Extended cross product in a 3-dimensional almost contact metric manifold with applications to curve theory

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Abstract

In this work, we define a new cross product in 3-dimensional almost contact metric manifold and we study the theory of curves using this new cross product in this manifold. Besides, in the works of Baikousis, Blair [1] and Cho et al. [4], we observe that some theorems are incomplete and excessively generalized are thus their alternative proofs presented.

Key Words: Sasakian Manifold, Legendre curve, cross product

1. Introduction

Let M be a $(2n+1)$-dimensional differentiable manifold. If there exist a 1-form η, such that $\eta \Lambda \left(d\eta \right)^n \neq 0$ on M, then (M, η) is called a *contact manifold* and η a *contact 1-form* [2]. A unique vector field ξ is called *Reeb vector* field (or characteristic vector field) where $\eta(\xi) = 1$ and $d\eta(\xi, .) = 0$ [2]. In a contact manifold, the contact distribution is defined by

$$D = \{X \in \chi(M) : \eta(X) = 0\}.$$

A $(2n + 1)$-dimensional differentiable manifold M is called an *almost contact manifold* if there is an almost contact structure (ϕ, ξ, η) consisting of a tensor field ϕ type $(1,1)$, a vector field ξ, and a 1-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \text{and (one of)} \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0. \tag{1.1}$$

If the induced almost complex structure J on the product manifold $M^{2n+1} \times \mathbb{R}$ defined by

$$J \left(X, f\frac{d}{dt} \right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt} \right)$$

is integrable then the structure (φ, ξ, η) is said to be normal, where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M^{2n+1} \times \mathbb{R}$ [2]. M becomes an *almost contact metric manifold* with an almost contact metric structure (ϕ, ξ, η, g), if

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

or equivalently,

$$g(X, \phi Y) = -g(\phi X, Y) \quad \text{and} \quad g(X, \xi) = \eta(X)$$

for all $X, Y \in TM$, where g is a Riemannian metric tensor of M [2]. For a 3-dimensional almost contact metric manifold, Z. Olszak proved that

$$(\nabla_X \phi) Y = g(\phi(\nabla_X \xi), Y)\xi - \eta(Y)\phi(\nabla_X \xi) \tag{1.2}$$

for all $X, Y \in TM$ [5].

An almost contact metric structure is called a *contact metric structure* if

$$g(X, \phi Y) = d\eta(X, Y)$$

holds on M for $X, Y \in TM$. A normal contact metric manifold is a *Sasakian manifold*. However an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi) Y = g(X, Y)\xi - \eta(Y)X, \quad X, Y \in TM, \tag{1.3}$$

where ∇ is Levi-Civita connection [2].

In any 3-dimensional contact metric manifold, the equation

$$(\nabla_X \phi) Y = g(X + hX, Y)\xi - \eta(Y)(X + hX) \tag{1.4}$$

is satisfied, where $h = \frac{1}{2}\pounds_\xi \phi$ and $\pounds$ denotes the Lie derivative [6].

A well known example of a Sasakian manifold in terms of $M = (R^3, \phi, \xi, \eta, g)$ can be given by

$$\phi := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & y & 0 \end{pmatrix}, \ \xi := 2\frac{\partial}{\partial z}, \ \eta := \frac{1}{4}(dz - ydx)$$

and $g := \frac{1}{4}(dx^2 + dy^2) + \eta \otimes \eta$. Here,

$$\varphi = \{e = 2\frac{\partial}{\partial y}, \ \phi(e) = 2(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}), \ \xi = 2\frac{\partial}{\partial z}\}$$

is an orthonormal basis. Furthermore, ϕ-sectional curvature of M is equal to -3. Therefore it appears to be a space form denoted by $R^3(-3)$ [1].

Let (M, η) be a 3-dimensional contact manifold and γ be a regular curve in this manifold. If $\eta(t) = 0$ (i.e. $t \in D$), we say that γ is a Legendre curve in M where $t = \gamma'$ [1]. Furthermore, in a 3-dimensional contact metric manifold, if the angle between tangent vector of the curve and the Reeb vector field is constant, then it is said that the curve is a slant curve [4].

