rangle,$$

where $m, n \in \mathbb{Z}^+ \setminus \{1\}$. Then the canonical form of S is

$$\{a^i \mid 1 \leqslant i \leqslant m-1\} \cup \{b^j \mid 1 \leqslant j \leqslant n-1\} \cup K,$$

where $K = \{a^i b^j \mid 1 \leqslant i \leqslant m-1, 1 \leqslant j \leqslant n-1\}$. Moreover, let $G = Gp(\pi)$. Then it is a well-known fact that K is a subgroup (and the unique minimal two-sided ideal) of S and also K is isomorphic to G. Moreover, to generate a and b we need an element from $\{a^i \mid 1 \leqslant i \leqslant m-1\}$ and an element from $\{b^j \mid 1 \leqslant j \leqslant n-1\}$ since Green $\mathcal{D}$-classes of a and b are subsets of $\{a^i \mid 1 \leqslant i \leqslant m-1\}$ and $\{b^j \mid 1 \leqslant j \leqslant n-1\}$, respectively.

(iii) A well-known theorem in the group theory states that for any finite groups H and K we have:

$$Aut(H) \times Aut(K) \hookrightarrow Aut(H \times K),$$

and if $gcd(|H|, |K|) = 1$, then

$$Aut(H) \times Aut(K) \cong Aut(H \times K).$$

Now we generalize the first statement to all semigroups and monoids and the second one to the direct product of monogenic semigroups and monoids. One may think about the validity of second assertion for all finite [4] semigroups and finite monoids. These are true, since a cross-product of automorphisms is also an automorphism. So, let S and T be any semigroups (or monoids). Then

$$Aut(S) \times Aut(T) \hookrightarrow Aut(S \times T).$$

This can be generalized to any finite direct product of semigroups.

(iv) Since the Green relations on semigroups are algebraic properties of them, thus every automorphism on a semigroup S preserves the Green relations on it. Particularly, it preserves the $\mathcal{D}$-classes of S.

2. The Proof of Theorem 1.1

Note that the semigroup presented by the presentation

$$\pi = \langle a, b \mid a^{m+1} = a,\ b^{n+1} = b,\ ab = ba \rangle,$$

is different from that of the direct product of two monogenic semigroups $S = \langle a \mid a^{m+1} = a \rangle$ and $T = \langle b \mid b^{n+1} = b \rangle$, we will express the relationship between the automorphism group of the direct product $S \times T$ and the derived results of Theorem 1.1 in Section 3, together with certain numerical examples.

Proof of Theorem 1.1 Let

$$\pi = \langle a, b \mid a^{m} = a,\ b^{n} = b,\ ab = ba \rangle,$$

where, $m, n \geqslant 2$ are positive integers, $S = Sg(\pi)$, $M = Mon(\pi)$ and $G = Gp(\pi)$. We know that $M \cong S^1 = S \cup \{1\}$.

(i) Let φ be any automorphism of S and $K = \{a^i b^j \mid 1 \leqslant i \leqslant m-1, 1 \leqslant j \leqslant n-1\}$. Then by the Remark 1.2, $\sigma : G \cong K$. Also, since $\varphi_{|K}$ is an automorphism of K, so $\sigma^{-1}\varphi_{|K}\sigma \in Aut(G)$. Thus, the map $\rho : Aut(S) \longrightarrow Aut(G)$ defined by $\rho(\varphi) = \sigma^{-1}\varphi_{|K}\sigma \in Aut(G)$, is a group automorphism and therefore,

$$Aut(S) \cong \rho(Aut(S)) \leqslant Aut(G).$$

Consequently, Cayley's Theorem yields that $|Aut(S)|$ divides $|Aut(G)|$.

(ii) If $m \neq n$, then by the Remark 1.2-(ii), $\theta \in Aut(S)$ if and only if $\theta(a) = a^r$, $\theta(b) = b^s$, such that $gcd(r, m-1) = 1$ and $gca(s, n-1) = 1$, and so

$$|Aut(S)| = \phi(m-1)\phi(n-1).$$

But if $m = n$ then, in addition to the above automorphisms we have the following:

$$\sigma(a) = b^s,\ \sigma(b) = a^r, \quad \forall r, s;\ gcd(r, m-1) = 1,\ (s, n-1) = 1.$$

So, $|Aut(S)| = 2\phi(m-1)\phi(n-1)$.

