Best simultaneous approximation in function and operator spaces

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Abstract

Let Z be a Banach space and G be a closed subspace of Z. For $f_1, f_2 \in Z$, the distance from f_1, f_2 to G is defined by $d(f_1, f_2, G) = \inf\limits_{f \in G} \max\{\|f_1 - f\|, \|f_2 - f\|\}$. An element $g^* \in G$ satisfying

$$\max\{\|f_1 - g^*\|, \|f_2 - g^*\|\} = \inf_{f \in G} \max\{\|f_1 - f\|, \|f_2 - f\|\}$$

is called a best simultaneous approximation for f_1, f_2 from G. In this paper, we study the problem of best simultananeous approximation in the space of all continuous X-valued functions on a compact Hausdorff space S; $C(S, X)$, and the space of all Bounded linear operators from a Banach space X into a Banach space Y; $L(X, Y)$.

Key Words: Simultaneous approximation, Banach spaces

1. Introduction

Let X be a Banach space and G be a closed subspace of X. For $x \in X$, we set

$$d(x, G) = \inf\{\|x - g\| : g \in G\}.$$

An element $g_0 \in G$, satisfying $\|x - g_0\| = d(x, G)$, is called a best approximant of x from G. If every element of X admits a best approximation from G, then G is said to be proximinal in X. The problem of best simultaneous approximation is a generalization of the the problem of best approximation and may be described as follows:

For a bounded subset $B \subset X$, we set

$$d(B, G) = \inf_{g \in G} \sup_{b \in B} \|b - g\|. \tag{1}$$

An element $g_0 \in G$ is said a best simultaneous approximant for B if

$$\sup_{b \in B} \|b - g^*\| = d(B, G).$$

The problem of best simultaneous approximation to a finite set of fuctions has recently been a subject of intensive study, see for example [1, 2, 6, 10, 11, 16] and for infite set see [17, 18]. The preveously metioned

works, except [1,2], have dealt with characterizaton and uniqueness of best simultaneous approximation. In contrast, studying subspaces of function spaces, which enjoy the property that any given finite set can be approximated simultaneously from them, has recieved little intrest; see for example [1, 2]. Subspaces that enjoy such properties are said to be simultaneously proximinal [14].

Let S be a compact Hausdorff space, Y a Banach space, and H be a closed subspace of Y. Let $\ell^\infty(S, Y)$ denote the Banach space of all bounded functions from S into Y with norm defined by

$$\|f\| = \sup_{s \in S} \|f(s)\|,$$

$C(S, H)$ to denote the subset of $\ell^\infty(S, Y)$ consists of all continuous functions, and $L(X, Y)$ to denote the Banach space of all bounded linear operatos from a Banach space X into Y. In [13] it is shown that, if Y is uniformly covex Banach space, then $C(S, Y)$ is simultaneously proximinal in $\ell^\infty(S, Y)$. In [2] it is shown that $C_0(S, Y)$, the subspace of $C(S, Y)$ that vanish at infinity, is simultaneously proximinal in $C(S, Y)$, also it is shown that if $K(X, Y)$ is the M-ideal of compact operators in $L(X, Y)$, then it is simultaneously proximinal in $L(X, Y)$.

The aim of this paper is to study simultaneous proximinality for other subspaces of the mentioned spaces. In particular, subspaces of the form $\ell^\infty(S, H), C(S, H)$, and $L(X, H)$ in $\ell^\infty(S, Y)$, $C(S, Y)$, and $L(X, Y)$, respectively. In this paper we will consider the problem of approximating a set of two elements $B = \{f_1, f_2\}$; and using the same procedure, it is possible to study the problem where $B = \{f_1, f_2, ..., f_n\}$ for any n.

2. Simultaneous proximinal subspaces

Throughout this section Y is a Banach space and H is a closed subspace of Y.