In a 3-dimensional Sasakian manifold, Baikuossis and Blair [1] stated the following proposition and theorems.

Proposition 1.1 *In a 3-dimensional Sasakian manifold, the torsion (k_2) of a Legendre curve is equal to* 1.

Theorem 1.1 *For a smooth curve γ in a 3-dimensional Sasakian manifold, set $\sigma = \eta(\dot\gamma)$. If $k_2 = 1$ and at one point $\sigma = \dot\sigma = 0$, then γ is a Legendre curve.*

Theorem 1.2 *If on a 3-dimensional contact metric manifold, the torsion (k_2) of Legendre is equal to 1, then the manifold is Sasakian.*

Cho et al. [4] have investigated slant curves in a Sasakian 3-manifold and stated the following theorem.

Theorem 1.3 *A non -geodesic curve in a Sasakian 3-manifold M is slant curve if and only if its ratio of $k_2 \pm 1$ and k_1 is constant. where k_1, k_2 are curvature and torsion of the curve, respectively.*

In this theorem, the necessary condition is not correct. We will present a counterexample which violates necessary condition of Theorem 1.3.

2. Cross product in 3-dimensional almost contact metric manifold

Definition 2.1 *Let $M^3 = (M, \phi, \xi, \eta, g)$ be a 3-dimensional almost contact metric manifold. We define a cross product $\wedge$ by*

$$X \wedge Y = -g(X, \phi Y)\xi - \eta(Y)\phi X + \eta(X)\phi Y, \tag{2.5}$$

where $X, Y \in TM$.

Example 2.1 *In a 3-dimensional Euclidean Space $R^3(x, y, z)$, if we define a subspace of $R^3(x, y, z)$ by $V = \{(x, y, 0) : x, y \in R\}$, there exist natural projection $\pi(x, y, z) = (x, y, 0)$, and almost complex map $J(x, y, 0) = (-y, x, 0)$ on V. If we define a map $\phi = J \circ \pi$, then it is seen that $(R^3(x, y, z), \phi, \xi, \eta, g)$ is an almost contact metric manifold where $\eta = dz$, $\xi = \frac{\partial}{\partial z}$ and g is standard Euclidean metric in 3-dimensional Euclidean space. As a result, we have*

$$X \wedge Y = -g(X, \phi Y)\xi - \eta(Y)\phi X + \eta(X)\phi Y,$$

where $X, Y \in TR^3$. In this case we have $X \wedge Y = X \times Y$ where $X \times Y$ is the usual cross product in R^3.

Theorem 2.1 *Let $M^3 = (M, \phi, \xi, \eta, g)$ be a 3-dimensional almost contact metric manifold. Then, for all $X, Y, Z \in TM$ the cross product has the following properties:*
a) The cross product is bilinear and antisymmetric (i.e. $X \wedge Y = -Y \wedge X$).
b) $X \wedge Y$ is perpendicular both of X and Y.
c)

$$Y \wedge \phi(X) = g(X, Y)\xi - \eta(Y)X, \tag{2.6}$$

$$\phi(X) = \xi \wedge X. \tag{2.7}$$

d) Define a mixed product by

$$(X, Y, Z) = g(X \wedge Y, Z),$$

then we have

$$(X, Y, Z) = -(g(X, \phi(Y))\eta(Z) + g(Y, \phi(Z))\eta(X) + g(Z, \phi(X))\eta(Y)) \tag{2.8}$$

and

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$$(X, Y, Z) = (Y, Z, X) = (Z, X, Y).$$

e)

$$g(X, \phi(Y))Z + g(Y, \phi(Z))X + g(Z, \phi(X))Y = -\det(X, Y, Z)\ \xi \tag{2.9}$$

$$(X \wedge Y) \wedge Z = g(X, Z)Y - g(Y, Z)X, \tag{2.10}$$

$$g(X \wedge Y, Z \wedge W) = g(X, Z)g(Y, W) - g(Y, Z)g(X, W), \tag{2.11}$$

$$\|X \wedge Y\|^2 = g(X, X)g(Y, Y) - g(X, Y)^2. \tag{2.12}$$

f)

$$(X \wedge Y) \wedge Z + (Y \wedge Z) \wedge X + (Z \wedge X) \wedge Y = 0,$$

Proof.   Proofs of (a), (b) are clear from equation (2.5).

