A generalization of Banach's contraction principle for some non-obviously contractive operators in a cone metric space

Yingxin Guo

Abstract

This paper investigates the fixed points for self-maps of a closed set in a space of abstract continuous functions. Our main results essentially extend and generalize some fixed point theorems in cone metric spaces. An application to differential equations is given.

Key Words: Cone metric space, fixed point, Ordered Banach space, self-maps of a closed set, iterative sequence

1. Introduction

Fixed point theory is a mixture of analysis, topology and geometry. The theory of existence of fixed points of maps has been revealed as a very powerful and important tool in the study of nonlinear phenomena [3, 5–9, 15–23, 27–29]. If a topological space is a metric space, or a linear topological space, then the fixed point theory in such spaces is very abundant. Cone metric spaces were introduced in [10]. The authors there described convergence in cone metric spaces and introduced completeness, then they proved some fixed point theorems of contractive mappings on cone metric spaces. Recently, in [1, 2, 4, 11–14, 16, 21, 24–25] some fixed point theorems were proved for maps on cone metric spaces. In particular, Du [26] showed that from each cone metric one can get the usual metric by using a scalarization function. Hence the results of Huang-Zhang [10] and the results of many other authors are obtained trivially by Du's method. But there is a paper, [15], on cone metric spaces in which Du's method is not applicable.

In this work we prove some fixed point theorems in cone metric spaces, including results which generalize those from Huang and Zhang's work. Given the fact that, in a cone, one has only a partial ordering, it is doubtful that their Theorem 2.1 can be further generalized.

The organization of this paper is as follows. In Section 2, problem formulation and preliminaries are given. In Section 3, some new results are given. Section 4 gives an example to illustrate the effectiveness of our results.

2. Preliminaries

Consistent with Guang and Xian [8], the following definitions and results will be needed in the sequel.

Let E be a real Banach space with norm $|\cdot|$. A subset P of E is called a cone if and only if:

(a) P is closed, nonempty and $P \neq 0$;

(b) $a, b \in R, a, b \geq 0, x, y \in P$ implies $ax + by \in P$;

(c) $x \in P$ and $x \notin P \Rightarrow x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ stands for $y - x \in \text{int}P$ (interior of P).

A cone P is called normal if there is a number $L > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq L\|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of P.

Remark 2.1 We note that there are no normal cones with normal constant $L < 1$ and for each $k > 1$ there are cones with normal constant $L > k$ by [18].

Definition 2.1 *Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies*

(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space. The concept of a cone metric space is more general than that of a metric space.

Definition 2.2 *Let (X, d) be a cone metric space. We say that $\{x_n\}$ is:*

(e1) a Cauchy sequence if for every c in E with $c \gg 0$, there is N such that for all $n, m > N, d(x_n, x_m) \ll c$;

(e2) a Convergent sequence if for every c in E with $c \gg 0$, there is N such that for all $n > N, d(x_n, x) \ll c$ for some fixed x in X.

When $\{x_n\}$ converges to x, we say x is the limit of $\{x_n\}$. We denote this by

$$\lim_{n \to \infty} x_n = x \quad \text{or} \quad x_n \to x \ (n \to \infty).$$

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. It is known that $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$. The limit of a convergent sequence is unique provided P is a normal cone with normal constant L (see [8] and [18, 20]).

3. Main results

Let (X, d) be a completely cone metric space, P be a normal cone. $I = [0, T](T > 0)$. Denote $C[I, X] = \{u : I \to X | u(t) \text{ is continuous on } I\}$. It is easy to see that $C[I, X]$ is a Banach space with the norm $\|u - v\| = \max_{\forall t \in I} |d(u(t), v(t))|$ for $u, v \in C[I, X]$.

Theorem 3.1 *Let F be a closed subset of $C[I, X]$ and $A : F \to F$ an operator. If there exist $\alpha, \beta \in [0, 1), M \in C(I, [0, \infty))$ such that for any $u, v \in F$,*

$$d(Au(t), Av(t)) \leq \beta d(u(t), v(t)) + \frac{M(t)}{t^\alpha} \int_0^t d(u(s), v(s))ds, \ \forall \ t \in (0, T], \tag{3.1}$$

then A has exactly one fixed point u^* in F. For any $x_0 \in F$, the iterative sequence $x_n = Ax_{n-1}(n = 1, 2, 3, \ldots)$ converges to u^* in F and for all $s > 0$,

$$\|x_n - u^*\| = o(n^{-s}) \ as \ n \to \infty.$$

Proof.   For any $u_0 \in F$, set $u_n = Au_{n-1}(n = 1, 2, 3, \ldots)$. By (3.1) we get