We set $Z = Y \underset{\infty}{\oplus} Y$ with $\|(x, y)\| = \max\{\|x\|, \|y\|\}$, and, $D(H) = \{(h, h) : h \in H\}$ with $\|(h, h)\| = \|h\|$. It is clear that Z is a Banach space and $D(H)$ is a closed subspace of Z. We say that H is simultaneously proximinal in Y if, for any pair $y_1, y_2 \in Y$, there exist $h_0 \in H$ such that

$$d(y_1, y_2, H) = \inf_{h \in H} \max\{\|x_1 - h\|, \|x_2 - h\|\} = \max\{\|x_1 - h_0\|, \|x_2 - h_0\|\}.$$

In this case, h_0 is called a best simultaneous approximant of y_1, y_2 from H. It is clear that $D(H)$ is proximinal in Z if and only if H is simultaneously proximinal in Y.

Definition 2.1 *Let H be simultaneously proximinal in Y. We say that H is simultaneously Chebyshev in Y if for any pair $y_1, y_2 \in Y$, there exist a unique best simultaneous approximant of y_1, y_2 from H.*

For $x, y \in Y$, we define

$$P_H(x, y) = \{h \in H : d(y_1, y_2, H) = \max\{\|y_1 - h\|, \|y_2 - h\|\}\}.$$

It is clear that if H is simultaneously proximinal in Y, then $P_H(x, y)$ is nonempty for every $x, y \in Y$.

Definition 2.2   If H is simultaneously proximinal in Y, then a simultaneous proximity map $\Pi_H : Y \underset{\infty}{\oplus} Y$
$\rightarrow H$ is a map that maps each element $(x, y) \in Y \underset{\infty}{\oplus} Y$ to some element in $P_H(x, y)$.

From the a bove definition the following assertions are obvious:

If H is simultaneously proximinal in Y and $\Pi_H : Y \underset{\infty}{\oplus} Y \rightarrow H$ is a simultaneous proximity map, then

1. $\Pi_H \left(\Pi_H \left(x, y \right), \ \Pi_H \left(x, y \right) \right) = \Pi_H \left(x, y \right)$ for any $x, y \in Y$.

2. $\Pi_H \left(x, y \right) \leq 2 \left\| (x, y) \right\|$ for any $x, y \in Y$.

3. Π_H is contineous at $(0, 0)$.

4. $\Pi_H \left((x, y) + \ (h, h) \right) = \Pi_H \left(x, y \right) + h$ for any $x, y \in Y$ and $h \in H$.

5. $\Pi_H \left(\alpha \left(x, y \right) \right) = \alpha \Pi_H \left(x, y \right)$ for any $x, y \in Y$ and α is a scalar.

Form (2) it is clear that if Π_H is linear, then Π_H is contineous.

Definition 2.3 Y is said to be uniformly convex if for any $\epsilon > 0$, there exists $\delta > 0$ such that, whenever
$x, y \in Y$, $\|x\| \leq 1$, $\|y\| \leq 1$,

$$\|x - y\| \geq \epsilon \Longrightarrow \left\| \frac{x + y}{2} \right\| \leq 1 - \delta. \tag{2}$$

If, for fixed $x \in Y$, $\|x\| \leq 1$, and $\epsilon > 0$, there exists $\delta(\epsilon, x) > 0$ such that (2) satisfied, then Y is said to be
locally uniformly convex.

We present some classes of of simultaneous proximinal subspaces.

Proposition 2.4 [1]   Let X be a Banach space and X^* be the dual space of X. If G is a w^*-closed
subspace of X^*, then G is simultaneously proximinal in X^*.

Example 2.5   If Y is uniformly convex and H is a closed subspace of Y, then H is simultaneously
Chebyshev in Y. This is, because uniform convexity of Y implies it is reflexive, hence H is simultaneously
proximinal in Y by propositin 2.4. The uniqueness of a best simultaeous approximant from H of each pair in
Y follows from [6].

Proposition 2.6   If Y is locally uniformly convex and H is simultaneously proximinal in Y, then H is

simultaneously Chebyshev in Y.

