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Groupoids, imaginaries and internal covers

Ehud Hrushovski

Abstract

Let T be a first-order theory. A correspondence is established between internal covers of models of T and definable groupoids within T. We also consider amalgamations of independent diagrams of algebraically closed substructures, and find strong relation between covers, uniqueness for 3-amalgamation, existence of 4-amalgamation, imaginaries of T^σ, and definable groupoids. As a corollary, we describe the imaginary elements of families of finite-dimensional vector spaces over pseudo-finite fields.

Key words and phrases: Internal cover, groupoid, higher amalgamation, elimination of imaginaries, pseudo-finite fields

The questions this manuscript addresses arose in the course of an investigation of the imaginary sorts in ultraproducts of p-adic fields. These were shown to be understandable given the imaginary sorts of certain finite-dimensional vector spaces over the residue field. The residue field is pseudo-finite, and the imaginary elements there were previously studied, and shown in fact to be eliminable over an appropriate base. It remains therefore to describe the imaginaries of finite-dimensional vector spaces over a field F, given those of F. I expected this step to be rather easy; but it turned out to become easy only after a number of issues, of interest in themselves, are made clear.

Let T be a first-order theory. A correspondence is established between internal covers of models of T and definable groupoids within T. Internal covers were recognized as central in the study of totally categorical structures, but nevertheless remained mysterious; it was not clear how to describe the possible T' from the point of view of T. We give an account of this here, in terms of groupoids in place of equivalence relations. This description permits the view of the cover as a generalized imaginary sort.

This seems to be a useful language even for finite covers, though there the situation is rather well-understood; cf. [9],[11]. We concentrate on finite generalized imaginaries, and describe a a connection between elimination of imaginaries and higher amalgamation principles within the algebraic closure of an independent n-tuple. The familiar imaginaries of T^{eq} correspond to 3-amalgamation, as was understood for some time for stable and simple theories, and finite generalized imaginaries correspond to 4-amalgamation. This brings out ideas present in some form in [8], [5], [10], [9]. In particular, 4-amalgamation always holds for stable theory T, if "algebraic closure" is taken to include generalized imaginaries. We also relate uniqueness of n-amalgamation to existence of $n+1$-amalgamation; using "all" finite imaginaries (not necessarily arising from groupoids) we show that n-amalgamation exists and is unique for all n.

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Adding an automorphism to the language to obtain a Robinson theory T^σ has the effect of shifting the amalgamation dimension by one; n-amalgamation in the expanded language corresponds to $n+1$-amalgamation for T. Thus ordinary imaginaries of T^σ can be understood, given generalized imaginaries of T.

We thus find a strong relation between four things: covers, failure of uniqueness for 3-amalgamation, imaginaries of T^σ, and definable groupoids. A clear continuation to $n = 4$ would be interesting.

Returning to the original motivation, we use these ideas to determine the imaginaries for systems of finite-dimensional vector spaces over fields, and especially over pseudo-finite fields (Theorem 5.10).

1. Preliminaries

Let T be a first-order theory, with universal domain $\mathbb{U}$. Def($\mathbb{U}$) is the category of $\mathbb{U}$-definable sets (with parameters) and maps between them.[1]

Let A, B be small subsets of $\mathbb{U}$. For each $b \in B$, we provide a new constant symbol c_b; and for each $a \in A$, a new variable x_a. We write $\mathrm{tp}(A/B)$ for the set of all formulas with these new variables and constants, true in $\mathbb{U}$ under the eponymous interpretation of constant symbols and assignment of variables. This is useful in expressions such as $\mathrm{tp}(A/B) \models \mathrm{tp}(A/B')$.

An ∞-definable set is the solution set of a partial type (of bounded size; say bounded by the cardinality of the language.) Morphisms between ∞ -definable sets are still induced by ordinary definable maps. If the partial type is allowed to have infinitely many variables, the set is called $\star$-definable instead. $\star$-definable sets can also be viewed as projective systems of definable sets and maps.

Dually, a $\bigvee$-definable set is the complement of an ∞-definable set.

When we say a set P is definable, we mean: without parameters. If we wish to speak about a set definable with parameters a, we will exhibit these parameters in the notation: P_a.

We will often consider two languages $L \subset L'$. The language L' may have more sorts than L. Let T' be a complete theory for L', $T = T'|L$. We say T is *embedded* if any relation of L is T'-equivalent to a formula of L'. We say the sorts of L are *stably embedded* if in any model $M' \models T'$, any M'-definable subset of $S_1 \times \times S_k$ (where the S_i are L-sorts) is also definable with parameters from $\cup S_i(M')$. This basic notion has various equivalent forms, see appendix to [6] and also [1].

Let D be a definable set of L'. We say D is *internal* to L if in some (or any) model M' of T', there exist sorts $S_1, \ldots, S_k$ of L and an M-definable map f whose domain is a subset of $S_1 \times \times S_k$, and whose image is D. See [15], appendix, where it is shown that internality is associated with definable automorphism groups; indeed, assuming T is embedded and stably embedded in T', and $L' \setminus L$ is finite for simplicity, and letting M denote the L'-sorts of M', there exists a definable group G such that $G(M')$ can be identified with $\mathrm{Aut}(D(M')/M)$. G is called the *liaison group*, a term due to Poizat. It is also shown in [15] that G is M-isomorphic to an M-definable group. In §2 we will prove a more precise, parameter-free version, using the notion of a *definable groupoid*.

We can immediately introduce one of the main notions of the paper.

[1] More generally we can work with a "Robinson theory", a universal theory with the amalgamation property for substructure; one then works with substructures of a universal domain, and takes "definable" to mean: quantifier-free definable. This was one of the "contexts" of [21]; I dubbed it "Robinson" when unaware of this reference, and the name stuck.

Definition 1.1 *Let N be a structure, M the union of some of the sorts of N. N is a* finite internal cover *of M if M is stably embedded in N, and* $\mathrm{Aut}(N/M)$ *is finite (uniformly in elementary extensions.) Equivalently, $N \subseteq dcl(M,b)$ for some finite $b \in acl(M)$.*

A finite internal cover is a special case of an *internal cover*, where we demand that N/M is internal in place of $\mathrm{Aut}(N/M)$ finite, and that $\mathrm{Aut}(N/M)$ is definable. In general, $\mathrm{Aut}(N/M)$ is an ∞-definable group of N, isomorphic over N to an ∞-definable group of M. cf. [15].

While there is no difficulty in treating the general case, we will assume for simplicity of language that $\mathrm{Aut}(N/M)$ is in fact *definable* in the internal covers considered in this paper. (Only the case of internal covers with *finite* automorphism groups is needed for our applications.)

Remark 1.2 Let T'' be a many-sorted expansion of T. For any $M'' \models T''$, let M be the restriction to the language of T, and let $\mathrm{Aut}(M'') \to \mathrm{Aut}(M)$ be the natural group homomorphism; let $K(M'', M)$ denote the kernel.

1. If $\mathrm{Aut}(M'') \to \mathrm{Aut}(M)$ is always surjective, then T is stably embedded in T''.

2. If in addition the kernel $K(M'', M) = Ker(\mathrm{Aut}(M'') \to \mathrm{Aut}(M))$ always has cardinality bounded in terms of M, then T'' is internal to T, i.e. all sorts of T'' are internal to the T-sorts.

3. If $K(M'', M)$ has cardinality bounded independently of M and M'', then T'' is T-internal with an ∞-definable liaison group which is bounded, hence finite. (Cf. [15], Appendix B.) Thus in this case each sort of T'' is a finitely imaginary sort of T.

4. If $\mathrm{Aut}(M'') \to \mathrm{Aut}(M)$ is always bijective, then $M'' \subset \mathrm{dcl}_{T''}(M)$.

The surjectivity implies that T' induces no new structure on the sorts of T, and also that T is stably embedded in T', cf. [6], Appendix. Injectivity of $\mathrm{Aut}(M') \to \mathrm{Aut}(M)$, implies that $M' \subseteq \mathrm{dcl}(M)$; for suppose $c \in M'$ and $c \notin \mathrm{dcl}(M)$. We may take M, M' to be sufficiently saturated and homogeneous. By stable embeddedness there exists a small subset A of M such that $\mathrm{tp}(c/A)$ implies $\mathrm{tp}(c/M)$. As $c \notin \mathrm{dcl}(M)$, there exists $c' \neq c$ with $\mathrm{tp}(c'/A) = \mathrm{tp}(c/A)$. So $\mathrm{tp}(c/M) = \mathrm{tp}(c'/M)$, and by stable embeddedness again there exists $\sigma \in \mathrm{Aut}(M'/M)$ with $\sigma(c') = c$; the restriction of σ to M is the identity, but σ is not, contradicting injectivity. For the rest see [15], Appendix B.

Lemma 1.3 *Let T' be a theory, T the restriction of T' to a subset of the sorts of T', T'' an expansion of T' on the same sorts as T'. Assume T is stably embedded in T', and for any $N'' \models T''$, if N', N are the restrictions to T', T respectively, the natural map* $\mathrm{Aut}(N''/N) \to \mathrm{Aut}(N'/N)$ *is surjective. Then T', T'' have the same definable relations.*

Proof. By the proof of Beth's implicit definability theorem, it suffices to show that $\mathrm{Aut}(N'') = \mathrm{Aut}(N')$ for any N''. This is clear from the exact sequences $1 \to \mathrm{Aut}(N'/N) \to \mathrm{Aut}(N') \to \mathrm{Aut}(N)$, $1 \to \mathrm{Aut}(N''/N) \to \mathrm{Aut}(N'') \to \mathrm{Aut}(N)$, and the equality $\mathrm{Aut}(N'/N) = Aut(N''/N)$. □

2. Definable groupoids

A *category* is a 2-sorted structure with sorts O, M, with maps $i_0, i_1 : M \to O$ (the morphism $m \in M$ goes from $i_0(m)$ to $i_1(m)$), and a partial composition $\circ : M \times_{i_1,i_0} M \to M$, and an identity map $Id : O \to M$ (so that $Id(x) : x \to x$ is the identity map), satisfying the usual associative laws. The language of categories is thus 2-sorted, with relation symbols $i_1, i_1, Id, \circ$.

A (Grothendieck) *groupoid* is a category $\mathcal{G} = (\mathrm{Ob}\mathcal{G}, \mathrm{Mor}\mathcal{G})$ where every morphism has a 2-sided inverse. For a groupoid $\mathcal{G}$, let $Iso_{\mathcal{G}}$ be the equivalence relation on $\mathrm{Ob}\mathcal{G}$: $\mathrm{Mor}G(c, c') \neq \emptyset$. On the other hand, for any $a \in \mathrm{Ob}\mathcal{G}$, we have a group $G_a = \mathrm{Mor}\mathcal{G}(a, a)$. These groups are isomorphic for $(a, b) \in Iso_{\mathcal{G}}$: : if $h \in \mathrm{Mor}\mathcal{G}(a, b)$, then $x \mapsto h^{-1}xh$ is an isomorphism $G_b \to G_a$. This isomorphism is well-defined up to conjugation. Thus groupoids generalize, at different extremes, both groups and equivalence relations: an equivalence relation is a groupoid with trivial groups, and a group is a groupoid with a single object.

We will assume in this section that $\mathcal{G}$ has a unique isomorphism type. (I.e. $\mathrm{Ob}\mathcal{G} \neq \emptyset$, and $\mathrm{Mor}\mathcal{G}(a, b) \neq \emptyset$ for all $a, b \in \mathrm{Ob}\mathcal{G}$.) Without this assumption, one obtains relative versions of the results, fibered over the set of objects; for instance in 2.1, the conclusion becomes that one can interpret a set S and a map $h : S \to T = \mathcal{G}/\equiv$, such that for $t \in T$, for any representative $b \in \mathrm{Ob}\mathcal{G}$ of t, $F(b)$ is definably isomorphic to $S_t = h^{-1}(t)$.

If X_a is a conjugation-invariant subset of some G_a, let $X_b = h^{-1}X_a h$, where $h \in \mathrm{Mor}\mathcal{G}(b, a)$; the choice of h does not matter.

In particular, if $N_a \triangleleft G_a$ is a normal subgroup, we obtain a system of normal subgroups $N_b \triangleleft G_b$. Moreover, we can define an equivalence relation N on $\mathrm{Mor}\mathcal{G}(a, b)$:

$$(f, g) \in N \ \leftrightarrow \ g^{-1}f \in N_a \ \leftrightarrow \ fg^{-1} \in N_b$$

This gives rise to a quotient groupoid with the same set of objects, and with $\mathrm{Mor}\mathcal{G}'(a, b) = \mathrm{Mor}\mathcal{G}(a, b)/N$.