$$d(u_2(t), u_1(t)) \leq (\beta + M(t)t^{1-\alpha})\|u_1 - u_0\| \leq (\beta + Kt^{1-\alpha})\|u_1 - u_0\|, \ \forall \ t \in (0, T],$$

where $K = \max\{M(t) | t \in I\}$. It follows by induction and (3.1) that, for any $t \in (0, T]$,

$$d(u_{n+1}(t), u_n(t)) \leq \Big(\beta^n + C_n^1 \beta^{n-1} Kt^{1-\alpha} + \frac{C_n^2 \beta^{n-2} K^2 t^{2-2\alpha}}{2 - \alpha} + \cdots$$

$$+ \frac{K^n t^{n-n\alpha}}{(2 - \alpha)(3 - 2\alpha)\cdots(n - (n-1)\alpha)}\Big)\|u_1 - u_0\|,$$

$$(n = 1, 2, 3, \ldots).$$

therefore

$$\|u_{n+1} - u_n\| \leq \Big(\beta^n + C_n^1 \beta^{n-1}h + \frac{C_n^2 \beta^{n-2}h^2}{2!} + \cdots + \frac{h^n}{n!}\Big)\|u_1 - u_0\|, \tag{3.2}$$

where $h = KT^{1-\alpha}(1 - \alpha)^{-1}$. For any n, set $n = km + j(0 \leq j < k)$, where $k(k \neq 1)$ is any given positive integer. Then whenever n is sufficiently large, it follows from the Stirling formula that

$$L_1 \equiv \beta^n + C_n^1 \beta^{n-1}h + \frac{C_n^2 \beta^{n-2}h^2}{2!} + \cdots + \frac{C_n^m \beta^{n-m}h^m}{m!}$$

$$\leq C_n^m \beta^{n-m}(1 + h + \frac{h^2}{2!} + \cdots + \frac{h^m}{m!})$$

$$= \frac{O(1)\beta^{n-m}n\Gamma(n)}{m(n-m)\Gamma(m)\Gamma(n-m)}$$

$$= \frac{O(1)n\beta^{n-m}}{m(n-m)} \cdot \frac{\sqrt{\frac{2\pi}{n}}(\frac{n}{e})^n(1 + O(\frac{1}{n}))}{\sqrt{\frac{2\pi}{m}}(\frac{m}{e})^m(1 + O(\frac{1}{m}))\sqrt{\frac{2\pi}{n-m}}(\frac{n-m}{e})^{n-m}(1 + O(\frac{1}{n-m}))}$$

$$= \frac{O(1)\beta^{n-m}\sqrt{2n\pi}n^n}{\sqrt{2m\pi}m^m(1 + O(\frac{1}{m}))\sqrt{2(n-m)\pi}(n-m)^{n-m}}$$

$$= O\left(\frac{k^m}{\sqrt{m}}\right)\left(\frac{\beta n}{n-m}\right)^{n-m} = O\left(\frac{\left(\beta^{k-1}k(\frac{k}{k-1})^{k-1}\right)^m}{\sqrt{m}}\right)$$

$$= O\left(\frac{(\beta^{k-1})^m}{\sqrt{m}}\right) = O\left(\frac{\beta^n}{\sqrt{n}}\right),$$

Similarly,

$$L_2 \equiv \frac{C_n^{m+1}\beta^{n-m-1}h^{m+1}}{(m+1)!} + \cdots + \frac{h^n}{n!}$$

$$\leq \frac{C_n^{[\frac{n}{2}]}}{(m+1)!}(\beta^{n-m-1}h^{m+1} + \cdots + h^n)$$

$$= o\left(\frac{1}{(m+1)^s}\right) = o(\frac{1}{n^s}) \quad (n \to \infty),$$

where $s > 1$ can be any real constant. Consequently, by (3.2) we have

$$\|u_{n+1} - u_n\| \leq (L_1 + L_2)\|u_1 - u_0\|$$

$$= O\left(\frac{\beta^n}{\sqrt{n}}\right) + o(\frac{1}{n^s}) = o(\frac{1}{n^s}) \quad (n \to \infty),$$

(3.3)

which implies that, for any fixed $s > 0$, there exists $n_0 > 0$ such that

$$\|u_{n+1} - u_n\| < \frac{1}{n^{s+1}}, \quad \forall \, n > n_0.$$

Therefore, for any positive integers $p > 0, n > n_0$, we have

$$\|u_{n+p} - u_n\| \leq \|u_{n+p} - u_{n+p-1}\| + \cdots + \|u_{n+1} - u_n\| < \sum_{i=n}^{\infty}\frac{1}{i^{s+1}}$$