Proof.   Let x, $y \in Y$ and $h_1, h_2 \in P_H(x, y)$. Let

$$d = \max \left\{ \|x - h_1\|, \|y - h_1\| \right\} = \max \left\{ \|x - h_2\|, \|y - h_2\| \right\}.$$

Assume that $\|h_1 - h_2\| = d\epsilon > 0$, then

$$\|h_1 - y - (h_2 - y)\| = \|h_1 - x - (h_2 - x)\| = d\epsilon,$$

and

$$\left\|\frac{h_1 - y}{d}\right\|, \ \left\|\frac{h_1 - x}{d}\right\| \leq 1.$$

By local uniform covexity, there exist $\delta_1, \delta_2 > 0$ such that $\|h_1 - y + (h_2 - y)\| \leq 2d(1 - \delta_1)$ and $\|h_1 - x + (h_2 - x)\| \leq 2d(1 - \delta_2)$, then

$$\max\left\{\left\|y - \frac{h_1 + h_2}{2}\right\|, \ \left\|x - \frac{h_1 + h_2}{2}\right\|\right\} \leq (1 - \min\{\delta_1 + \delta_2\})d < d.$$

Which is a contradiction. Thus H is simultaneously Chebyshev in Y. $\qquad\square$

Example 2.7 [8]   *Let Y be a strictly convex Bannach space and H be a finite- dimesional subspace, then H is simultaneously Chebyshev in Y.*

Definition 2.8   *Let X be a Banach space. A linear projection P is called an M-projection if $\|x\| = \max\{\|P(x)\|, \ \|x - P(x)\|\}$ for all $x \in X$. A closed subspace $J \subseteq X$ is called an M-summand if it is the range of an M-projection.*

Proposition 2.9   *Let H be an M-summand of Y. Then H is simultaneously proximinal in Y, moreover H has a linear simultaneous proximity map.*

Proof.   Let $x, y \in Y$ and let $\theta = \frac{P(x) + P(y)}{2}$. Then

$$\begin{aligned}
\|x - \theta\| &= \left\|x - P(x) + \frac{1}{2}(P(x) - P(y))\right\| \\
&= \max\left\{\|x - P(x)\|, \ \frac{1}{2}\|P(x) - P(y)\|\right\}.
\end{aligned}$$

Similarly,

$$\|y - \theta\| = \max\left\{\|y - P(y)\|, \ \frac{1}{2}\|P(x) - P(y)\|\right\}.$$

Then,

$$\max\{\|x - \theta\|, \ \|y - \theta\|\} \leq \max\left\{\|x - P(x)\|, \ \|y - P(y)\|, \ \frac{1}{2}\|P(x) - P(y)\|\right\}.$$

Now, let $z \in H$, then

$$\begin{aligned}
\frac{1}{2}\|P(x) - P(y)\| &\leq \frac{1}{2}\|P(x) - z\| + \frac{1}{2}\|P(y) - z\| \\
&\leq \max\{\|P(x) - z\| + \|P(y) - z\|\}.
\end{aligned}$$

Thus, we have:

$$
\begin{aligned}
\max\left\{\|x-\theta\|,\ \|y-\theta\|\right\} &\leq \max\left\{ \begin{array}{c} \max\left\{\|x-P(x)\|,\ \|P(x)-z\|\right\}, \\ \max\left\{\|y-P(y)\|,\ \|P(y)-z\|\right\} \end{array} \right\} \\
&= \max\left\{\|x-z\|,\ \|y-z\|\right\}.
\end{aligned}
$$

Since $z \in H$ was arbitrary, then

$$
\max\left\{\|x-\theta\|,\ \|y-\theta\|\right\} \leq \max\left\{\|x-z\|,\ \|y-z\|\right\} \quad \text{for all } z \in H.
$$

Hence, θ is a best simultaneous approximatnt of x, y and H is simultaneously proximinal in Y. Now, define

$$
\pi : Y \underset{\infty}{\oplus} Y \to H, \qquad \text{by} \quad \pi(x,y) = \frac{P(x)+P(y)}{2}.
$$

It's clear that π is a simultaneous proximity map. The linearity of π follows from the linearity of P. $\square$

Corollary 2.10   *If H is an M-summand of Y, then H has a continuous simultaneous proximity map.*

Proof.   The result follows directly from Proposition 2.9 and the fact that the linear simultaneous proximity map is contineous. $\square$

3.   Main result

Let S be a compact Hausdorff space, X be a Banach space. we denote $C(S, X)$ to the Banach space of all X-valued contuous functions on S equipped with supremum norm, and $\ell^{\infty}(S, X)$ to the Banach space of all X-valued bounded functions on S equipped with supremum norm.