Now we prove the other cases:

c) Replacing X by ϕX in equation (2.5), we have

$$\begin{aligned} \phi(X) \wedge Y &= -g(\phi X, \phi Y)\xi - \eta(Y)\phi^2 X \\ &= -(g(X, Y) + \eta(X)\eta(Y))\,\xi - \eta(Y)\,(-X + \eta(X)\xi) \\ &= -(g(X, Y)\xi - \eta(Y)X). \end{aligned}$$

Then we get

$$Y \wedge \phi(X) = g(X, Y)\xi - \eta(Y)X.$$

Replacing Y by ξ in equation (2.5), we obtain $\phi X = X \wedge \xi$.

d) If we calculate $g(X \wedge Y, Z)$, then we find

$$\begin{aligned} (X, Y, Z) &= g(X \wedge Y, Z) \\ &= -g(X, \phi Y)g(\xi, Z) - \eta(Y)g(\phi X, Z) + \eta(X)g(\phi Y, Z) \\ &= -(g(X, \phi(Y))\eta(Z) + g(Y, \phi(Z))\eta(X) + g(Z, \phi(X))\eta(Y)). \end{aligned}$$

Then we can easily see that

$$(X, Y, Z) = (Y, Z, X) = (Z, X, Y).$$

e) We know that the dimension of $D = \left\{ X \in \chi(M^3) : \eta(X) = 0 \right\}$ is 2. If e is a unit vector in D, then $\varphi = \{e, \phi e, \xi\}$ is an orthonormal basis for $\chi(M)$. Then we can write $X = X_1 e + X_2 \phi e + X_3 \xi$, $Y = Y_1 e + Y_2 \phi e + Y_3 \xi$, $Z = Z_1 e + Z_2 \phi e + Y_3 \xi$ where $X, Y, Z \in \chi(M^3)$. Since $\phi Y = -Y_2 e + Y_1 \phi e$, we have

$$\begin{aligned} g(X, \phi Y) &= g(X_1 e + X_2 \phi e + X_3 \xi, -Y_2 e + Y_1 \phi e) \\ &= X_2 Y_1 - X_1 Y_2. \end{aligned}$$

If we calculate $g(Y, \phi(Z))$ and $g(Z, \phi(X))$, we obtain

$$g(X, \phi(Y))Z + g(Y, \phi(Z))X + g(Z, \phi(X))Y = -\det(X, Y, Z)\ \xi.$$

Using equation (2.5), (2.6) and (2.7) we have

$$
\begin{aligned}
(X \wedge Y) \wedge Z &= -\left(X_2 Y_1 - X_1 Y_2\right)\left(-Z_2 e + Z_1 \phi e\right) \\
&\quad + \left(Z_1(Y_3 X_1 - Y_1 X_3) + Z_2(Y_3 X_2 - Y_2 X_3)\right)\xi \\
&\quad - Z_3\left((Y_3 X_1 - Y_1 X_3)e + (Y_3 X_2 - Y_2 X_3)\phi e\right) \\
&= \left(X_1 Z_1 + X_2 Z_2 + X_3 Z_3\right)\left(Y_1 e + Y_2 \phi e + Y_3 \xi\right) \\
&\quad - \left(Y_1 Z_1 + Y_2 Z_2 + Y_3 Z_3\right)\left(X_1 e + X_2 \phi e + X_3 \xi\right) \\
&= g(X, Z)Y - g(Y, Z)X.
\end{aligned}
$$