It makes sense to speak of Abelian or solvable groupoids (meaning each G_a is that).

If $\mathrm{Ob}\mathcal{G}$ and $\mathrm{Mor}\mathcal{G}$ are defined by formulas in some structure $\mathbb{U}$, as well as the domain and range maps $\mathrm{Mor}\mathcal{G} \to \mathrm{Ob}\mathcal{G}$ and the composition, we say that $\mathcal{G}$ is a *definable groupoid* in $\mathbb{U}$.

A sub-groupoid is *full* if it consists of a subset of the objects, with all morphisms between them.

Let $F : \mathcal{G} \to Def(\mathbb{U})$ be a functor. We say that F is *definable* if $\{(a, d) : a \in \mathrm{Ob}\mathcal{G}, d \in F(a)\}$ is definable, as well as $\{(a, b, c, d, e) : a, b \in \mathrm{Ob}\mathcal{G}, c \in \mathrm{Mor}\mathcal{G}(a, b), d \in F(a), e \in F(b), F(c)(d) = e\}$.

Similarly for $\star$-definable (= Pro-definable) or $\bigvee$-definable (see §1). But if there exist a definable relation F_1 and definable function F_2 such that for $a \in \mathrm{Ob}\mathcal{G}$, $F(a) = F_1(a)$, and for $a, b \in \mathrm{Ob}\mathcal{G}$, $c \in \mathrm{Mor}\mathcal{G}(a, b)$, $F(c) = F_2(c)$, we will say that F is a (relatively) definable functor (even if if $\mathcal{G}$ is only $\star$-definable.)

Example 2.1 *Suppose each G_a is trivial. Then for each $a, b \in \mathrm{Ob}\mathcal{G}$ $\mathrm{Mor}\mathcal{G}(a, b)$ consists of a unique morphism. In this case if $F : \mathcal{G} \to Def(U)$ is a definable functor, one can interpret without parameters a set S, definably isomorphic to each $F(a)$.*

Let

$$E_S = \{(a, b, a', b') : a, a' \in \mathrm{Ob}\mathcal{G}, b \in F(a), b' \in F(a'), \exists c \in \mathrm{Mor}\mathcal{G}(a, a').\, F(c)(b) = b'\}$$

$$S = \{(a, b) : a \in \mathrm{Ob}\mathcal{G}, b \in F(a)\}/E_S$$

Example 2.2 *If $\mathcal{G}$ is Abelian, then the G_a are all canonically isomorphic, and one can interpret without parameters a single group, isomorphic to all G_a.*

Proof. As in 2.1: the maps $G_a \to G_b$, being unique up to conjugacy, are in this case in fact unique. □

From ⋆-definable to definable groupoids.

Lemma 2.3 *Let $\mathcal{G}^0$ be a groupoid, with a distinguished element $* \in \mathrm{Ob}\mathcal{G}^0$. Suppose $G^0_* = \mathrm{Mor}\mathcal{G}^0(*,*)$ is a subgroup of a group G. Then $\mathcal{G}^0$ extends canonically to a groupoid $\mathcal{G}$ with the same objects, and with* $\mathrm{Mor}\mathcal{G}(*,*) = G$.

Proof. In other words, the natural map $\mathcal{G} \mapsto \mathrm{Mor}\mathcal{G}(*,*)$, from supergroupoids $\mathcal{G}$ of $\mathcal{G}^0$ with the same object set, to supergroups G of G^0_*, is surjective. If $\mathcal{G}^0, G$ are ⋆-definable, so is $\mathcal{G}$.

Construction: Let

$$(\mathrm{Mor}\mathcal{G}(a,b) = \mathrm{Mor}\mathcal{G}^0(*,b) \times G \times \mathrm{Mor}\mathcal{G}^0(a,*))/\sim,$$

where $(f,g,h) \sim (f',g',h')$ iff $((f')^{-1}f)g = g'(h'h^{-1})$. Note that the expression makes sense, since $((f')^{-1}f), (h'h^{-1}) \in \mathrm{Mor}\mathcal{G}^0(*,*) \leq G$. It defines an equivalence relation: for instance, transitivity: if $((f')^{-1}f)g = g'(h'h^{-1})$ and $((f'')^{-1}f')g' = g''(h''(h')^{-1}))$, then

$$((f'')^{-1}f)g = ((f'')^{-1}f')((f')^{-1}f)g = ((f'')^{-1}f')g'(h'h^{-1}) = g''(h''(h')^{-1}))(h'h^{-1}) = g''(h''h^{-1}).$$

Define composition

$$\mathrm{Mor}\mathcal{G}(b,c) \times \mathrm{Mor}\mathcal{G}(a,b) \to \mathrm{Mor}\mathcal{G}(a,c)$$

by: $(j',g',f')/\sim) \circ ((f,g,h)/\sim) = (j', g'(f'f)g, h)/\sim$.

The verifications are left to the reader. □

Lemma 2.4 *Let $\mathcal{G}^0$ be a ⋆-definable groupoid, with G^0_a definable. Then $\mathcal{G}^0$ extends to a definable groupoid $\mathcal{G}$, with $\mathrm{Mor}\mathcal{G}(a,b) = \mathrm{Mor}\mathcal{G}^0(a,b)$ for $a,b \in \mathrm{Ob}\mathcal{G}^0$.*

If $\sim$ an $\bigvee$-definable equivalence relation, and all $a,b \in \mathrm{Ob}\mathcal{G}^0$ are $\sim$-equivalent, we can obtain the same for $\mathcal{G}$.

Proof. The hypothesis is intended to read: G^0_a is definable uniformly in a (or equivalently, that the statement is true in any model.) It follows that $\mathrm{Mor}\mathcal{G}^0(a,b)$ is definable, for any $a,b \in \mathrm{Ob}\mathcal{G}^0$. (This set is a torsor over G^0_a, so it is definable with parameters; being ⋆-definable with parameters a,b, it must be definable uniformly in these parameters.) The definition of $\mathrm{Mor}\mathcal{G}^0(a,b)$ must extend over all a,b in some definable set S_0 containing $\mathrm{Ob}\mathcal{G}^0$. The groupoid properties are certain universal axioms holding for all $a,b,c \in \mathrm{Ob}\mathcal{G}^0$; by compactness they must hold for all $a,b,c \in S_1$ (some definable S_1, with $\mathrm{Ob}\mathcal{G}^0 \subset S_1 \subset S_0$.) Let $\mathrm{Ob}\mathcal{G} = S_1$, and use the definable function above to define $\mathcal{G}$.

The two additional statements are also immediate consequences of compactness. □

Some theories, notably stable ones (cf. [16]), theories of finite S1 rank ([14]), and more generally supersimple theories ([25]), have the property that every ⋆-definable group is a projective limit $\lim_{\leftarrow n} G_n$, where $G_1 \leftarrow G_2 \leftarrow \cdots$ is a sequence of definable groups and maps. As soon as this holds for one of the groups G_a, we can use the two lemmas above to pass from an $\star$-definable groupoid to a definable one.

The liaison groupoid. Let $\mathbb{U}$ be a universal domain for a theory admitting elimination of quantifiers and elimination of imaginaries.

Let $\mathbb{V}$ be a union of sorts of $\mathbb{U}$, closed under images of definable maps. So $\mathbb{V}$ also admits elimination of imaginaries . We will obtain a $\star$-definable groupoid; the set of objects will be $\star$-definable, and the image of an ∞-definable set of $\mathbb{U}$; the sets of morphisms ∞-definable. In the situation of 2.4, this will be the limit of definable groupoids.

Proposition 2.5 *Assume $\mathbb{V}$ is stably embedded in $\mathbb{U}$.*

Let Q be a definable set of $\mathbb{U}$, internal to $\mathbb{V}$.

There exists $\star$-definable groupoids $\mathcal{G}$ in $\mathbb{U}$ and $\mathcal{G}_{\mathbb{V}}$ in $\mathbb{V}$, and definable functors $F : \mathcal{G} \to Def(\mathbb{U})$ and $F_{\mathbb{V}} : \mathcal{G}_{\mathbb{V}} \to Def(\mathbb{V})$, such that $\mathcal{G}_{\mathbb{V}}$ is a full subgroupoid of $\mathcal{G}$, $F_{\mathbb{V}} = F|\mathcal{G}_{\mathbb{V}}$, $\mathrm{Ob}\mathcal{G} = \mathrm{Ob}\mathcal{G}_{\mathbb{V}} \cup \{*\}$, and $F(*) = Q$.

We have $F(G_*) = \mathrm{Aut}(Q/\mathbb{V})$.

Proof. By internality of Q, and using elimination of imaginaries in $\mathbb{V}$, there exists Q_b definable over b in $\mathbb{V}$, and a $\mathbb{U}$-definable bijection $f_c : Q \to Q_b$. Since $\mathbb{V}$ is stably embedded, $\mathrm{tp}(c/b') \vdash \mathrm{tp}(c/\mathbb{V})$ for some b'; increasing b, we may assume $b = b'$. (It is here that we must allow b' to be a tuple with an infinite index set.)

Let $\mathrm{Ob}\mathcal{G}_{\mathbb{V}}$ be the set of solutions of $\mathrm{tp}(b)$, and let $\mathrm{Ob}\mathcal{G} = \mathrm{Ob}\mathcal{G}_{\mathbb{V}} \cup \{*\}$ (a formal element.) Let $\mathrm{Mor}\mathcal{G}(*, b') = \{c' : \mathrm{tp}(bc) = \mathrm{tp}(b'c')\}$ (the morphism is viewed as identical with the map $f_{c'} : Q \to Q_{b'}$.) Let $\mathrm{Mor}\mathcal{G}(b', *)$ be the same set of codes, but each code viewed now as coding the inverse map $Q_{b'} \to Q$. Let $\mathrm{Mor}\mathcal{G}(*, *) = \mathrm{Aut}(Q/\mathbb{V})$.

Observe the coherence of what has been defined so far: if $c, d \in \mathrm{Mor}\mathcal{G}(*, b')$, then $\mathrm{tp}(c/b') = \mathrm{tp}(d/b')$. Thus $\mathrm{tp}(c/\mathbb{V}) = \mathrm{tp}(d/\mathbb{V})$. Since $\mathbb{V}$ is stably embedded, there exists $\sigma \in \mathrm{Aut}(\mathbb{U}/\mathbb{V})$ with $\sigma(c) = d$. Let $g = \sigma|Q$. If $a \in Q$, then $\sigma(f_c(a)) = f_c(a)$ (since σ fixes $\mathbb{V}$) but also $\sigma(f_c(a)) = f_{\sigma(c)}(\sigma(a)) = f_d(g(a))$. Thus $f_c = f_d \circ g$. Conversely, if $\tau \in \mathrm{Aut}(\mathbb{U}/\mathbb{V})$ is arbitrary, $h = \tau|Q$, then $f_d \circ h = f_{\sigma^{-1}(d)}$.

For $b', b'' \in \mathrm{Ob}\mathcal{G}_{\mathbb{V}}$, let $\mathrm{Mor}\mathcal{G}(b', b'')$ be the set of maps $Q_{b'} \to Q_{b''}$ of the form $f_{c''} \circ {f_{c'}}^{-1}$, where $c' \in \mathrm{Mor}\mathcal{G}(*, b')$, $c'' \in \mathrm{Mor}\mathcal{G}(*, b'')$.

Note that if $\sigma(c'') = d''$, $g = \sigma|Q$, then $f_{c''} = f_{d''} \circ g$, $f_{c'} = f_{d'} \circ g$ for some d' $(=\sigma(c'))$, so $f_{c''} \circ {f_{c'}}^{-1} = f_{d''} \circ {f_{d'}}^{-1}$. Since $\mathrm{Aut}(Q/\mathbb{V})$ is transitive on $\mathrm{Mor}\mathcal{G}(*, b'')$, fixing some $d'' \in \mathrm{Mor}\mathcal{G}(*, b'')$, an arbitrary element of $\mathrm{Mor}\mathcal{G}(b', b'')$ can be written $f_{d''} \circ {f_{d'}}^{-1}$. Similarly, an arbitrary element of $\mathrm{Mor}\mathcal{G}(b'', b''')$ can be written $f_{d'''} \circ {f_{d''}}^{-1}$. So the composition of an arbitrary element of $\mathrm{Mor}\mathcal{G}(b', b'')$ with one of $\mathrm{Mor}\mathcal{G}(b'', b''')$ is an element of $\mathrm{Mor}\mathcal{G}(b', b''')$. It follows that we have indeed a groupoid.