Theorem 3.1   *Let H be a closed subspace of a Banach space X, and S be a compact Hausdorff space. For any $f_1, f_2 \in C(S, X)$, we have*

$$
d(f_1,\ f_2, C(S, H)) = d(f_1,\ f_2, \ell^{\infty}(S, H)) = \sup_{t \in S} d(f_1(t),\ f_2(t), H).
$$

Proof.   Since $C(S, H) \subseteq \ell^{\infty}(S, X)$, it is clear that

$$
d(f_1,\ f_2, C(S, H)) \geq d(f_1,\ f_2, \ell^{\infty}(S, H)).
$$

If $g \in \ell^{\infty}(S, H)$, then

$$
\max\left\{\|f_1(t)-g(t)\|,\ \|f_2(t)-g(t)\|\right\} \geq d(f_1(t),\ f_2(t), H) \qquad \text{for all} \quad t \in S.
$$

On applying *sup* to both sides, we get

$$
\max\left\{\|f_1-g\|,\ \|f_2-g\|\right\} \geq \sup d(f_1(t),\ f_2(t), H).
$$

Since $g \in \ell^\infty (S, H)$ was arbitrary, then

$$d (f_1, \ f_2, \ell^\infty (S, H)) \geq \sup d (f_1 (t), \ f_2 (t), H).$$

Now let $\lambda > \sup d (f_1 (t), \ f_2 (t), H)$. For $t \in S$, define

$$\Phi (t) = \{h \in H : \max \{\|f_1 (t) - h\|, \ \|f_2 (t) - h\|\} \leq \lambda\}.$$

The Φ is nonempty closed subset of H. We shall prove that $\Phi (t)$ is convex for every $t \in S$ and Φ is lower simicontinuous. Let $t \in S$, $h_1, h_2 \in \Phi (t)$, and $0 \leq \alpha \leq 1$.

$$\max \{\|f_1 (t) - \alpha h_1 - (1 - \alpha) h_2\|, \ \|f_2 (t) - \alpha h_1 - (1 - \alpha) h_2\|\}$$

$$\leq \max \{\alpha \|f_1 (t) - h_1\| + (1 - \alpha) \|f_1 (t) - h_2\|, \ \alpha \|f_2 (t) - h_1\| + (1 - \alpha) \|f_2 (t) - h_2\|\}$$

$$\leq \alpha \max \{\|f_1 (t) - h_1\|, \ \|f_2 (t) - h_1\|\} + (1 - \alpha) \max \{\|f_1 (t) - h_2\|, \ \|f_2 (t) - h_2\|\}$$

$$\leq \alpha \lambda + (1 - \alpha) \lambda = \lambda.$$

To show that Φ is lower simicontinuous, let ∂ be an open set in H and put

$$\partial^* = \{t \in S : \Phi (t) \cap \partial \neq \phi\}.$$

It is to be shown that ∂^* is open. Let $\sigma \in \partial^*$, then $\Phi (\sigma) \cap \partial \neq \phi$. Hence, there exists an $h \in \partial$ such that

$$\max \{\|f_1 (\sigma) - h\|, \ \|f_2 (\sigma) - h\|\} \leq \lambda.$$

By the definition of λ, $\lambda > \inf_{y \in H} \max \{\|f_1 (\sigma) - y\|, \ \|f_2 (\sigma) - y\|\}$, there exists $h^{'} \in H$ such that

$$\max \left\{\left\|f_1 (\sigma) - h^{'}\right\|, \ \left\|f_2 (\sigma) - h^{'}\right\|\right\} < \lambda.$$