Then we get

$$
\begin{aligned}
g(X \wedge Y, Z \wedge W) &= (X \wedge Y, Z, W) \\
&= g((X \wedge Y) \wedge Z, W) \\
&= g(X, Z)g(Y, W) - g(Y, Z)g(X, W)
\end{aligned}
$$

and

$$
\begin{aligned}
\|X \wedge Y\|^2 &= g(X \wedge Y, X \wedge Y) \\
&= g(X, X)g(Y, Y) - g(X, Y)^2.
\end{aligned}
$$

f) From equation (2.10), we can easily see that

$$
(X \wedge Y) \wedge Z + (Y \wedge Z) \wedge X + (Z \wedge X) \wedge Y = 0.
$$

$\square$

Theorem 2.2 *Let* $M^3 = (M, \phi, \xi, \eta, g)$ *be a 3-dimensional almost contact metric manifold. Then for all* $X, Y, Z \in TM$ *we have*

$$
\nabla_Z(X \wedge Y) = (\nabla_Z X) \wedge Y + X \wedge (\nabla_Z Y) \tag{2.13}
$$

where ∇ *is Levi-Civita connection on* M^3.

Proof. Using equation (2.5), we have

$$
\begin{aligned}
\nabla_Z(X \wedge Y) &= -g(\nabla_Z X, \phi Y)\xi - \eta(Y)\phi(\nabla_Z X) + \eta(\nabla_Z X)\phi(Y) \\
&\quad - g(X, \phi(\nabla_Z Y))\xi - \eta(\nabla_Z Y)\phi X + \eta(X)\phi(\nabla_Z Y) \\
&\quad - \eta(Y)\left(\nabla_Z \phi X - \phi(\nabla_Z X)\right) - g(X, \nabla_Z \phi Y - \phi(\nabla_Z Y))\xi \\
&\quad + \eta(X)\left(\nabla_Z \phi Y - \phi(\nabla_Z Y)\right) \\
&\quad + \left(g(\nabla_Z X, \xi) + g(X, \nabla_Z \xi)\right)\phi Y + \eta(X)\nabla_Z \phi Y,
\end{aligned}
$$

so we get

$$
\begin{aligned}
\nabla_Z(X \wedge Y) &= (\nabla_Z X) \wedge Y + X \wedge (\nabla_Z Y) \\
&\quad - g(X, (\nabla_Z \phi)Y)\xi - \eta(Y)(\nabla_Z \phi)X + \eta(X)(\nabla_Z \phi)Y \\
&\quad - g(X, \phi Y)\nabla_Z \xi - g(Y, \nabla_Z \xi)\phi X + g(X, \nabla_Z \xi)\phi Y.
\end{aligned} \tag{2.14}
$$

From equation (1.2), we have

$$-g(X, (\nabla_Z \phi)Y)\xi - \eta(Y)(\nabla_Z \phi)X + \eta(X)(\nabla_Z \phi)Y = 0.$$

In 3-dimensional almost contact metric manifold, $\{\phi Z, \phi^2 Z, \xi\}$ is orthogonal basis. Addition, $\nabla_Z \xi$ is orthogonal ξ. Thus we have

$$\nabla_Z \xi = a\phi Z + b\phi^2 Z. \tag{2.15}$$

From equation (2.15) and (2.9), we have

$$-g(X, \phi Y)\nabla_Z \xi - g(Y, \nabla_Z \xi)\phi X + g(X, \nabla_Z \xi)\phi Y = 0.$$

Using the equation (2.14), proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3. A curve theory 3-dimensional almost contact metric nanifold

In 3-dimensional almost-contact metric manifold there exist two types of Frenet frames. Let $M^3 = (M, \phi, \xi, \eta, g)$ be a 3-dimensional almost contact metric manifold and γ be a regular curve in M^3 parametrized by arc length.

First Type Frenet Frame (Usual Type Frenet Frame).