Also, by expressing $\mathrm{Mor}\mathcal{G}(*, *)$ as $\mathrm{Mor}\mathcal{G}(b, *) \circ \mathrm{Mor}\mathcal{G}(*, b)$ for some b, it follows that $\mathrm{Mor}\mathcal{G}(*, *)$ is an ∞-definable set of permutations of Q (over b, but a posteriori over $\emptyset$, since at all events it is invariant.)

Define the functor F by $F(*) = Q$, $F(b') = Q_{b'}$, and define F on morphisms tautologically.

Let $\mathcal{G}_{\mathbb{V}}$ be the restriction of $\mathcal{G}$ to $\mathrm{Ob}\mathcal{G}_{\mathbb{V}}$, and $F_V = F|\mathcal{G}_{\mathbb{V}}$. All the properties are then clear. □

Remark 2.6 *1. There exist definable maps $f_i : Q \to \mathbb{V}$ $(i \in I)$ such that $F(G_*)$ is transitive on each fiber of $f = (f_i)_{i \in I}$.*

2. *Assume $G_* = \cap_n G_n$ where $G_1 \geq G_2 \geq \cdots$ are definable groups. Then one can find a* definable *groupoid $\mathcal{G}$ and a finite I satisfying Theorem 2.5* except *the last statement and (1) above. (And we still have $F(G_*) \supset \mathrm{Aut}(Q/\mathbb{V})$).*

Proof.

1. Let $f = (f_i)_{i \in I}$ enumerate all definable maps $Q \to \mathbb{V}$. Then $c, d \in Q$ are $\mathrm{Aut}(Q/\mathbb{V})$-conjugate iff $\mathrm{tp}(c/\mathbb{V}) = \mathrm{tp}(d/\mathbb{V})$ (by stable embeddedness) iff $f(c) = f(d)$.
2. By 2.3, 2.4.

□

3. Generalized imaginaries

The notion of an imaginary sort for a theory T can be described as follows. Let T' be an extension of T in a language containing the language of T, and having an additional sort S. A universal domain $\mathbb{U}'$ for T' thus has the form $\mathbb{U}, S(\mathbb{U}')$. S is an *imaginary sort of* T if every model $M \models T$ expands to a model $M' = (M, S_{M'})$ of T' with $S_{M'} \subseteq \mathrm{dcl}(M)$; equivalently (as noted above), for any such M',

$$\mathrm{Aut}(M') \to \mathrm{Aut}(M)$$

is a group isomorphism.

We will now consider a slight generalization. A *finite generalized imaginary* sort is defined as above, except that the homomorphism

$$\mathrm{Aut}(M') \to \mathrm{Aut}(M)$$

is allowed to have finite kernel. It is still assumed to be surjective. More generally, S is called an *internal generalized imaginary sort* if the language of T' is finite relative to the language of T (i.e finitely many relation symbols are added), and T' is internal to T. In this case, $\mathrm{Aut}(S_{M'}/M)$ is isomorphic to $G(M)$ for some definable group G. It makes sense to consider generalized sorts S relative to a sort D of T, meaning that a definable map $S \to \bar{S}$ is given, and each fiber is an internal imaginary sort. But in this paper we will consider internal generalized imaginary sorts almost exclusively, and will omit the adjective "internal".

An equivalent, more concrete definition of (ordinary) imaginaries can be given in terms of equivalence relations (cf. [22]). Let E be a definable equivalence relation on a set S; then S/E is added as a new sort, together with the canonical map $S \to S/E$.

This is used to find canonical parameters for definable families. For $s \in S$, let $\delta(s)$ be a definable set; such that $\delta(s) = \delta(s')$ iff $(s, s') \in E$. Then the image of s in s/E serves as a canonical parameter for $\delta(s)$.

More generally, in place of equality, one often has a definable bijection $f_{s,s'} : \delta(s) \to \delta(s')$, forming a commuting system. Then for $\bar{s} \in S/E$ one introduces $\delta(\bar{s})$ as the quotient of the $(\delta(s) : s/E = \bar{s})$ by the system $f_{s,s'}$, obtaining a canonical family $\delta(\bar{s}) : \bar{s} \in S/E)$. This can still be treated using equivalence-relation imaginaries, by an appropriate equivalence relation on $\dot{\bigcup}_s \delta(s)$. However, if the system has more than one definable bijection $\delta(s) \to \delta(s')$, this fails. We now generalize the above construction to more general groupoids.

A *concrete definable category* of T is a triple $\mathcal{G} = (\mathrm{Ob}\mathcal{G}, \mathrm{Mor}\mathcal{G}, \delta_{\mathcal{G}})$ with $(\mathrm{Ob}\mathcal{G}, \mathrm{Mor}\mathcal{G})$ a category interpretable in T, and $\delta_{\mathcal{G}} : \mathrm{Ob}\mathcal{G} \to Def(\mathbb{U})$ a faithful definable functor.

An *embedding* of $\mathcal{G}_1$ into $\mathcal{G}_2$ is a 0-definable fully faithful functor $h : \mathrm{Ob}\mathcal{G}_1 \to \mathrm{Ob}\mathcal{G}_2$, together with a 0-definable system of definable bijections $h_c : \delta_1(c) \to \delta_2(h(c))$ for $c \in \mathrm{Ob}\mathcal{G}_1$, such that $h_{c'} \circ \mathrm{Mor}\mathcal{G}_1(c, c') = \mathrm{Mor}\mathcal{G}_2(h(c), h(c'))\circ h_c$. In particular, conjugation by h_c induces a group isomorphism $\mathrm{Mor}\mathcal{G}_1(c, c) \to \mathrm{Mor}\mathcal{G}_2(h(c), h(c))$.

A *concrete groupoid* is a concrete category that is a groupoid.

A groupoid $\mathcal{G}$ is *canonical* if $Iso_{\mathcal{G}}$ is the identity, i.e. two isomorphic objects of $\mathcal{G}$ are equal. A (concrete) groupoid $\mathcal{G}$ is *a group (action)* if $\mathrm{Ob}\mathcal{G}$ has a single element.

Two concrete groupoids $\mathcal{G}_1, \mathcal{G}_2$ of T are *equivalent* if there exist 0-definable embeddings $h_i : \mathcal{G}_i \to \mathcal{G}$ for some concrete groupoid $\mathcal{G}$, such that the image of h_i meets every isomorphism class of $\mathcal{G}$ (thus h_i is an equivalence of categories.) In this case, $\mathrm{Ob}\mathcal{G}$ may be taken to be $\mathrm{Ob}\mathcal{G}_1 \dot{\bigcup} \mathrm{Ob}\mathcal{G}_2$, and the embeddings may be taken to be the identity maps. If $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_2, \mathcal{G}_3$ are equivalent, via concrete groupoid structures on $\mathcal{G}_1 \dot{\bigcup} \mathcal{G}_2$ and on $\mathcal{G}_2 \dot{\bigcup} \mathcal{G}_3$, one may take the concrete groupoid generated by the union of these (with objects $\mathrm{Ob}\mathcal{G}_1 \dot{\bigcup} \mathrm{Ob}\mathcal{G}_2 \dot{\bigcup} \mathrm{Ob}\mathcal{G}_3$) to see that $\mathcal{G}_1, \mathcal{G}_3$ are equivalent.

The cover associated to a definable groupoid. We describe a canonical cover of a theory T, associated with a definable groupoid $\mathcal{G}$. The new theory adds a distinguished object to each isomorphism class of $\mathcal{G}$. The cover will be internal if the groupoid has a single isomorphism class. A general groupoid $\mathcal{G}$ can be viewed as a disjoint union over $\nu \in \mathrm{Ob}\mathcal{G}/Iso_{\mathcal{G}}$ of the full sub-groupoid $\mathcal{G}(\nu)$ whose objects are those of the isomorphism class ν. The cover $T'_{\mathcal{G}}$ will then be the the free union of the covers $T'_{\mathcal{G}(\nu)}$. The construction extends to the case of concrete groupoids.

Let T be a theory, $\mathcal{G}$ a definable groupoid, and $\delta : \mathcal{G} \to Def(T)$ a definable functor. We construct a theory $T' = T'_{\mathcal{G},\delta}$ extending T. The sorts of T' are those of T, along with three new sorts O, M, D. The language of T' is the language of T expanded by relations $i'_0, i'_1, \circ', Id'$ for the language of categories on O, M (with O the objects, M the morphisms), and maps $r : D \to O$, $d : D \times_{r,i_0} M \to D$. We will make (O, M) into a concrete groupoid $\mathcal{G}'$ with functor δ' by letting $\delta'(x) = r^{-1}(x)$, and $\delta'(f)$ be the restriction of d to $r^{-1}(x) \times \{f\}$. Finally the language has function symbols j for a functor $(\mathcal{G}, \delta) \to (O, M, \delta')$ of concrete categories.

The axioms of T' are those of T, together with the statement that (O, M, δ') is a concrete groupoid, j is an embedding of concrete categories $\mathcal{G} \to (O, M, \delta')$; and O has a unique element outside the image of j, in each isomorphism class.

As usual, we will write $Hom_{\mathcal{G}'}(a, b)$ for $\{m \in : i_0(m) = a, i_1(m) = b\}$. In particular $Hom_{\mathcal{G}'}(a, a)$ forms a group, denoted $\mathrm{Aut}_{\mathcal{G}'}(a)$.

Lemma 3.1 *T' is complete (relative to T). T' induced no new structure on the sorts of T. Each model M of T extends to a unique model M' of T', up to isomorphism over M. For any $a \in O(M')$, $\delta'(a)$ is internal to the sorts of T, and* $\mathrm{Aut}(\delta'(a)/T) = \mathrm{Aut}_{\mathcal{G}'}(a)$.

Proof. Given $M \models T$, choose a representative r_ν of each isomorphism class ν of $\mathrm{Ob}_{\mathcal{G}}(M)$. Let O_0 be a copy of $\mathrm{Ob}_{\mathcal{G}}$, with $j : \mathrm{Ob}_{\mathcal{G}} \to O$ a bijection; $*_\nu$ be a new element, $O = O_0 \cup \{*_\nu : \nu\}$, and define a groupoid structure in such a way that j is an isomorphism of categories from $\mathcal{G}$ the the sub-groupoid with objects O_0, and each $*_\nu$ is isomorphic to each element of ν. It is easy to see that this can be done, and uniquely so up to M-isomorphism. In effect to construct O one adds to each isomorphism class a new copy $*_\nu$ of r_ν, and let $\mathrm{Mor}(*_\nu, y)$ be a copy of $\mathrm{Mor}(r_\nu, y)$ for any $y \in \mathrm{Ob}\mathcal{G}$, and $\mathrm{Mor}(*_\nu, *_\eta) = \mathrm{Mor}(r_\nu, r_\eta)$. Similarly $\delta'(*_\nu)$ is a copy of $\delta(r_\nu)$. For uniqueness, given two versions O, O', for any ν pick an isomorphism $f_\nu \in \mathrm{Mor}(r_\nu, *_\nu)$, $f'_\nu \in \mathrm{Mor}(r_\nu, *'_\nu)$, and conjugate using f from $*_\nu$ to r_ν, then using f' from r_ν to $*'_\nu$, to obtain isomorphisms $Iso(*_\nu) \to Iso(*'_\nu)$; compose $\delta'(f')$ with $\delta'(f)^{-1}$ to obtain maps $\delta'(*_\nu) \to \delta'(*'_\nu)$; etc.

Completeness of T' follows from the uniqueness of M'.

Any element of $\mathrm{Aut}_{\mathcal{G}'}(a)$ acts on $\delta'(a)$, and also acts on any nonempty $\mathrm{Mor}_{\mathcal{G}'}(a,b)$ by conjugation; these combine to give a concrete groupoid automorphism fixing the image of j, hence an automorphism fixing the T-sorts. Given any automorphism $\sigma \in Aut(T'/T)$, let $a \in M' \setminus M$ and pick $b \in \mathrm{Ob}\mathcal{G}(M)$ with a, b isomorphic in $\mathcal{G}$. we have $\sigma(a) = a$ since $\sigma(b) = b$ and a is the unique element outside the image of j and isomorphic to $j(b)$. Pick an isomorphism $r \in \mathrm{Mor}(a, b)$, Then $\sigma(r)^{-1} r$ is an $\mathcal{G}'$-isomorphism of a, σ coincides on $\delta'(a)$ and on any $\mathrm{Mor}(a, c)$ with the action of and conjugation by this element.

□

Remark The cover constructed above is 1-analyzable, i.e. relatively internal over a set interpretable in T, namely the set $Iso\mathcal{G}$ of isomorphism classes of $\mathcal{G}$; moreover and has no relations among the fibers over $Iso\mathcal{G}$. In general, a 1-analyzable cover $f : C \to D$ may have relations among fibers of f, not sensed by the associated groupoid. However any relation concerns finitely many fibers, so between them the groupoids associated to the induced covers $f^n : C^n \to D^n$ for each n do capture the information, and the cover may by coded by a definable simplicial groupoid.