Now, $h \in \partial$, then there exists $\epsilon > 0$ such that

$$B (h, \epsilon) = \{y \in H : \|y - h\| < \epsilon\} \subseteq \partial.$$

Let $\delta = \{. \frac{\epsilon}{2 \|h - h^{'}\|}$ if $\left\|h - h^{'}\right\| \geq 1 \frac{\epsilon}{2}$ if $\left\|h - h^{'}\right\| \leq 1$; it is clear that $0 \leq \delta \leq 1$. Let $h^{''} = (1 - \delta) h + \delta h^{'}$, then $\left\|h^{''} - h\right\| = \delta \left\|h - h^{'}\right\| < \epsilon$, hence $h^{''} \in \partial$. By the covexity of $\Phi (\sigma)$, $h^{''} \in \Phi (\sigma)$ and

$$\max \left\{\left\|f_1 (\sigma) - h^{''}\right\|, \ \left\|f_2 (\sigma) - h^{''}\right\|\right\} < \lambda.$$

Now, let N be a neighbohood of σ such that

$$\max \{\|f_1 (\sigma) - f_1 (t)\|, \ \|f_2 (\sigma) - f_2 (t)\|\} < \lambda - \max \left\{\left\|f_1 (\sigma) - h^{''}\right\|, \ \left\|f_2 (\sigma) - h^{''}\right\|\right\}.$$

For any $t \in N$ we have

$$\max \left\{ \left\| f_1 \left(t \right) - h^{''} \right\|, \ \left\| f_2 \left(t \right) - h^{''} \right\| \right\}$$

$$\leq \quad \max \left\{ \left\| f_1 \left(t \right) - f_1 \left(\sigma \right) \right\| + \left\| f_1 \left(\sigma \right) - h^{''} \right\|, \ \left\| f_2 \left(t \right) - f_2 \left(\sigma \right) \right\| + \left\| f_2 \left(\sigma \right) - h^{''} \right\| \right\}$$

$$\leq \quad \max \left\{ \left\| f_1 \left(t \right) - f_1 \left(\sigma \right) \right\|, \ \left\| f_2 \left(t \right) - f_2 \left(\sigma \right) \right\| \right\} + \max \left\{ \left\| f_1 \left(\sigma \right) - h^{''} \right\|, \ \left\| f_2 \left(\sigma \right) - h^{''} \right\| \right\}$$

$$\leq \quad \lambda.$$

Hence, $h^{''} \in \Phi \left(t \right) \cap \eth$, $t \in \eth^*$, $N \subseteq \eth^*$, and $\eth$ is open. By Michael Selection Theorem there exists $g \in C \left(S, X \right)$ such that $g \left(t \right) \in \Phi \left(t \right)$ for all $t \in S$. Hence

$$\max \left\{ \left\| f_1 \left(t \right) - g \left(t \right) \right\|, \ \left\| f_2 \left(t \right) - g \left(t \right) \right\| \right\} \leq \lambda \quad \text{ for all } \ t \in S,$$

and

$$\max \left\{ \left\| f_1 - g \right\|, \ \left\| f_2 - g \right\| \right\} \leq \lambda.$$

Then

$$d \left(f_1, f_2, \ C \left(S, H \right) \right) \leq \lambda,$$

thus

$$d \left(f_1, f_2, \ C \left(S, H \right) \right) \leq \sup d \left(f_1 \left(t \right), f_2 \left(t \right), \ H \right).$$

$\square$

Lemma 3.2 *Let S be a compact Hausdorff space, Y be a Banach space, and H , G be closed subspaces of Y. Then*

$$C \left(S, \ H \underset{\infty}{\oplus} G \right) = C \left(S, H \right) \underset{\infty}{\oplus} C \left(S, G \right).$$

Proof. Fof $f \in C \left(S, \ H \underset{\infty}{\oplus} G \right)$, let $f_1 : S \to H$ and $f_2 : S \to G$ be such that $f \left(t \right) = \left(f_1 \left(t \right), f_2 \left(t \right) \right)$ for all $t \in S$. It is clear that $f_1 \in C \left(S, H \right)$ and , $f_2 \in C \left(S, G \right)$. Define

$$\psi : C \left(S, \ H \underset{\infty}{\oplus} G \right) \to C \left(S, H \right) \underset{\infty}{\oplus} C \left(S, G \right),$$

by $\psi \left(f \right) = \left(f_1, f_2 \right)$. It is clear that ψ is onto isometry, noting that

$$\left\| \psi \left(f \right) \right\| = \max \left\{ \left\| f_1 \right\|, \left\| f_2 \right\| \right\} = \sup \max \left\{ \left\| f_1 \left(t \right) \right\|, \left\| f_2 \left(t \right) \right\| \right\} = \sup \left\| f \left(t \right) \right\| = \left\| f \right\|.$$