$$= \frac{1}{s(n-1)^s} + o\left(\frac{1}{(n-1)^{s+1}}\right) \ (\text{see}[2])$$

$$= O\left(\frac{1}{n^s}\right) \quad (n \to \infty),$$

where $s > 0$. Hence $\{u_n\}$ is a Cauchy sequence and there exists $u^* \in F$ such that $\|u_n - u^*\| \to 0$ as $n \to \infty$. By (3.1),

$$d(Au^*(t), u^*(t)) \leq d(Au^*(t), Au_n(t)) + d(Au_n(t), u^*(t))$$

$$\leq (\beta + Kt^{1-\alpha})\|u_n - u^*\| + \|u_{n+1} - u^*\|,$$

for any $\forall \, t \in (0, T]$. Then

$$\|Au^*(t) - u^*(t)\| \leq \|(\beta + Kt^{1-\alpha})\|u_n - u^*\| + \|u_{n+1} - u^*\|,$$

which implies by $\|u_n - u^*\| \to 0 \ (n \to \infty)$ that $Au^* = u^*$.

For any $x_0 \in F$, set $x_n = Ax_{n+1}(n = 1, 2, 3, \cdots)$. By (3.1) and using a similar way as establishing (3.3) we can get, for any $s > 0$,

$$\|x_n - u^*\| = o(n^{-s}) \ as \ n \to \infty,$$

which means that u^* is the unique fixed point of A since $x_0 \in F$ is arbitrary. This completes the proof. □

Remark 3.1 We show that Theorem 3.1 is a generalization of the Theorems 1-4 of [10] in $C[I, E]$.

On one hand, it is easy to give some self-maps of a closed subset of $C[I, E]$, which satisfy (3.1) but are not contractions on a complete cone metric space (see theorem 1 in [10]). For example, operator $A : C[J, E] \to C[J, E](J = [0, 1])$ defined by

$$Au(t) = \frac{2}{3} \int_0^1 u(t)dt + 3t^{-\frac{1}{2}} \int_0^t u(s)ds, \ \forall \ t \in (0, 1], \ Au(0) = \frac{2}{3}u_0$$

is such a map.

On the other hand, if F is a closed subset of a complete cone metric space (X, d) operator $A : F \to F$ satisfies

$$d(Au, Av) \leq \alpha d(u, v) \ \forall \ u, v \in F, \tag{3.4}$$

where $\alpha \in [0, 1)$. Then Theorem 1 of [10] shows that A has exactly one fixed point in F. We assert that this conclusion can also be obtained by Theorem 3.1. In fact, we can embed F into $C[I, X]$ by regarding the elements of F as constant-value functions of $C[I, X]$. Then F is a closed set in $C[I, X]$ and $A : F \to F$ can be regarded as a map in $C[I, X]$. So (3.4) implies that A satisfies (3.1) for $K = 0$ and then, in the subset F of $C[I, E]$, A has exactly one fixed point by Theorem 3.1, which is the unique fixed point of A in the subset F of X.

As we proved Theorem 3.1, we can similarly prove

Theorem 3.2 Let F be a closed subset of $C[I, X]$ and $A : F \to F$ an operator. If there exist $\alpha, \beta \in [0, 1), M \in C(I, [0, \infty))$, where α satisfies $(-1)^\alpha = -1$, such that, for some fixed $\eta \in I = [0, T]$ and for any $u, v \in F$,

$$d(Au(t), Av(t)) \leq \beta d(u(t), v(t)) + \frac{M(t)}{(t - \eta)^\alpha} \int_\eta^t d(u(s), v(s))ds, \ \forall t \in (\eta, T], \tag{3.5}$$

then the conclusions of Theorem 3.1 hold.

4. An example

In this section, an example is used to demonstrate that the method presented in this paper is effective.

Example 4.1 *Consider the following third-order three-point problem:*

$$\begin{cases} \mathbf{D}^{3/2}u(t) = u\dfrac{3t^2 \cos t}{2\sqrt{1-t}} + t^3 + \sin t, & 0 < t < 1, \\ u(0) = 0, u(1) = \beta u(\dfrac{1}{2}). \end{cases} \tag{4.1}$$

where $\mathbf{D}^\alpha$ is the Riemann-Liouville differential operator of order $\frac{3}{2}$.