Remark Assume $\mathcal{G}$ has a single isomorphism class. if one fixes a parameter $b \in \mathrm{Ob}\mathcal{G}$, one may interpret the new element of O by doubling b. (Add one new object b', and let $\mathrm{Mor}(b', c)$ be a copy of $\mathrm{Mor}(b, c)$, etc., with the obvious rules.) In this case, the corresponding groupoid imaginary is interpretable with parameters. However, unlike the groupoid imaginary sort, this interpretation is incompatible with the automorphism group of the original structure.

Internal covers and concrete groupoids. Two generalized imaginary sorts S', S'' of T (with theories T', T'') are *equivalent* if they are bijectively bi-interpretable over T, i.e. whenever $N' \models T', N'' \models T''$ are two models with the same restriction M to the T-sorts, there exists a bijection $f : S_N \to S_{N'}$ such that $f \cup Id_M$ preserves the class of 0-definable relations.

Theorem 3.2 *There is a bijective correspondence between internal imaginary sorts of T and definable concrete groupoids with a single isomorphism class (both up to equivalence.)*

Proof. Given the concrete definable groupoid $\mathcal{G}$ with functor F, let $T'_{\mathcal{G}}$ be the theory described in Lemma 3.1. Since $\mathcal{G}$ has a single isomorphism class, there is a single element $*$ of O outside the image of $\mathrm{Ob}_{\mathcal{G}}$. The sort $S_{\mathcal{G}}$ is taken to be $\delta'(*)$, with the structure induced from T'. (Note that the rest of T' is definable over the sorts of T and $S_{\mathcal{G}}$, using stable embeddedness.)

Conversely, given an internal cover N, we obtain a *-definable concrete groupoid by Proposition 2.5. (The liaison groupoid of N.) Since the number of sorts and generating relations is finite, it is clear that

$\mathrm{Aut}(S_N/M)$ is definable rather than $*$-definable. By Lemma 2.4 we can take the groupoid $\mathcal{G}$ definable. It is clearly a concrete groupoid of M, well-defined up to equivalence.

By Lemma 3.1, the liaison groupoid of $S_{\mathcal{G}}$ is (equivalent to) $\mathcal{G}$. Conversely, if we begin with N and let $\mathcal{G}$ be the liaison groupoid of N, then $S_{\mathcal{G}}$ can be identified with N, though a priori N may have more relations; but by construction $S_{\mathcal{G}}, N$ have the same automorphism group over N, so by Lemma 1.3 their definable sets coincide. □

In particular, a finite internal cover of N may be realized as a generalized imaginary sort, where the groupoid has a single isomorphism class, and finite isomorphism group at each point.

Example 3.3 Let M be a finite structure, i.e. finitely generated, with finitely many elements of each sort. Then any finite extension $1 \to K \to \widetilde{G} \to G \to 1$ of $G = \mathrm{Aut}(M)$ is the automorphism group of some finite internal extension $\widetilde{M}$ of M. The same holds in the $\aleph_0$-categorical setting, if the topology on G is taken into account; see [1] for proofs and [2] for good examples.

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Lemma 3.4 *Let N be a finite internal cover of M, whose corresponding concrete groupoid is equivalent to a (0-definable) group action. Then the sequence*

$$1 \to \mathrm{Aut}(N/M) \to \mathrm{Aut}(N) \to \mathrm{Aut}(M) \to 1$$

is split.

Proof. In this case, the construction beginning with $\mathcal{G}$ yields a structure interpretable in M: if $\mathrm{Ob}\mathcal{G} = \{1\}$, the new structure has new sorts $\delta(*)$ and $\delta(\mathrm{Mor}(*,1))$; by choosing a point of $\delta(\mathrm{Mor}(*,1))$ one obtains M together with a copy $\delta(*)$ of $\delta(1)$ and a copy $\delta(\mathrm{Mor}(*,1))$ of $\delta(\mathrm{Mor}(1,1))$. This interpretation yields a group homomorphism $\mathrm{Aut}(M) \to \mathrm{Aut}(N)$ splitting the sequence.

This provides examples of structures that do not eliminate groupoid imaginaries. □

Lemma 3.5 *Conversely, let N be a finite internal extension of M, with associated concrete groupoid $\mathcal{G}$; and suppose the exact sequence of automorphism groups is split functorially. Then $\mathcal{G}$ is equivalent to a group action.*

Proof. By assumption, there exists a subgroup $H \leq \mathrm{Aut}(N)$, varying functorially when (M,N) is replaced by an elementary extension, such that $H \to \mathrm{Aut}(M)$ is an isomorphism. Let $\mathbf{N}$ be the expansion of N by all H-invariant relations. Then $\mathbf{N}$ is bi-interpretable with M. It follows that the concrete groupoid corresponding to $\mathbf{N}$ is equivalent to a group action. □

Definition 3.6 *A finite internal cover T' of T is* almost split *if whenever $N' \models T'$, with N the restriction to the sorts of T, for some finite 0-definable set C of imaginaries of N', $N' \subseteq dcl(N \cup C)$. If $\mathrm{Aut}(N'/C) \to \mathrm{Aut}(N)$ is surjective, we say that the cover is* split.

Thus "T' almost split over T" is the same as: "$T'_{\mathrm{acl}(\emptyset)}$ is split, over $\mathrm{acl}(\emptyset)$."

Definition 3.7 *T eliminates (finite, strict) generalized imaginaries if every concrete groupoid (with finite automorphism groups, with one isomorphism class) $\mathcal{G}$ is equivalent to a canonical one.*

Note that ordinary elimination of imaginaries holds iff every groupoid with trivial groups is equivalent to a canonical one.

Lemma 3.8 *T eliminates finite generalized imaginaries iff T eliminates finite imaginaries, and every finite internal cover of T is split.*

Proof. We use Theorem 3.2. T eliminates finite generalized imaginaries iff every concrete groupoid $\mathcal{G}$ with finite automorphism groups and one isomorphism class is equivalent to a group action. If $\mathcal{G}$ is a group action, the finite internal cover corresponding to $\mathcal{G}$ is clearly split. Conversely if the cover T' of T is split, it has an expansion T'' bi-interpretable with T. T'' is still a finite internal cover, and by Theorem 3.2 corresponds to a sub-groupoid $\mathcal{G}''$ of $\mathcal{G}$, with one isomorphism class and trivial automorphism groups. Let $*$ be a formal element corresponding to the isomorphism class of $\mathcal{G}$. We may assume $\mathrm{Ob}\mathcal{G} = \mathrm{Ob}\mathcal{G}'$. For $a \in \mathrm{Ob}\mathcal{G}$, let $\mathrm{Mor}(a,*) = \mathrm{Mor}_{\mathcal{G}}(a,a)$. Given $a,b \in \mathrm{Ob}_{\mathcal{G}}$, there is a unique $f_{a,b} \in \mathrm{Mor}_{\mathcal{G}'}(a,b)$. Use $\delta(f_{a,b})$ to identify $\delta(a), \delta(b)$, and let $\delta(*)$ be the quotient. Also use $f_{a,b}$ to identify $\mathrm{Mor}(a,a)$ and $\mathrm{Mor}(b,b)$, by composition, and let $\mathrm{Mor}(*,*)$ be the quotient. We have found a common extension of the group action of $\mathrm{Mor}(*,*)$ on $\delta(*)$, and of the concrete groupoid $(\mathcal{G},\delta)$. □

Remark 3.9 *If algebraic points form an elementary submodel of M, then every finite internal cover is almost split.* Indeed by definition, a finite internal cover N satisfies $N \subset \mathrm{dcl}(M,C)$ with C a finite, M-definable set. As $\mathrm{acl}(\emptyset) \prec M$, we can choose $C \subset \mathrm{acl}(\emptyset)$.

A definable group homomorphism $f : \widetilde{H} \to H$ is a *definable central extension* if f is surjective and $\ker f$ is contained in the center of H. We now relate finite internal covers of internal covers of a theory τ to definable central extensions of the liaison group of the latter. Assumption (3) below says that finite generalized imaginaries of M arising from definable finite central extensions of groups are eliminable; the conclusion is that all finite generalized imaginaries are.

Proposition 3.10 *Let T be a theory with a distinguished stably embedded sort $\mathbf{k}$, $\tau = Th(\mathbf{k})$. Let $M \models T$. Assume T eliminates imaginaries, and:*

1. *Every finite internal cover of $\mathbf{k}$ is almost split.*
2. *Let D be a T-definable set. Then D is $\mathbf{k}$-internal, and $\mathrm{Aut}(D/\mathbf{k})$ is T_M-definably isomorphic to a τ-definable group H.*
3. *Let F_0 be a finite definable set of imaginaries of T, D a T_{F_0}-definable set, $h : \mathrm{Aut}(D/\mathbf{k}, F_0) \to H$ an M_T-definable group isomorphism (as in (2)). Let $f : \widetilde{H} \to H$ be a τ-definable central extension. Then there exists a finite T-definable F containing F_0, and a T_F- definable $\widetilde{D}$ containing D, and injective T_M-definable group homomorphisms $\tilde{h} : \mathrm{Aut}(\widetilde{D}/k, F) \to \widetilde{H}$, $h : \mathrm{Aut}(D/k, F) \to H$, with images of finite index, and $f\tilde{h} = h$.*

4. *For any τ-definable group H, and any finite Abelian group K, the group of M-definable homomorphisms $H \to K$ is finite.*

Then any finite internal cover of M is almost split.

Proof. Let $M' = M \cup \{C\}$ be a finite internal cover of M, with C definable; $T' = Th(M')$. We have $G := \mathrm{Aut}(M'/M) = \mathrm{Aut}(C/M) = \mathrm{Aut}(C/D)$ for some definable set D of T; G is a finite group. We may enlarge C so that $D \subseteq C$.

Two preliminary remarks:

If F is a finite definable set of imaginaries of T', there exist finite definable sets F_T, F_τ of imaginaries of $M, \mathbf{k}$ respectively, such that for for any M, M' as above, $textrmdcl(F) \cap M^{eq} = \mathrm{dcl}(F_T) \cap M^{eq}$, $\mathrm{dcl}(F) \cap \mathbf{k}^{eq} = \mathrm{dcl}(F_\tau) \cap \mathbf{k}^{eq}$. Thus $\mathrm{Aut}(M'/F) \to \mathrm{Aut}(M'/F_T) \to \mathrm{Aut}(\mathbf{k}_M/F_\tau)$ are defined and surjective. Thus to show that T' is split, it suffices to prove the same for T'_F. In particular, taking F to be the finite group G, we may assume each element of G is 0-definable. In this case, G is central in $\mathrm{Aut}(M'/\mathbf{k})$. We have a central extension

$$1 \to G \to \mathrm{Aut}(C/\mathbf{k}) \to \mathrm{Aut}(D/\mathbf{k}) \to 1$$

By internality, the sequence is isomorphic to a central extension

$$1 \to K \to \widetilde{H} \to H \to 1$$

of τ-definable groups, via an M'-definable map $\tilde{f} : \mathrm{Aut}(C/\mathbf{k}) \to \widetilde{H}$, and an M-definable map $f : \mathrm{Aut}(D/\mathbf{k}) \to H$.

The condition in (3) is stated for central extensions with prime cyclic kernel; by iteration it is closed under all finite central extensions.

Hence, after naming parameters for a further finite definable set, and passing to corresponding subgroups of finite index in $\widetilde{H}, H$, there exists a T-definable $\widetilde{D}$ (containing D) such that $\mathrm{Aut}(\widetilde{D}/\mathbf{k}) \to \mathrm{Aut}(D/\mathbf{k})$ is isomorphic to $\widetilde{H} \to H$, by T-definable maps $\tilde{h}, h$; and $h = f$.

Now $\widetilde{H} \times_H \widetilde{H}$ has a subgroup of finite index isomorphic to $\widetilde{H}$, namely the diagonal subgroup $\Delta_{\widetilde{H}}$. $\Delta_{\widetilde{H}}$ is invariant under any M-definable automorphism of $\widetilde{H} \times_H \widetilde{H}$ of the form $(\alpha \times_\beta \alpha)$, with $\alpha : \widetilde{H} \to \widetilde{H}$ an automorphism lying over $\beta : H \to H$. But any automorphism of $\widetilde{H}$ over H has the form $x \mapsto z(x)x$ for some homomorphism $z : \widetilde{H} \to K$. By (4), there are only finitely many such definable homomorphisms, and so the group of M-definable automorphisms of $\widetilde{H}$ over H is finite, and hence the automorphisms $(\alpha \times_\beta \alpha)$ have finite index within the group of all M-definable automorphisms of $\widetilde{H} \to H$. So $\Delta_{\widetilde{H}}$ has finitely many conjugates by such automorphisms. Taking their intersection, we find a subgroup S of $\widetilde{H} \times_H \widetilde{H}$ of finite index, mapping injectively to each factor $\widetilde{H}$, and invariant under all M-definable automorphisms of $(\widetilde{H}, H)$.