$\square$

Theorem 3.3 *Let S be a compact Hausdorff and H be a closed subspace of a Banach space Y. Then*

1. *If $C(S, H)$ is simultaneously proximinal in $C(S, Y)$, then H is simultaneously proximinal Y.*

2. *If H has a continuous simultaneously proximity map, then $C(S, H)$ is simultaneously proximinal in $C(S, Y)$ and has a continuous simultaneously proximity map.*

Proof.

(1) : Let $x, y \in X$. Define $f_y : S \to Y$ and $f_x : S \to Y$ by

$$f_y(s) = y, \quad f_x(s) = x \qquad \text{for all} \quad s \in S.$$

Since $C(S, H)$ is simultaneously proximinal in $C(S, Y)$, there exists $g \in C(S, H)$ such that

$$
\begin{aligned}
\max\{\|f_y - g\|, \ \|f_x - g\|\} \ &= \ d(f_y, f_x, C(S, H)) \\
&= \ \sup d(f_y(s), f_x(s), \ H) \qquad (\text{ Theorem 3.1 }) \\
&\leq \ d(x, y, \ H).
\end{aligned}
$$

Then, for some $s_0 \in S$, we have

$$\max\{\|f_y(s_0) - g(s_0)\|, \ \|f_x(s_0) - g(s_0)\|\} \leq d(x, y, \ H).$$

Hence $g(s_0)$ is a best simultaneous approximation for x, y from H.

(2) Let $A : Y \underset{\infty}{\oplus} Y \to H$ be a continuous simultaneously proximity map for H. Define

$$A' : C\left(S, Y \underset{\infty}{\oplus} Y\right) \to C(S, H)$$

by $A'(f) = A \circ f$. By lemma 3.2, A' can be redefined as

$$A' : C(S, Y) \underset{\infty}{\oplus} C(S, Y) \to C(S, H), \quad \text{and} \quad A'(f_1, f_2)(s) = A(f_1(s), f_2(s)) \quad \text{for all } s \in S.$$

It is clear that $A'(f_1, f_2) \in C(S, H)$. Let $g \in C(S, H)$, then

$$
\begin{aligned}
&\max\{\|f_1(s) - A(f_1(s), f_2(s))\|, \ \|f_2(s) - A(f_1(s), f_2(s))\|\} \\
&\leq \ \max\{\|f_1(s) - g(s)\|, \ \|f_2(s) - g(s)\|\},
\end{aligned}
$$

for all $s \in S$, then

$$\max\{\|f_1 - A(f_1, f_2)\|, \ \|f_2 - A(f_1, f_2)\|\} \leq \max\{\|f_1 - g\|, \ \|f_2 - g\|\}.$$

Thus, $A(f_1, f_2)$ is a best simultaneous approximation for f_1, f_2 from $\mathrm{C}(S, H)$ and then $C(S, H)$ is simultaneously proximinal in $C(S, Y)$. It is clear that

$$A' : C(S, Y) \underset{\infty}{\oplus} C(S, Y) \to C(S, H)$$

is a continuous simultaneous proximity map. $\qquad\square$

Corollary 3.4   *If H is an M-summand of a Banach space Y, then $C(S, H)$ is simultaneously proximinal in $C(S, Y)$ and has a continuous simultaneously proximity map.*

Proof.   The result follows from Corollary 2.10 and Theorem 3.3.   □

Let X, Y be Banach spaces. $L(X, Y)$ is denoted to the space of all bounded linear operators from X into Y.