In this space, it is well known $t = \gamma'(s)$, $\kappa = \|\nabla_t t\|$, $n = \frac{\nabla_t t}{\kappa}$. Using the new cross product, we have $b = t \wedge n$, which is orthogonal to two independent directions. In this way we can define the third vector in the Frenet frame to be cross product of the first two unit vectors and the torsion to be the component of $\nabla_t n$ in the third direction. So we have $\tau = g(\nabla_t n, b)$, where $\{t, n, b\}$ is a usual Frenet frame and κ, τ are the curvature and the torsion of the curve, respectively, where ∇ is Levi-Civita connection on M^3.

Second Type Frenet Frame.

From Gram-Schmidt procedure, we have

$$E_1 = t$$

$$E_2 = \nabla_t t - g(\nabla_t t, V_1)V_1$$

$$E_2 = \nabla_t^2 t - g(\nabla_t^2 t, V_1)V_1 - g(\nabla_t^2 t, V_2)V_2,$$

where $t = \gamma'(s)$, $V_1 = \frac{E_1}{\|E_1\|} = t$, $V_2 = \frac{E_2}{\|E_2\|}$ and $V_3 = \frac{E_3}{\|E_3\|}$. Thus curvatures of the curve are obtained by $k_1 = g(\nabla_t V_1, V_2)$ and $k_2 = g(\nabla_t V_2, V_3)$.

There are some difference between the usual type Frenet Frame and second type Frenet Frame. In the second type Frenet Frame, H. Gluck [3] proved that curvatures of the curve are given by

$$k_1 = \frac{\|E_2\|}{\|E_1\|}$$

and

$$k_2 = \frac{\|E_3\|}{\|E_2\|}.$$

Thus k_1 and k_2 are not negative. In the paper by Baikuossis and Blair [1], they use second type Frenet Frame, in which they assumed that curvatures of the curve are positive. In this paper, we use the usual type Frenet Frame.

Relations between the usual type Frenet Frame and the second type Frenet Frame are given by

$$t = V_1, n = V_2, b = \varepsilon V_3, \kappa = k_1, \tau = \varepsilon k_2, \tag{3.16}$$

where $\varepsilon = det(V_1, V_2, V_3) = \pm 1$ is the orientation of the curve. Using the equations in (3.16), we state the following corollaries.

Corollary 3.1 *i)* $\tau > 0$ *if and only if* $\varepsilon = 1$

ii) $\tau < 0$ *if and only if* $\varepsilon = -1$.

Corollary 3.2 $k_2 = 1$ *if and only if one the following condition are satisfied*

Case 1: $\tau = 1$ *($\varepsilon = 1$);*

Case 2: $\tau = -1$ *($\varepsilon = -1$).*

Considering the notations above we state the following equations and propositions.

$$\eta(t)t + \eta(n)n + \eta(b)b = \xi. \tag{3.17}$$

Using equation (3.17), we have

$$\eta(t)^2 + \eta(n)^2 + \eta(b)^2 = 1 \tag{3.18}$$

and

$$t \wedge n = b, n \wedge b = t, b \wedge t = n. \tag{3.19}$$

Proposition 3.1 *Let* $M^3 = (M, \phi, \xi, \eta, g)$ *be a 3-dimensional almost contact metric manifold and* γ *a regular curve in* M^3 *parametrized by arc length. Then the following equations are satisfied:*

$$\phi t = \eta(b)n - \eta(n)b, \tag{3.20}$$

$$\phi n = \eta(t)b - \eta(b)t \tag{3.21}$$

and

$$\phi b = \eta(n)t - \eta(t)n. \tag{3.22}$$

Proof. Using equations (3.17) and (3.19), we have

$$\begin{aligned} \phi t &= \xi \wedge t \\ &= \eta(t)t \wedge t + \eta(n)n \wedge t + \eta(b)b \wedge t \\ &= \eta(b)n - \eta(n)b. \end{aligned}$$