It follows that the pullback S' of S under $(\tilde{h}, \tilde{f}, f)$ does not depend on the choice of the triple $(\tilde{h}, \tilde{f}, f)$. It is thus a definable subgroup of $\mathrm{Aut}(\widetilde{D}/\mathbf{k}) \times_{\mathrm{Aut}(D/\mathbf{k})} \mathrm{Aut}(C/\mathbf{k})$. Hence $\mathrm{Aut}(\widetilde{D}, C/\mathbf{k})$ also has a definable subgroup of finite index S'' mapping injectively to $\mathrm{Aut}(\widetilde{D}/\mathbf{k})$.

By [15], there exists a T'-definable set Q with $\mathrm{Aut}(Q/\mathbf{k}) = \mathrm{Aut}(\widetilde{D}/\mathbf{k})$, and such that $\mathrm{Aut}(\widetilde{D}/\mathbf{k})$ acts transitively on Q, with trivial point stabilizer. The quotient Q/S'' is a finite internal cover of $\mathbf{k}$. By (1), for some 0-definable finite set F' we have $Q/S'' \subseteq \text{dcl}(F')$. So $\text{Aut}(M'/F')$ is contained in S''. Hence $\text{Aut}(M'/F', M) \subseteq \text{Aut}(M'/F', \widetilde{D}) = 1$. So M' is an almost split extension of M. □

Groupoids in ACF. Consider ACF_L, the theory of algebraically closed fields containing a field L. Let L^a be the algebraic closure of L. Every concrete groupoid is equivalent to a a subgroupoid with finitely many objects in each equivalence class. The question essentially reduces to concrete groupoids with finitely many objects.

If L is real-closed, the Galois group $\text{Aut}(L^a/L)$ is $\mathbb{Z}/2\mathbb{Z}$, and admits nontrivial central extensions $1 \to Z \to E^a \to \text{Aut}(L^a/L) \to 1$. E^a lifts to an extension E of $\text{Aut}(K/L)$ (where $K \models ACF_L$.) The sequence $1 \to Z \to E \to \text{Aut}(K) \to 1$ is not split, any more than the E^a-sequence. As in Example 3.3 there exists a finite internal cover M of ACF_L with $\text{Aut}(M) = E$. The concrete groupoid corresponding to M cannot be equivalent to a canonical one.

On the other hand, if L is PAC, then (cf. [12]) $\text{Aut}(L^a/L)$ is a projective profinite group. In this case every finite concrete groupoid should be equivalent to a canonical one.

Problem 3.11 *Give a geometric description of the groupoid-imaginaries when when L is a finitely generated extension of an algebraically closed field.*

4. Higher amalgamation

Let T be a theory (or Robinson theory), for simplicity with quantifier elimination. A T-structure is an algebraically closed substructure of a model of T. Let $\mathcal{C}_T$ be the category of algebraically closed T-structures. A partially ordered set P can also be viewed as a category, and we will consider functors $P \to \mathcal{C}_T$. Specifically let $P(N)$ be the partially ordered set of all subsets of $\{1, \ldots, N\}$, and let $P(N)^-$ be the sub-poset of proper subsets.

By an *N-amalgamation problem* we will mean a functor $A : P(N)^- \to \mathcal{C}_T$. A *solution* is a functor $\bar{A} : P(N) \to \mathcal{C}_T$, where $P(N)$ is the partially ordered set of all subsets of $P(N)$, extending A. We will demand for both $a = A, \bar{A}$ that $a(s) = \text{acl}\{a(i) : i \in s\}$ (this is by no means essential, but simplifies the definitions of independence-preservation and of uniqueness of solutions below.)

We assume T is given with a notion of *canonical* 2*-amalgamation*. That is, we are given a functorial solution of all 2-amalgamation problems. Equivalently, we have a notion of *independence* of two substructures of a model of T, over a third; or again, a functorial extension process $p \mapsto p|B$ of types over A to types over B, where $A \leq B \in \mathcal{C}_T$. We assume that this notion of independence is symmetric and transitive, cf. [4]. When for any B, $p|B$ is an A- definable type, we will say that amalgamation is definable at p. (This is always the case for stable theories, cf. [22].)

(The "uniqueness of non-forking extensions" comes with the presentation here; some of the considerations below generalize easily to the case of a canonical *set* of solutions rather than one.)

A functor $A : P \to \mathcal{C}_T$ is *(2)-independence-preserving* if it is compatible with the given canonical 2-amalgamation; i.e. whenever $s = s_1 \cap s_2 \subset s' \in P$, $A(s_1), A(s_2)$ are independent over $A(s)$ within $A(s')$.

At this point, we consider the problem of *independent amalgamation.* An independent amalgamation problem (or solution) is a functor $A : P \to \mathcal{C}_T$ (where $P = P(N)^-$, respectively $P = P(N)$) compatible with the given canonical 2-amalgamation; i.e. such that whenever $s = s_1 \cap s_2 \subset s' \in P$, $A(s_1), A(s_2)$ are independent over $A(s)$ within $A(s')$. We will also demand: $A(\emptyset) = \mathrm{acl}(\emptyset)$.

Let us say that T has n-uniqueness (existence, exactness) if every independent n-amalgamation problem has at most one (at least one, exactly one) solution, up to isomorphism.

Similar diagrams appear in work of Shelah in various contexts, cf. e.g. [23]. Elimination of imaginaries was introduced in [22] precisely in order to obtain 2-exactness for stable theories. 3-existence follows, but 4-existence, and 3-uniqueness, can fail: cf. [17]. We will see below, however, that with generalized imaginaries taken into account, stable theories are 3-exact.

Occasionally we will also require $(n-1, n+k)$-existence for $k \geq 1$. This means that a solution exists to every *partial* independent amalgamation problem $(a(u) : u \subset \{1, \ldots, n+k\}, |u| < n\}$. We have however:

Lemma 4.1 *Assume T_A has n-existence for all A. Then*

(1) T has $(n-1, n+k)$-existence for any $k \geq 1$.

(2) If n-uniqueness holds, so does $n+1$-existence.

Proof. (1) An easy induction. For instance take $k = 1$, and assume given $(a(u) : u \subset \{1, \ldots, n+1\}, |u| < n)$. Let $U = \{u \subset \{1, \ldots, n+1\} : |u| \leq n, (n+1) \in u\}$. For $u \in U$, by $(n-1, n)$-existence, one can find $a^*(u)$ extending $a(v)$ for $v \subset u, |v| < n$. But U is isomorphic to the set of subsets of $\{1, \ldots, n\}$ of size $< n$, so by another use of $(n-1, n)$-existence $(a^*(u) : u \in U)$ admits a solution $(b(u) : u \subseteq \{1, \ldots, n+1\})$; this also solves the original problem a.

(2). We will use $(n-1, n+1)$-amalgamation. Given an $n+1$-independent amalgamation problem b, let a be the restriction to the faces u with $|u| < n$. This problem has a solution c; for each u with $|u| = n$, c restricts to a solution $c(u)$ of the problem $(a(v) : v \subset u)$; by n-uniqueness, these solutions must be isomorphic to the original solutions $b(u)$. By means of these isomorphisms $b(u) \to c(u)$ $(|u| = n)$, c provide a solution to the problem b. □

Lemma 4.2 *Let T be a theory with a canonical 2-amalgamation, admitting elimination of imaginaries. For any independence-preserving functor $a : P(3) \to \mathcal{C}_T$, the following two conditions are equivalent:*

(1) $a(12) \cap \mathrm{dcl}(a(13), a(23)) = dcl(a(1), a(2))$

(2) If $c \in a(12) = \mathrm{acl}(a(1), a(2))$, then $tp(c/a(1), a(2))$ implies $tp(c/a(13), a(23))$.

Moreover, 3-uniqueness is equivalent to the truth of (1,2) for all such a.

Proof. Assume (1) holds. In the situation of (2), the solution set X of $\mathrm{tp}(c/a(13), a(23))$ is a finite set, hence coded in $a(12)$, and defined over $a(13)+a(23)$; thus by (1), X is defined over $a(1), a(2)$; being consistent with $\mathrm{tp}(c/a(1), a(2))$, it must coincide with it.

Conversely, if $c \in a(12) \cap \mathrm{dcl}(a(13), a(23))$, then $\mathrm{tp}(c/a(13), a(23))$ has the unique solution c, so if (2) holds then the same is true of $\mathrm{tp}(c/a(1), a(2))$, and hence $c \in \mathrm{dcl}(a(1), a(2))$.

Suppose (1) fails. Then the restriction map

$$\mathrm{Aut}(a(12)/a(13), a(23)) \to \mathrm{Aut}(a(12)/a(1), a(2))$$

is not surjective. Let $\sigma \in \mathrm{Aut}(a(12)/a(1), a(2))$ be an automorphism that does not extend to $a(13)a(23)$. Let a' be the same as a on subsets of $\{1, 2, 3\}$, and also the same on morphisms except for the inclusion

$i : \{1,2\} \to \{1,2,3\}$; and let $a'(i) = a(i) \circ \sigma$. Then a' is a solution to the independent amalgamation problem $a|P(3)^-$, and is not isomorphic to a. So 3-uniqueness fails.

Conversely, suppose we are given an independent amalgamation problem $a : P(3)^- \to \mathcal{C}_T$, and two solutions a', a'' on $P(3)$. We may take a' to take the morphisms to inclusion maps. Then all $a(ij) = a'(ij)$, and are embedded in $A = a'(123)$. We can identify $a''(123)$ with $A = \mathrm{acl}(a_1, a_2, a_3)$ also. Then the additional data in a'' consists of isomorphisms $a(i,j) \to a(i,j)$, compatible with the inclusions of the $a(i)$. By 2-uniqueness, we may further assume that these isomorphisms are the identity on $a(2,3)$ and $a(1,2)$; so that a'' reduces to an automorphism f of $a(1,3)$, over $a(1), a(3)$. By (1), f extends to an elementary map fixing $a(12), a(13)$. This further extends to an automorphism F of $a(1,2,3)$. F shows that the two solutions a', a'' of the problem a are isomorphic. □

Proposition 4.3 *Let T be a stable theory admitting elimination of quantifiers and of imaginaries. Assume every finite internal cover of T_A almost splits over A. Then T has 3-uniqueness.*

In place of stability, we can assume T is given with a notion of 2-amalgamation, and show Lemma 4.2 (1) holds whenever the amalgamation is definable at one of the vertices of the triangle in question.

We will see later that 4-existence is equivalent to 3-uniqueness.

Compare [5], where a finite internal cover was constructed in the same way; the purpose there was to interpret a group from the group configuration, in a stable theory. This is also done for simple theories in [17], where 4-existence is assumed. In hindsight, it all coheres.

Proof. Since by definition 3-uniqueness for $T_{\mathrm{acl}(\emptyset)}$ implies 3-uniqueness for T, we may assume $\mathrm{acl}(\emptyset) = \mathrm{dcl}(\emptyset)$.

Let $a : P(3) \to \mathcal{C}_T$ be an independence-preserving functor, with notation as in Lemma 4.2. Replacing T by $T_{a(\emptyset)}$ we may assume $a(\emptyset) = \mathrm{acl}(\emptyset) = \mathrm{dcl}(\emptyset)$. Fix an enumeration of $a(i)$. We will describe a finite internal cover T^+ of T, associated with a.

Let FU be the set of formulas $S(x_1, x_2; u)$ such that whenever $M \models S(a_1, a_2; c)$,

1. $c \in \mathrm{acl}(a_1, a_2)$
2. If $\mathrm{tp}(a_3) = \mathrm{tp}(a(3))$ and a_3 is independent from $\mathrm{acl}(a_1, a_2)$, then $c \in \mathrm{dcl}(\mathrm{acl}(a_1, a_3), \mathrm{acl}(a_2, a_3))$.

If a_i enumerates $a(i)$, then by definability of the canonical extension of $\mathrm{tp}(a(3))$, for any $c \in a(12) \cap \mathrm{dcl}(a(13), a(23))$ there exists $S \in FU$ with $S(a_1, a_2, c)$.