Theorem 4.2 *If $\beta > \frac{8\sqrt{2}+8\sqrt{2\pi}}{8\sqrt{\pi}-\sqrt{2}}$, then (4.1) has exactly one nontrivial solution.*

Proof. By Lemma 3.1 in [27], problem (4.1) has a solution $u = u(t)$ if and only if u is a solution in C[J, R](J=[0,1]) of the operator equation

$$(Tu)(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(u(s)\frac{3s^2 \cos s}{2\sqrt{1-s}} + s^3 + \sin s \right) ds$$

$$+ \frac{t^{\alpha-1}}{1 - \beta\eta^{\alpha-1}} \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \left(u(s)\frac{3s^2 \cos s}{2\sqrt{1-s}} + s^3 + \sin s \right) ds$$

$$- \frac{\beta t^{\alpha-1}}{1 - \beta\eta^{\alpha-1}} \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} \left(u(s)\frac{3s^2 \cos s}{2\sqrt{1-s}} + s^3 + \sin s \right) ds,$$

where $\eta = \frac{1}{2}$. So we only need to seek a fixed point of T in C[J, R]. For any $u_1, u_2 \in C[J, R]$,

$$|(Tu_1)(t) - (Tu_2)(t)|$$

$$= \left| -\frac{1}{\Gamma(\alpha)} \int_0^t \sqrt{t-s}(u_1(s) - u_2(s))\frac{3s^2 \cos s}{2\sqrt{1-s}} ds \right.$$

$$+ \frac{t^{\alpha-1}}{1 - \beta\eta^{\alpha-1}} \frac{1}{\Gamma(\alpha)} \int_0^1 \sqrt{1-s}(u_1(s) - u_2(s))\frac{3s^2 \cos s}{2\sqrt{1-s}} ds$$

$$\left. - \frac{\beta t^{\alpha-1}}{1 - \beta\eta^{\alpha-1}} \frac{1}{\Gamma(\alpha)} \int_0^\eta \sqrt{\eta-s}(u_1(s) - u_2(s))\frac{3s^2 \cos s}{2\sqrt{1-s}} \right|$$

$$\leq \frac{D}{\Gamma(\alpha)} \int_0^t |u_1(s) - u_2(s)| ds \left(\text{here } D = \int_0^1 \sqrt{1-s}\frac{3s^2 \cos s}{2\sqrt{1-s}} ds \right)$$

$$+ |L_1(u_1(t) - u_2(t))| + |L_2(u_1(t) - u_2(t))|.$$

Where

$$L_1 u(t) = \frac{t}{|1 - \beta\eta^{\alpha-1}|} \frac{1}{\Gamma(\alpha)} \int_0^1 \sqrt{1-s}\frac{3s^2 \cos s}{2\sqrt{1-s}} |u(s)| ds,$$

and

$$L_2 u(t) = \frac{|\beta|}{|1 - \beta\eta^{\alpha-1}|} \frac{t}{\Gamma(\alpha)} \int_0^\eta (\sqrt{\eta-s}\frac{3s^2 \cos s}{2\sqrt{1-s}} |u(s)| ds.$$

We have that

$$\|L_1\| + \|L_2\| \leq \frac{1}{|1 - \beta\eta^{\alpha-1}|} \frac{1}{\Gamma(\alpha)} \int_0^1 \sqrt{1-s}\frac{3s^2 \cos s}{2\sqrt{1-s}} ds$$

$$+ \frac{|\beta|}{|1 - \beta\eta^{\alpha-1}|} \frac{1}{\Gamma(\alpha)} \int_0^\eta \sqrt{\eta-s}\frac{3s^2 \cos s}{2\sqrt{1-s}} ds.$$

Since $\beta > \frac{8\sqrt{2}+8\sqrt{2\pi}}{8\sqrt{\pi}-\sqrt{2}}$, we have that

$$\|L_1\| + \|L_2\|$$

$$\leq \frac{1}{\Gamma(3/2)} \left[\frac{\sqrt{2}}{\beta - \sqrt{2}} \int_0^1 \sqrt{1-s}\frac{3s^2}{\sqrt{1-s}} ds + \frac{\sqrt{2}\beta}{\beta - \sqrt{2}} \int_0^{1/2} \sqrt{1/2-s}\frac{3s^2}{\sqrt{1-s}} ds \right]$$

$$< \frac{1}{\Gamma(3/2)} \cdot \left[\frac{\sqrt{2}}{\beta - \sqrt{2}} + \frac{\sqrt{2}\beta}{\beta - \sqrt{2}} \cdot \frac{1}{8} \right] < 1.$$

Then, by Theorem 3.1, we know Equation (4.1) has a nontrivial solution. This completes the proof. $\square$

Remark 4.1 However, when $\frac{8\sqrt{2}+8\sqrt{2\pi}}{8\sqrt{\pi}-\sqrt{2}} < \beta \leq \frac{34\sqrt{2}}{9}$, we don't know if Equation (4.1) has a nontrivial solution by Theorem 3.2 in [27] (see [27, Example 4.1]). Therefore, our results in this paper extend and improve them in [27].

Acknowledgments

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Yingxin GUO
College of Control Science and Engineering,
Shandong University, Jinan,
Shandong 250100, PR CHINA
Department of Mathematics,
Qufu Normal University, Qufu, Shandong,
273165, Peoples Republic of CHINA
e-mail: yxguo312@163.com