Lemma 3.5 [2]   *Let X, Y be Banach spaces, then*

$$L\left(X, Y \underset{\infty}{\oplus} Y\right) = L(X, Y) \underset{\infty}{\oplus} L(X, Y).$$

Theorem 3.6   *Let H be a simultaneous proximinal subspace of a Banach space Y. If H has a linear proximity map, then $L(X, H)$ is simultaneously proximinal in $L(X, Y)$ and has a linear simultaneous proximity map.*

Proof.   Let $\pi : Y \underset{\infty}{\oplus} Y \to H$ be a linear simultaneously proximity map for H. Define $A : L\left(X, Y \underset{\infty}{\oplus} Y\right) \to L(X, H)$ by $A(f) = \pi \circ f$. By lemma 3.5, we may write

$$A : L(X, Y) \underset{\infty}{\oplus} L(X, Y) \to L(X, H),$$

define by $A(f_1, f_2) = \pi \circ (f_1, f_2)$, $\pi(f_1, f_2)(x) = \pi(f_1(x), f_2(x))$ for all $x \in X$. It is clear that

$$A(\alpha(f_1, f_2) + \beta(g_1, g_2)) = \alpha A(f_1, f_2) + \beta A(g_1, g_2),$$

and $A(f_1, f_2)$ is a linear simultaneously proximity map for $L(X, H)$.   □

Corollary 3.7   *If H is an M-summand of a Banach space Y, then $L(X, H)$ is simultaneously proximinal in $L(X, Y)$ and has a continuous simultaneous proximity map.*

Proof.   The result follows from Corollary 2.11 and Theorem 3.6.   □

Let X be a Banach space. We denote X^* to the dual space of X.

Theorem 3.8   *Let H be a closed subspace of a Banach space Y. If $L(X, H)$ is simultaneously proximinal in $L(X, Y)$, then H is simultaneously proximinal in Y.*

Proof.   Let $y_1, y_2 \in Y$ and $0 \neq x_0 \in X$. By Hanh-Banach theorem, there exists $x^* \in X^*$ such that $x^*(x_0) = 1 = \|x^*\|$. Consider the operators

$$x^* \otimes y_1, \ x^* \otimes y_2 \ : X \to Y$$

defined by $(x^* \otimes y_i)(x) = x^*(x) y_i$ for all $x \in X$ and for $i = 1, 2$. It is clear that $x^* \otimes y_1, \ x^* \otimes y_2 \in L(X, Y)$. Since $L(X, H)$ is simultaneously proximinal in $L(X, Y)$, there exists $T \in L(X, H)$ such that

$$\max\left\{\|x^* \otimes y_1 - T\|, \ \|x^* \otimes y_2 - T\|\right\} \leq \max\left\{\|x^* \otimes y_1 - B\|, \ \|x^* \otimes y_2 - B\|\right\}$$

for all $B \in L(X, H)$. Let B runs over all operators of the form $x^* \otimes h, \ h \in H$. Then

$$\max\{\|x^* \otimes y_1 - T\|, \ \|x^* \otimes y_2 - T\|\} \leq \max\{\|x^* \otimes y_1 - x^* \otimes h\|, \ \|x^* \otimes y_2 - x^* \otimes h\|\}$$
$$\leq \|x^*\| \max\{\|y_1 - h\|, \ \|y_2 - h\|\},$$

for all $h \in H$. Then,

$$\max\{\|(x^* \otimes y_1)(x_0) - T(x_0)\|, \ \|(x^* \otimes y_1)(x_0) - T(x_0)\|\} \leq \max\{\|y_1 - h\|, \ \|y_2 - h\|\},$$
$$\max\{\|y_1 - T(x_0)\|, \ \|y_1 - T(x_0)\|\} \leq \max\{\|y_1 - h\|, \ \|y_2 - h\|\},$$

for all $h \in H$. Then, $T(x_0)$ is a best simultaneous approximant for y_1, y_2 from H.