Let $S, S', S'', \ldots \in FU$. By definability of the canonical extension of $\mathrm{tp}(a(1))$, for any formula $\phi(x, y, z, y', z', y'', z'', \ldots, w)$ there exists a formula $\phi^*(y, z, y', z', \ldots, w)$ (depending on ϕ and on the sequence $S, S', \ldots$) such that for any $b, c, b', c', \ldots, d$, any any $a \models \mathrm{tp}(a(1))$ with a independent from $\{b, c, b', c', \ldots, d\}$ and such that $S(a, b, c), S'(a, b', c'), \ldots$,

$$\phi(a, b, c, b', c', \ldots, d) \iff \phi^*(b, c, b', c', \ldots, d)$$

We construct a many-sorted cover T' of T as follows.

Let L^+ be a language containing L, as well as a new sort $NS(u)$ for any $S \in FU$; and a definable map f_S; and for each $S, S', S'' \ldots \in FU$ and each $\phi(x, y, z, y', z', \ldots, w)$, a relation $N\phi(z, z', \ldots, w)$. Given $M \models T$, we construct an L^+-structure M^+ as follows. Within some elementary extension M^* of M, let $a \models \mathrm{tp}(a(1))|M$. Let

$$NS(M^+) = \{(b, c) : b \in M, M^* \models S(a, b, c)\}$$

and let $f_S(y, z) = y$. For $d \in M, c \in NS(M^+), c' \in NS'(M^+), \ldots$, interpret $N\phi$ so that

$$N\phi((b, c), (b', c'), \ldots, d) \iff \phi(a, b, c, b', c', \ldots, c)$$

Using the definability of $\mathrm{tp}(a(1))|M$, one sees that $T^+ = Th(M^+)$ does not depend on any of the choices made.

We now use Remark 1.2. Each sort of T^+ will be seen to be a finite internal cover of T, as soon as we show:

Claim. T^+ is a bounded internal cover of T.

Proof. Given $M \models T$, we constructed an expansion $M^+ \models T^+$ of the same cardinality, such that $\mathrm{Aut}(M^+) \to \mathrm{Aut}(M)$ is surjective. It remains to show that the kernel is bounded. M^+ can be constructed as follows. We have $M \models T$. Let M_3 be an elementary extension of M, with $a_3 \in M_3$, $a_3 \models \mathrm{tp}(a(3))|M$. Let M^* be an elementary extension of M_3, with $a_1 \in M^*$, $a_1 \models \mathrm{tp}(a(1))|M_3$. We can construct M^+ using M, a_1, so that $M^+ \subseteq \mathrm{acl}(M, a_1)$; actually $M^+ \subseteq \cup_{a_2 \in M} \mathrm{acl}(a_1, a_2) \cap \mathrm{dcl}(\mathrm{acl}(a_2, a_3), \mathrm{acl}(a_1, a_3))$. Any automorphism of M^+ over M lifts to an automorphism of M_3^+ over M_3, which in turn is an elementary automorphism of M_3^+ (viewed as a subset of M^*) over $M_3(a_1)$. Thus the homomorphism $\mathrm{Aut}(\mathrm{dcl}(acl(a_1, a_3), M_3)/M_3, a_1, a_3) \to \mathrm{Aut}(M^+)$ is surjective. But the first group is clearly bounded. □

Now by assumption, every finite internal cover of T is almost split. Let M be a model of T containing $a(2)$, and let $a(1)$ be independent from M, in some elementary extension M^* of M. Then M^+ can be embedded into $\mathrm{dcl}(a(1), M)$. Let $c \in a(12) \cap \mathrm{dcl}(a(13), a(23))$, so that $c \in M^+$. Then $c \in \mathrm{dcl}(a(1), M)$. But $\mathrm{tp}(a(1), c/M)$ is $a(2)$-definable, since $\mathrm{tp}(a(1)/M)$ is definable, and $c \in \mathrm{acl}(a(1), a(2))$, and $a(2)$ is algebraically closed. Thus $c \in \mathrm{dcl}(a(1), a(2))$. This proves the property of Lemma 4.2 (1). □

4.4. Adding an automorphism

We include a general lemma on adding an automorphism to a stable theory, that will aid in describing the linear imaginaries of pseudo-finite fields. This was the route taken in [14] to the imaginaries of the pseudo-finite fields themselves; it appears best to repeat it from scratch in the linear context. In [14], as here, only the fixed field was actually needed. The imaginaries for the full theory were considered (and eliminated) in [7] for strongly minimal T (in [6] for $T = ACFA$). An unpublished example of Chatzidakis and Pillay shows that it is not true in general. We show however that the principle is correct if generalized imaginaries are taken into account.

Let T be a theory with elimination of quantifiers and elimination of imaginaries. (In our application, T will be a linear extension of the theory of algebraically closed fields.)

Let $\widetilde{\mathcal{C}} = \{(A,\sigma) : A \in \mathcal{C}_T, \sigma \in \mathrm{Aut}(A)\}$. We define independence for $\widetilde{\mathcal{C}}$ by ignoring the automorphism σ. In the present framework, 2-uniqueness will not hold; this is because of the choice involved in extending an automorphism from $\mathrm{dcl}(A_1 \cup A_2)$ to $\mathrm{acl}(A_1 \cup A_2)$.

Consider pairs (A,σ), with A an algebraically closed substructure of a model of T, and σ an automorphism of A. This is the class of models of a theory $T_\sigma^\forall$, in a language where quantifiers over T-definable finite sets are still viewed as quantifier-free. Under certain conditions, including the application in §4 to linear theories over ACF, $T_\sigma^\forall$ has a model completion, a theory $\widetilde{T_\sigma}$ whose models are the existentially closed models of $T_\sigma^\forall$. $\widetilde{T_\sigma}$ is unique if it exists. At all events, $\widetilde{\mathcal{C}}$ amalgamates to a universal domain, and can be viewed as a Robinson theory.

Proposition 4.5 *Let T be a theory with a canonical 2-amalgamation, admitting elimination of imaginaries. Assume T_A has n-existence Then conditions (1)–(4) are equivalent.*

1. *n-uniqueness.*
2. *n-existence for $\widetilde{\mathcal{C}}$*
3. *Let $a : P(n)^- \to \mathcal{C}_T$ be independence-preserving. Let $u_0 = \{1,\dots,n-1\}$; let $a(<v) = \mathrm{dcl}(\{a(v') : v' \subset v, v' \neq v\})$, $a(\not\geq v) = \mathrm{dcl}(\{a(v') : v \not\subseteq v'\})$. then*
$$a(u_0) \cap a(\not\geq u_0) = a(< u_0)$$
4. *With a, u_0 as in (3),*
$$\mathrm{Aut}(a(u_0)/a(\not\geq u_0)) = \mathrm{Aut}(a(u_0)/a(< u_0))$$

Proof. (1) $\implies$ (3) is proved as in Lemma 4.2.

(3) $\iff$ (4): Using imaginary Galois theory, cf. [20]).

(4) $\implies$ (1): Let $a : P(n)^- \to \mathcal{C}_T$ be an n-amalgamation problem, and let a', a'' be two solutions. As in Lemma 4.2 we may assume that for each $u \subset n$, $a'(Id_u)$ is the inclusion of $a(u)$ in $\mathrm{acl}(a(1),\dots,a(n))$. Now for $i \in \{1,\dots,n\}$, $u^i = \{1,\dots,n\} \setminus \{i\}$, Id_{u^i} the inclusion of u^i in $\{1,\dots,n\}$, $a''(Id_{u^i})$ is an isomorphism $a''(u^i) \to a''(\{1,\dots,n\})$, i.e. an automorphism f^i of $a(u^i) = \mathrm{acl}(a(j) : j \in u^i)$; and since a'' extends a, $f^i \in \mathrm{Aut}(a(u^i)/a(< u^i))$. By (4), $f^i \in \mathrm{Aut}(a(u^i)/a(\not\geq u^i))$. So f^i extends to an automorphism F^i of $a(\{1,\dots,n\})$ fixing $a(\not\geq u^i))$. Let F be the product of the F^i (choose any ordering.) Then $F|a(u^i) = f^i$. So F shows that a'', a' are isomorphic.

(1) $\implies$ (2): consider an independent n-amalgamation problem for $\widetilde{\mathcal{C}}$; it consists of an independent n-amalgamation problem $a = (a(u) : u \subset \{1,\dots,n\})$ and a compatible system of automorphisms $\sigma_u \in \mathrm{Aut}(a(u))$. Using n-existence, extend a to a solution; it is a system $(b(u) : u \subseteq \{1,\dots,n\})$, and compatible embeddings $f_u : a(u) \to b(u)$. Now let $g_u = f_u \circ \sigma(u) : a(u) \to b(u)$. Then (b,g) is another solution. By n-uniqueness, the two solutions must be isomorphic; so there exists $\sigma : b(\{1,\dots,n\}) \to b\{1,\dots,n\})$ such that $g_u = \sigma f_u$. This shows exactly that (b,σ) is a solution to the original automorphic problem, via f.

(2) $\implies$ (3): Let $\sigma_0 \in \mathrm{Aut}(a(u_0)/a(< u_0))$, let $\sigma_u = Id_{a(u)}$ for every $u \not\geq u_0$. View this data as an independent amalgamation problem for $\widetilde{\mathcal{C}}$. By (3), it has a solution a'. We use 2-uniqueness to note that

$\mathrm{tp}(a'(u_0), a'(n)) = \mathrm{tp}(a(u_0), a(n))$. Thus σ_0 has an extension to $\mathrm{Aut}(a(\{1, \ldots, n\}))$ fixing $a(\not\geq u_0)$. So σ_0 fixes $a(u_0) \cap a(\not\geq u_0)$. By imaginary Galois theory again, $a(u_0) \cap a(\not\geq u_0) \subseteq a(< u_0)$.

□

Remark 4.6 *If n-uniqueness fails, it fails already for the Abelian algebraic closure. For the Abelian algebraic closure, a formulation in terms of homological algebra becomes possible.*

Fix types $p_1, p_2, \ldots$ or more simply one type p. Let $a_1, \ldots, a_k$ be k independent realizations of p. Let G_k be the Abelianization of $\mathrm{Aut}(\mathrm{acl}(a_1, \ldots, a_k)/\mathrm{dcl}(a_1, \ldots, a_k))$. There is a natural homomorphism $G_{k+1} \to G_k^{k+1}$, restricting to each k-face. In terms of this basic data, one can describe homologically the questions of n-existence and uniqueness. The point is that in the independence preserving functors a, the $a(u)$ can be taken to be standard objects $\mathrm{acl}(a_i : i \in u)$, so only the image of morphisms under the functor matters.

The proof of Proposition 4.7 below is follows the same outline as [14],[6], [7]. It may be possible to give a proof based on minimizing $\mathrm{tp}(a/A)$ in the fundamental order, subject to consistency with $\mathrm{tp}_{\widetilde{T_\sigma}}(a/Ae)$; this would be even closer to the original proof.

Proposition 4.7 *Let T be a stable theory admitting elimination of quantifiers, and let $\widetilde{T_\sigma}$ be the theory described above. Assume T eliminates imaginaries and, and for any $A = \mathrm{acl}(A)$, T_A eliminates finite generalized imaginaries. Then $\widetilde{T_\sigma}$ admits elimination of imaginaries.*

Remark 4.8 *The stability condition can be weakened. It suffices to assume $\mathbb{U}$ carries a notion of independence with 2-existence and uniqueness, and the following characterization of independence: if (a_i) is an indiscernible sequence over A, $A_w = \mathrm{acl}(A \cup \{a_i : i \in w\})$, and $A_w \cap A_{w'} = A$ for $w < w'$, then the (a_i) are independent over A.*

Proof. By Proposition 4.3, T has 3-uniqueness. It follows that $\widetilde{\mathcal{C}}$ has 3-existence (Proposition 4.5). Let $\widetilde{\mathbb{U}}$ be a saturated model of $\widetilde{T_\sigma}$.

Part of the assumption is that finite sets are codes in T; hence also in $\widetilde{T_\sigma}$. Thus it suffices to prove elimination of imaginaries to the level of finite sets; in other words we have to show: if e is an imaginary element of $\widetilde{\mathbb{U}}$, and A is the set of real elements of $\mathrm{acl}_{\widetilde{T_\sigma}}(e)$, then $e \in \mathrm{dcl}_{\widetilde{T_\sigma}}(A)$.

We have $e \in \mathrm{dcl}_{\widetilde{T_\sigma}}(Aa)$ for some real tuple a; $e = a/E$ for some $\widetilde{\mathbb{U}}$-definable equivalence relation E. Let B be the set of real elements of $\mathrm{acl}_{\widetilde{T_\sigma}}(Aa)$.