(iii) Let $\theta \in Aut(G)$ and suppose that $\theta(a) = a^r b^s$. Since the order of a and $a^r b^s$ are the same, i.e.; $| a | = | a^r b^s |$ then, $m - 1 = m_1 n_1$ where, $| a^r | = m_1$ and $| b^s | = n_1$. It follows that $m - 1 = m_1$ and $n_1 = 1$ since $gcd(m-1, n-1) = 1$ and n_1 divides $n - 1$. Thus $\theta(a) = a^r$, and so $gcd(r, m-1) = 1$. In a similar way we may get $\theta(b) = b^s$, and so $gcd(s, n-1) = 1$. Therefore $\theta \in Aut(S)$. □

3. Conclusion

The direct product of the monogenic semigroups $S = \langle a \mid a^{m+1} = a \rangle$ and $T = \langle b \mid b^{n+1} = b \rangle$ may be presented as

$$P = \langle a, b \mid a^{m+1} = a,\ b^{n+1} = b,\ ab = ba,\ a^m = b^n \rangle.$$

Considering the relation $Aut(Sg(P)) \leq Aut(Sg(\pi))$ we deduce that if $gcd(m,n) = 1$ then $\mid Aut(Sg(P)) \mid$ divides $\mid Aut(Gp(\pi)) \mid$. We conclude this section by giving certain examples, considering calculations.

Example 2 *On the condition of the Theorem 1.1-(iii) we are able to determine the structure of* $Aut(Sg(\pi))$*. Indeed,* $Aut(Sg(\pi))$ *is abelian if* $gcd(m-1, n-1) = 1$*, for, this condition shows that the group* $Gp(\pi)$ *is abelian and so is* $Aut(Gp(\pi))$*.*

Example 3 *For every integer* $k \geq 2$ *and the presentation*

$$\pi = \langle a, b \mid a^3 = a, b^{2^k+1} = b, ab = ba \rangle,\ (k \geqslant 2),$$

we have $|Aut(Sg(\pi))| = \frac{1}{4}|Aut(Gp(\pi))|$*. This is a result of the Theorem 1.1-(ii), for, we may easily see that*

$$|Aut(Gp(\pi))| = |C_2 \times C_{2^k}| = 2^{k+1} = |Gp(\pi)|,$$

and get

$$|Aut(Sg(\pi))| = (2^k - 2^{k-1}) = 2^{k-1}.$$

(Here, $Gp(\pi)$ *is an example of abelian group such that* $Aut(Gp(\pi))$ *is not abelian.)*

Example 4 *For every integer* $k \geq 2$ *and the presentation*

$$\pi = \langle a, b \mid a^3 = a,\ b^{2\cdot 3^k+1} = b,\ ab = ba \rangle,\ (k \geqslant 1),$$

we have $Aut(Sg(\pi))| = \phi(2 \cdot 3^k) = (3^k - 3^{k-1}) = 2 \cdot 3^{k-1}$*. This is because of the equations* $|Gp(\pi)| = |C_2 \times C_{2\cdot 3^k}| = 2^2 \cdot 3^k$ *and*

$$\begin{aligned} |Aut(Gp(\pi))| &= |Aut(C_2 \times C_2)| \cdot |Aut(C_{3^k})| \\ &= 6 \cdot (3^k - 3^{k-1}) = 2 \cdot 3^k(3-1) = 2^2 \cdot 3^k. \end{aligned}$$

which show that $|Aut(Gp(\pi))| = |Gp(\pi)|$*. On the other hand, we get*

$$Aut(Sg(\pi))| = \phi(2 \cdot 3^k) = (3^k - 3^{k-1}) = 2 \cdot 3^{k-1}.$$

So the result follows at once.

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