Let X be a Banach space and H be a closed subspace Y. for $f, g \in Y$, it has been shown in Theorem 3.1 that

$$d(f, g, \ell^\infty(S, H)) = \sup_{s \in S} d(f(s), g(s), H)$$

for any compact Hausdorff space S. In fact we have the following result for any nonempty set S. $\qquad \square$

Lemma 3.9   *Let X be a Banach space, H be a closed subspace of X, S be any nonempty set. For any $f, g \in \ell^\infty(S, Y)$ we have*

$$d(f, g, \ell^\infty(S, H)) = \sup_s d(f(s), g(s), H).$$

Proof.   Let $h \in \ell^\infty(S, H)$, then $\max\{\|f(s) - h(s)\|, \ \|g(s) - h(s)\|\} \geq d(f(s), g(s), H)$, for all $s \in S$. Then

$$\max\{\|f - h\|, \|g - h\|\} \geq \sup_n (f(s), g(s), H).$$

Since $h \in \ell^\infty(S, H)$ was arbitury, then

$$d(f, g, \ell^\infty(S, H)) \geq \sup_s d(f(s), g(s), H).$$

For the reverse inequality, let $\epsilon > 0$, then for all $s \in S$, there exists $k(s) \in H$ such that

$$\max\{\|f(s) - k(s)\|, \ \| g(s) - k(s)\|\} < d(f(n), g(n), H) + \epsilon.$$

Using Axiom of Choice, we may define $h_0(s) = k(s)$ for all $s \in S$. It is clear that $h_0 \in \ell^\infty(S, H)$. Then

$$d(f, g, \ell^\infty(S, H)) \leq \max\{\|f - h_0\|, \ \| g - h_0\|\} < \sup_s d(f(s), g(s), H) + \epsilon,$$
$$d(f, g, \ell^\infty(S, H)) \leq \sup_s d(f(s), g(s), H) + \epsilon.$$

Since $\epsilon > 0$ was arbitury, then $d(f, g, \ell^\infty(S, H)) \leq \sup d(f(s), g(s), H)$. $\qquad \square$

Theorem 3.10   *Let Y be a Banach space, H be a closed subspace of Y, and S be any nonempty subset. Then the following are equivalent:*

1. H is simultaneously proximinal in Y.

2. $\ell^\infty(S,H)$ is simultaneously proximinal in $\ell^\infty(S,Y)$.

Proof. $(\mathbf{1}) \to (\mathbf{2})$

Let $f, g \in \ell^\infty(S,Y)$. Since H is simultaneously proximinal in Y, then for any $s \in S$ there exists $k(s) \in H$ such that

$$\max\{\|f(s) - k(s)\|, \| g(s) - k(s) \|\} \le \max\{\|f(s) - z\|, \| g(s) - z \|\}, \qquad (3)$$

for all $z \in G$. In particular it holds for any $z(s) \in G$, $z \in \ell^\infty(S,H)$. By Axiom of Choice, there exists $k \in \ell^\infty(S,H)$ satisfying (3). Hence

$$\max\{\|f - k\|, \| g - k \|\} \le \max\{\|f - z\|, \| g - z \|\},$$

for all $z \in \ell^\infty(S,H)$. Then

$$\max\{\|f - k\|, \| g - k \|\} = d(f, g, \ell^\infty(S,H)),$$

which implies that $\ell^\infty(S,H)$ is simultaneously proximinal in $\ell^\infty(S,Y)$. $\qquad\square$

$(2) \to (1)$.   Let $x, y \in X$. Set $f_x : S \to Y$, $f_y : S \to Y$, defined by $f_x(s) = x$, $f_y(s) = y$ for all $s \in S$. By Lemma 3.9,

$$d(f_y, f_x, \ell^\infty(S,H)) = \sup_s d(f_y(s), f_x(s), H) = d(x, y, H).$$

Since $\ell^\infty(S,H)$ is simultaneuosly proximinal in $\ell^\infty(S,Y)$, then there exists $h \in \ell^\infty(S,H)$ such that

$$\max\{\|f_x - h\|, \| f_y - h \|\} = d(x, y, H).$$

Choose $s_0 \in S$ such that

$$\max\{\|x - h(s_0)\|, \| y - h(s_0) \|\} \le d(x, y, G).$$

Then $h(s_0)$ is a best simultaneous approximation for x and y form H.

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