If $b \in B \setminus A$, then $b \notin \mathrm{acl}(Ae)$, so $\mathrm{Aut}(\mathbb{U}/Aeb)$ has infinite index in $\mathrm{Aut}(\mathbb{U}/Ae)$. It follows that $\{g : g(b) = b\}$ is a subgroup of $\mathrm{Aut}(\mathbb{U}/Ae)$ of infinite index. By Neumann's Lemma [19, Lemma 2.3] , and compactness, it follows that there exists $g \in \mathrm{Aut}(\mathbb{U}/Ae)$ such that $g(b) \neq b'$ for any $b, b' \in B \setminus A$.

Hence there exists a conjugate $a' = g(a)$ of a over Ae such that any real element of $\mathrm{acl}(Aa) \cap \mathrm{acl}(Aa')$ lies in A.

Let $a_1 = a, a_2 = a'$, and define a_n inductively so that $\mathrm{tp}(a_n, a_{n+1}/A) = \mathrm{tp}(a_1, a_2/A)$ and a_{n+1} is independent from $a_1, \ldots, a_{n-1}$ over $A(a_n)$. It is then easy to see that $\mathrm{acl}(A, a_w) \cap \mathrm{acl}(A, a_{w'}) = A$ for any two sets of indices w, w' with $w < w'$. (Observe first that $\mathrm{acl}(A, a_w) \cap \mathrm{acl}(A, a_{w'}) \subset \mathrm{acl}(A, a_m) \cap \mathrm{acl}(A, a_{m+1})$, where m is the maximal element of w.) This remains true if we extract (using Ramsey and compactness) an indiscernible sequence a_i. Thus by the property in Remark 4.8, $a_w, a_{w'}$ are A- independent as tuples of $\mathbb{U}$, the T-restriction of $\widetilde{\mathbb{U}}$. Hence by definition they are A-independent in $\widetilde{\mathbb{U}}$.

We have found an A-independent pair of E-equivalent realizations of $\mathrm{tp}(a/A)$, namely a_1, a_2. On the other hand, one easily obtains E-inequivalent independent elements realizing $\mathrm{tp}(a/A)$. Namely let c, c' be any two realizations of $\mathrm{tp}(a/A)$ with $(c, c') \notin E$. (E.g. $c = a, c' = g(a)$ where $g \in \mathrm{Aut}_A(\widetilde{\mathbb{U}})$ and $g(e \neq e)$.) Let $b \models \mathrm{tp}(a/A)$ be such that $(c, c'), b$ are A-independent. Then either (c, b) or $(c', b) \notin E$.

But a triangle with two equivalent and one inequivalent side cannot exist. This contradicts 3-existence for $\widetilde{\mathcal{C}}$. □

Proposition 4.9 *Let T be a stable theory admitting elimination of imaginaries. Then T has 4-existence iff (with $A = \mathrm{acl}(\emptyset)$) T_A eliminates finite generalized imaginaries.*

Proof. Since T is stable, T_A has 2-uniqueness and hence 3-existence over any algebraically closed set A'. By Proposition 4.3, T has 3-uniqueness; by Lemma 4.1, it has 4-existence.

Conversely, assume T has 4-existence. Let G be a definable concrete groupoid with finite automorphism groups, defined in T_A. Fix a type p of elements of G, and let tp be the set of types of triples (a, b, d) with $a, b \models p$, a, b independent over A, and $c \in Mor_G(a, b)$. Consider $(q_{12}, q_{23}, q_{13}) \in \mathrm{tp}$ such that there exist independent $a_1, a_2, a_3 \models p$ and c_{ij} with $(a_i, a_j, c_{ij}) \models q_{ij}$ for $i < j$, and such that $c_{12} = c_{23}^{-1} c_{13}$. We can take $q_{23} = q_{13}$. Such triples can be 4-amalgamated. It follows easily that for *any* independent a_1, a_2, a_3 and c_{ij} with $(a_i, a_j, c_{ij}) \models q_{ij}$ for $i < j$, one has $c_{12} = c_{23}^{-1} c_{13}$. (Otherwise, 3 triples with this property and 1 triple without it could not be 4-amalgamated.) Pick $(q_{12}, q_{23}, q_{13}) \in \mathrm{tp}$ with $q_{23} = q_{13}$. It follows that for any independent $a_1, a_2 \models p$ there exists a *unique* $c \in Mor_G(a_1, a_2)$ with $q_{12}(a_1, a_2, c)$. Moreover, we have a sub-groupoid G' of G with the same objects and such that $Mor_{G'}(a_1, a_2)$ is the unique realization of q_{12}. For any functor F on G into definable sets, we now obtain an equivalence relation on the disjoint union of the objects of G, identifying $e \in F(a), e' \in F(a')$ if $h(e) = e'$ for the unique $h \in \mathrm{Mor}_{G'}(a, a')$. Using elimination of imaginaries, it is now easy to construct a finite group action equivalent to G. □

Corollary 4.10 *For stable T, the following are equivalent: 4-existence, 3-uniqueness, elimination of finite generalized imaginaries, 3-existence for $\widetilde{\mathcal{C}}$.*

Proof. By Proposition 4.9, Proposition 4.5, Lemma 4.1, Proposition 4.3. □

Discussion. In many proofs regarding stable theories, there is no harm in passing to a theory T' with more sorts, as long as T remains stably embedded and with the same induced structure; especially if $M' = \mathrm{acl}(M'|L(T))$ for $M' \models T'$. In this situation, by interpreting algebraic closure more widely in such extensions T', the 3-uniqueness or 4-existence property for amalgamation holds.

A generalization of the above proof for $n > 3$ using an appropriate notion of higher groupoids, would be very interesting. [2]

[2] I have recently become aware of Jacob Lurie's work [18], which may hold the key to this. Note the apparent resonance between Lurie's main theorem 6.1.0.6 there, and our Theorem 3.2.

We do not at present have a concrete description of the requisite sorts (analogous to equivalence relations or groupoids), but can at least prove their existence.

Proposition 4.11 *Let T be a theory with a canonical 2-amalgamation. There exists an expansion T^* of T to a language with additional sorts, such that:*

(1) T is stably embedded in T^*, and the induced structure from T^* on the T-sorts is the structure of T. Each sort of T^* admits a 0-definable map to a sort of T, with finite fibers.

(2) T^* has existence and uniqueness for n-amalgamation.

We sketch the proof.

Condition (1) is equivalent to:

(1') If $N^* \models T^*$ and N is the restriction to the sorts of T, then $\mathrm{Aut}(N^*) \to \mathrm{Aut}(N)$ is surjective, with profinite kernel.

For T with canonical 2-amalgamation, and p a type of T over $\mathrm{acl}(\emptyset)$, consider an expansion T_p of T as in Proposition 4.3: the points of a model M_p of T_p correspond to $\mathrm{acl}(a, M)$ where M is the restriction to T of M_p, $a \models p$, and a, M are embedded in some bigger model of T via canonical 2-amalgamation.

The proof of Proposition 4.3 shows that if each T_p has unique $(n-1, n+1)$-amalgamation then T has unique $(n, n+1)$-amalgamation.

To prove the proposition, construct first an expansion T^* of T with property (1), such that (U) any expansion of T together with finitely many sorts of T^* with property (1) is equivalent to a sort of T^*.

Note that T_p^* enjoys the same property; since a relatively finite cover of T_p^*, fibered over a sort Y_a, arises from a relatively finite cover of T^*, fibered over Y.

Suppose that unique $(n, n+1)$-amalgamation fails for a theory T with the universal property (U). Take n minimal. Then uniqueness at $(n-1, n+1)$ fails for T_p for appropriate p. By Lemma 4.1, T' does not have $(n-1, n)$-uniqueness. But T_p also has (U). This contradicts the minimality of n.

Problem 4.12 *For T^*, prove an analog of §4.4 for n commuting automorphisms.*

5. Linear imaginaries

We first discuss linear imaginaries in general; then restrict attention to the triangular imaginaries that we will need.

Definition 5.1 *Let $\mathbf{t}$ be a theory of fields (possibly with additional structure.)*

A $\mathbf{t}$-linear structure A is a structure with a sort $\mathbf{k}$ for a model of $\mathbf{t}$, and additional sorts V_i ($i \in I = I(\mathsf{A})$) denoting finite-dimensional vector spaces. Each V_i has (at least) a $\mathbf{k}$-vector space structure, and $\dim_i V_i < \infty$.

We assume:

1. *$\mathbf{k}$ is stably embedded;*
2. *the induced structure on $\mathbf{k}$ is precisely given by $\mathbf{t}$;*
3. *the V_i are closed under tensor products and duals.*

Explanation The language includes the language of $\mathbf{t}$ (applying to $\mathbf{k}$), and for each i, a symbol for addition $+: V_i \times V_i \to V_i$, and scalar multiplication $\cdot : k \times V_i \to V_i$.

Given i, j, for some $\mathbf{k}$, the language includes bilinear map $b : V_i \times V_j \to V_k$, inducing an isomorphism $V_i \otimes V_j \to V_k$.

For each i, there is j and a function symbol for a pairing $V_i \times V_j \to \mathbf{k}$, inducing an isomorphism $\check{V_i} \to V_j$.

Additional structure is permitted, subject to the embeddedness conditions (1,2).

Note that the tensor product $V \otimes W$ and dual $\check{V}$ are at all events interpretable in (k, V, W); so the conditions (3) can be viewed as (partial) elimination of imaginaries conditions.

For any finite tuple $s = (s_1, \ldots, s_k)$ of indices, let $V_s = V_{s_1} \oplus \cdots \oplus V_{s_k}$, and let P_s be the projectivization $V_s \setminus (0)/k^*$. These can also clearly be viewed as imaginary sorts of A.

Proposition 5.2 *Let* $\mathbf{k}$ *be an algebraically closed field. Then any* $\mathbf{k}$*-linear theory eliminates imaginaries to the level of the projective spaces* P_s.

Proof. This goes back to the 19th century (cf. references to Darboux in [24]) and occurs also in [13], Proposition 2.6.3, but not in easily quotable form. If W is a direct sum of some of the V_i, note first that the elements of the exterior powers $\Lambda^i W$ can be coded. Indeed an element of $\Lambda_i W$ can be viewed as a certain multilinear map on $\check{W}$, thus as an element of $W^{\otimes i}$. This is again a direct sum of some of the V_i.

A d-dimensional subspace of such a W corresponds to a certain 1-dimensional subspace of $\Lambda^k W$, and hence of $W^{\otimes d}$. Hence it can always be coded as an element of the projectivization P_s.

Now any Zariski closed subset Z of W is determined, for some l, by the space of polynomials of degree $\leq l$ vanishing on Z. This is a subspace of $\bigoplus_{i \leq l} (\check{W})^{\otimes i}$. Hence it is coded.

It follows by induction on dimension that every definable subset of W is coded (code the Zariski closure, and then the complement.) □

A couple of remarks via the following lemmas.

Lemma 5.3 *Let* $A = (k, V_i)_{i \in I(A)}$ *be a* $\mathbf{t}$ *linear structure, and let* k^* *be an elementary extension of* $\mathbf{k}$. *Let* $V_i^* = k^* \otimes_k V_i$, *and let* $A^* = (k^*, V_i^*)_{i \in I}$. *Then* A^* *expands uniquely to an elementary extension of* A. *(And every elementary extension of* A *is obtained in this way.)*

Proof. Clearly, if $A \prec A^*$, then $\mathbf{k} \prec k^*$ and (by the finite dimension) $V_i^* = k^* \otimes_k V_i$.

Also, any V_i has a basis b_i in A. There is a b_i-definable bijection $f : V_i \to k^{n_i}$. If R is a relation on V_i, or among several V_i, then fR is a relation S on $\mathbf{k}$, and in any elementary extension one must have: $R = f^{-1}S$. Thus uniqueness of the expansion is clear, and it remains to show that this prescription always does yield an elementary extension. We may fix constants for each b_i. But then $V_i \subseteq \mathrm{dcl}(k)$, and the assertion is immediate. □

Lemma 5.4 *Let* $\mathfrak{T}$ *be a theory, internal to a predicate* $\mathbf{k}$, *and with elimination of imaginaries. Let* $\mathfrak{T}'$ *be an expansion of* $\mathfrak{T}$, *such that every subset of* k^m, $\mathfrak{T}'$ *-definable with parameters, is* $\mathbf{k}$*-definable with parameters. Then* $\mathfrak{T}'$ *admits elimination of imaginaries.*

Proof. **Claim** If X is a definable subset of $\mathfrak{T}'$ (with parameters), then X is also $\mathfrak{T}$ -definable (with parameters). $\square$

Proof. By internality, there exists a $\mathfrak{T}$ definable (with parameters) map f on $\mathbf{k}^n$, whose image contains X. $f^{-1}(X)$ is $\mathfrak{T}$-definable (with parameters) by the assumption regarding new structure on $\mathbf{k}$. Thus $X = ff^{-1}X$ is $\mathfrak{T}$-definable (with parameters.)

Hence any $\mathfrak{T}'$-definable set can be written $X = Y_c$ where c is a canonical parameter for Y_c in $\mathfrak{T}$. It follows that Y_c is $\mathfrak{T}'$-definable and c is a canonical parameter for Y_c in $\mathfrak{T}'$. $\square$

Linear structures with flags.

We now consider *flagged spaces.* For us this will mean: a finite dimensional vector space V together with a filtration $V_1 \subset \ldots \subset V_n = V$ by subspaces, with $\dim V_i = i$.

Given V, we form the dual $\check{V}$ with the natural filtration $\check{V}_i$. If V, W are filtered spaces, take the tensor product with the filtration $V_1 \otimes W_1 \subset V_1 \otimes W_2 \subset \cdots \subset V_1 \otimes W \subset V_2 \otimes W_1 + V_1 \otimes W \subset \cdots \subset V_2 \otimes W \subset \cdots \subset V \otimes W$.

Thus a family of flagged spaces can be closed under tensors and duals, without losing the flag property.

Definition 5.5 *A* $\mathbf{t}$*-linear structure* A has flags *if:*

(*) For any i *with* $\dim(V_i) > 1$, *for some* j, k *with* $\dim(V_j) = \dim(V_i) - 1$, $\dim V_k = 1$, *there exists a* 0*-definable exact sequence* $0 \to V_k \to V_i \to V_j \to 0$.

Lemma 5.6 *(Elimination of projective imaginaries.) Let* A *be a flagged* $\mathbf{t}$*-linear structure. Then elements of projectivizations of the vector spaces of* A *can be coded in* A. *In particular if* $\mathbf{k}$ *is an algebraically closed field,* A *admits elimination of imaginaries.*

Proof. Lemma 5.2 applies here: $(k, V_1 \subset \ldots \subset V_n)$ can be viewed as an expansion of $(k, V_1 \oplus \ldots \oplus V_n)$. By Proposition 5.2 and Lemma 5.4, when $\mathbf{k}$ is algebraically closed, all imaginaries of A are coded by elements of projective spaces.

So it suffices to show in general how to code the projectivizations of the vector spaces W of A. Say $\dim(W) = d$. W comes with a 0-definable filtration, including $W_0 \subset W_1 \subset \cdots W_{\dim W} = W$ by subspaces W_i with $\dim(W_i) = i$. Thus it suffices to code a 1-dimensional subspace U of a filtered vector space W. Say $U \subset W_{k+1}, U \not\subseteq W_k$. Let $f : W_{k+1} \to Y = W_{k+1}/W_k$ be the natural map. Given U, one obtains $f|U$ and hence $(f|U)^{-1} : Y \to W$. But this is an *element* rather than a subspace of $\mathrm{Hom}(Y, W) = \check{Y} \otimes W$, and hence is coded. $\square$

Linear structures with roots.

We say that a linear structure A has roots if for any one-dimensional $V = V_i$, and any $m \geq 2$, there exists $W = V_j$ and 0-definable $\mathbf{k}$-linear embeddings $f : W^{\otimes m} \to V_l$ and $g : V \to V_l$, with $g(V) \subset f(W)$.

A *good* linear structure is one with flags and roots.

Proposition 5.7 *Let* A *be a linear structure with flags and roots, for an algebraically closed field* $\mathbf{k}$ *of characteristic* 0. *Then every finite internal cover of* A *almost splits.*

Proof. We may expand the theory by algebraic points; in particular we may assume that the definable points of $\mathbf{k}$ form an algebraically closed field. For $\mathbf{k}$ itself, the lemma follows from Remark 3.9. For A, we use Proposition 3.10. Let D a 0-definable set of A. $\mathrm{Aut}(D/k)$ definably isomorphic to the $\mathbf{k}$-linear group H. Since A has flags, for each sort W of A, $\mathrm{Aut}(W/k)$ preserves a flag, so it is a solvable group, and hence so is H. Thus $H = UT$ where U is the unipotent part of H. Let $f : \widetilde{H} \to H$ be a central extension of H with prime cyclic kernel $Z = \mathbb{Z}/l\mathbb{Z}$. We have to show that $\widetilde{H}$ is represented as $\mathrm{Aut}(\widetilde{D}/k)$ for some $\widetilde{D}$.

We may take H and $\widetilde{H}$ connected. Then $\widetilde{H}$ is solvable. So $\widetilde{H} = \tilde{U}\tilde{T}$ with $\tilde{U}$ unipotent. $f|\tilde{U}$ has finite kernel, hence trivial kernel (unipotent groups in characteristic zero have no nontrivial torsion elements). We obtain an induced map $\tilde{T} \to \widetilde{H}/\tilde{U} \to H/U = T$, and see that $\widetilde{H} \to H$ is induced from a central extension $0 \to Z \to \tilde{T} \to T \to 0$. Now $T \cong (G_m)^n$, and every central extension of T with kernel Z is induced from the extension $G_m \to G_m$, $x \mapsto x^l$, via some rational character $\chi : T \to G_m$.

Let D' be the set of bases (of the $\mathbf{k}$-space spanned by D) contained in D; then $\mathrm{Aut}(D'/k) = H$ also; let $D'' = D'/Ker(\chi)$; by Lemma 5.2 and Lemma 5.6, A admits elimination of imaginaries, so D'' can be viewed as a 0-definable set in A. Now $\mathrm{Aut}(D''/k) = G_m$, and we have to represent the finite central cover $x \mapsto x^l$.

D'' lies in some vector space W in A. Since $\mathrm{Aut}(W/k) = G_m$, $W = \oplus W_i$, and G_m acts on W_i via a character χ_i. Each 1-dimensional subspace S of W_i is $\mathrm{Aut}(W/k)$-invariant.

At the same time, W contains a 0-definable one-dimensional subspace W_1; and W/W_1 is 0-definable. Either $S = W_1$, or S embeds into W/W_1. Continuing this way we find a 0-definable 1-dimensional space V such that $\mathrm{Aut}(V/k) = \mathrm{Aut}(S/k)$. Doing this for each S in some some decomposition of W into one-dimensional $\mathrm{Aut}(W/k)$-invariant subspaces as above, we find 0-definable one-dimensional $V_1, \ldots, V_j$ with $\mathrm{Aut}(W/k) = \mathrm{Aut}(V_1, \ldots, V_j/k)$.

Now by taking roots of the V_i we can find $\widetilde{V_i}$ and maps $(\widetilde{V_i})^{\otimes l} \to V_i$ such that $\mathrm{Aut}(\widetilde{V_i}/k) \to \mathrm{Aut}(V_i/k)$ is isomorphic to $x \mapsto x^l, G_m \to G_m$. Pulling back to $\mathrm{Aut}(\widetilde{V}_1, \ldots, \widetilde{V}_j/k)$ we succeed in splitting the cover. □

Note that a 1-dimensional vector-space V amounts to the same thing as a set $V^* = V \setminus (0)$ and a regular action of k^* on V^*. (Given such an action, recover the vector space structure on $V = V^* \cup \{0\}$: $0 \cdot v = 0$, $\alpha u + \beta u = (\alpha + \beta)u$.) So one has the usual H^1 formalism; the tensor product corresponds to the sum in H^1.

An m'th root W of V, i.e. a one-dimensional vector space with an isomorphism $f : W^{\otimes m} \to V$, yields a $(k^*)^m$ $T = T(W) \subset V^*$, i.e. a class of the equivalence relation of $(k^*)^m$-conjugacy. Namely, let $t(w) = f(w \otimes \cdots \otimes w)$, and $T = t(W)$.

Note that (because of t and the $\mathbf{k}$- linear structure on W), $\mathrm{Aut}(W/V, k) \leq \mu_m(k)$, the group of m'th roots of unity in $\mathbf{k}$. Thus W can be regarded as a finite internal cover of (V, k), and in particular as a generalized imaginary sort.

Remark 5.8 *Assume* $\mathbf{t}$ *admits linear elimination of imaginaries for structures with flags and roots, and let* A *be a* $\mathbf{t}$*-linear structure with flags. Assume that for any 1-dimensional* V *of* A*, and any* m*,* V *has a distinguished* $(k^*)^m$*-class (over* $\emptyset$*.) Then* A *admits elimination of imaginaries.*

Proof. We can arrive at a structure with EI by successively adding roots to one -dimensional vector spaces V of A. This involves adding an m'th root W, all tensor powers $W^{\otimes n}$ for $n \in \mathbb{Z}$, and all tensor products

$W^{\otimes n} \otimes U$ for U a vector space of A. If $\mu_m(k) = 1$, then W can be identified with a $(k^*)^m$-class in V, and $W^{\otimes n} \otimes U$ embeds into $V^{\otimes n} \otimes U$ via this class. Otherwise, $\mu_m(k)$ acts on the new structure, fixing $\mathbf{k}$ and A, and one sees that if an element $w \otimes u \in W^{\otimes n} \otimes U$ is fixed by $\mu_m(k)$, then $n = 0$, or $w = 0$. Thus any imaginary of A coded in the new structure already lies in A. Applying this iteratively, we see that A has EI. □

5.9. Pseudo-finite fields

A *pseudo-finite* field is a perfect PAC field (i.e. every irreducible variety over F has an F-point), with Galois group $\hat{\mathbb{Z}}$. Ax showed that F is pseudo-finite iff every sentence true in F is true in infinitely many finite fields. ([3], [12], [14].) *We take F to come together with an isomorphism $\hat{\mathbb{Z}} \to Gal(F)$.* In terms of language, this means that we fix, for each $d \in \mathbb{N}$, an imaginary element coding a generator of $\mathrm{Gal}(F_d/F)$ (where F_d is the extension of F of order d.) (This is a little more than the pure field structure; but all finite fields do have a canonical generator of Galois, so perhaps it's only fair that the pseudo-finite ones should.) When $char(k) = 0$, this is equivalent to fixing a surjective group homomorphism $k[\mu_d]^* \to \mu_d$; cf. [6]. As noted there, in this language, F admits elimination of imaginaries. (I take this opportunity to note an mistake: it is also asserted there that F admits elimination of imaginaries in the field language, up to sorts coding elements of the Galois group; but this slightly stronger statement is incorrect.)

Theorem 5.10 *Let F be a pseudo-finite field of characteristic* 0 *(with fixed generator of Galois). Then a good linear structure over F admits elimination of imaginaries.*

Proof. Let $\mathbf{t} = Th(F)$, and let A be a good $\mathbf{t}$-linear structure. If A' is a reduct of A, with the same sorts, with the full structure on the field sort, and remembering the F-linear structure of each vector space in A and all 0-definable linear maps among them, then A' is also good. Moreover A admits EI if A' does (Lemma 5.4.) Thus we may assume the structure on A consists just of the structure on $\mathbf{k}$, the $\mathbf{k}$-linear structures and the 0-definable $\mathbf{k}$-linear maps among them.

Let A^a be the linear structure over the algebraic closure F^a, and with vector spaces $V^a = F^a \otimes_F V$ for each vector space V of A. Any 0-definable linear map in A extends uniquely to a 0-definable linear map in A^a, and we take this to define a structure on A^a. Since A is good, so is A^a. Let $T_1 = Th(F^a, V^a)_{V \in \mathsf{A}}$.

1. T_1 admits elimination of imaginaries (Lemma 5.2) and of generalized finite imaginaries, and is stable (every sort has finite Morley rank.)
2. Let T_2 be the model companion of the theory of pairs (M, σ) where $M \models T_1$ and $\sigma \in \mathrm{Aut}(M)$. Then T_2 admits elimination of imaginaries. (Lemma 4.7.)
3. Let $(K, V, \sigma) \models T_2$. Let (F, V) be the fixed field and fixed vector spaces of σ. Then $\dim_F V = d$. F is pseudo-finite, and K can be chosen so that the fixed field will have the same theory as the original F. (cf. [6].)
4. Any imaginary of (F, V) is in particular an imaginary of (K, V, σ), and thus can be coded by a tuple of elements of (K, V). Each of these elements must be fixed by σ.
5. (F, V_F) coincides with the σ-fixed part of (K, V). This follows from the fact that V has a σ-fixed basis.

□

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Ehud HRUSHOVSKI
Institute of Mathematics,
the Hebrew University of Jerusalem,
Givat Ram, Jerusalem, 91904, ISRIEL
e-mail: ehud@math.huji.ac.il

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