The proofs of the equations (3.21) and (3.22) are similar. This proves the proposition. □

Proposition 3.2 *Let γ be a regular curve in a 3-dimensional contact metric manifold $M^3 = (M, \phi, \xi, \eta, g)$ parameterized by arc length. Then the following equations hold:*

$$\sigma'_t = \kappa\sigma_n - g(t, \phi ht) \tag{3.23}$$

$$\sigma'_n = -\kappa\sigma_t + (\tau - 1)\sigma_b - g(n, \phi ht) \tag{3.24}$$

$$\sigma'_b = -(\tau - 1)\sigma_n - g(b, \phi ht), \tag{3.25}$$

where $\sigma(s) = \sigma_t(s) = \eta(t) = g(t, \xi)$, $\sigma_n(s) = \eta(n) = g(n, \xi)$ and $\sigma_b(s) = \eta(b) = g(b, \xi)$.

Proof.   It is known that

$$\nabla_X \xi = -\phi(X) - \phi h(X) \tag{3.26}$$

in a contact metric manifold [2]. From (3.26), it is easily seen that

$$\sigma'_t = g(\nabla_t t, \xi) + g(t, \nabla_t \xi) = \kappa n - g(t, \phi ht).$$

The proofs of the equation (3.24) and (3.25) are similar. So our proposition is proved.   □

4.   A curve theory in 3-dimensional sasakian manifold

Theorem 4.1 *Let γ be a regular curve in a 3-dimensional Sasakian manifold $M^3 = (M, \phi, \xi, \eta, g)$ parametrized by arc length. Then the equations below hold*

$$\sigma'_t = \kappa\sigma_n \tag{4.27}$$

$$\sigma'_n = -\kappa\sigma_t + (\tau - 1)\sigma_b \tag{4.28}$$

$$\sigma'_b = -(\tau - 1)\sigma_n. \tag{4.29}$$

Proof.   In a 3-dimensional Sasakian manifold, it is known that $h = \frac{1}{2}\mathcal{L}_\xi\phi = 0$ [1]. Considering equations (3.23), (3.24) and (3.25), we easily obtain equations (4.27), (4.28) and (4.29).   □

Remark 4.1 *Let $M^3 = (M, \phi, \xi, \eta, g)$ be a 3-dimensional Sasakian manifold and γ be a regular Legendre curve in this manifold parameterized by arc length. In this case, Baikousiss and Blair [1] found that $t = \gamma'(s)$, $n = \pm\phi t$, and $k_2 = 1$. If $n = \phi t$, then using equations (2.7), (2.10) and (3.19), we have $b = \xi$ and $\tau = 1$. If $n = -\phi t$, then by similar method we have $b = -\xi$ and $\tau = 1(\varepsilon = 1)$. They assumed that the torsion k_2 of the curve is a positive quantity, while in the current notation we assume that τ can be positive or negative. In [1], Baikousiss and Blair assumed that the torsion (k_2) of the curve is equal to 1, while in the current notation torsion (τ) is equal to 1 ($\varepsilon = 1$) or -1 ($\varepsilon = -1$). Because of the notation in [1], Proposition 1.1 and Theorem 1.2 seems to be incomplete and Theorem 1.1 appears to be (incorrectly) too much generalized (in [1], they used absolute value of τ). In order to solve these problems, we present another proof of Proposition 1.1.*

Proof. If the curve is a Legendre curve, then we have $\sigma = \sigma_t = 0$. Using equations (4.27), (4.28) and (4.29), we have

$$\sigma_n = 0 \text{ and } (\tau - 1)\sigma_b = 0$$

and using equation (3.18), we obtain $\sigma_b = \pm 1 \neq 0$. Then we have $\tau = 1 \ (\varepsilon = 1)$. $\qquad \square$

We give an example for Proposition 1.1.

Example 4.1 *In $R^3(-3)$ Sasakian space, from [1], it is known that*

$$\nabla_e \phi e = \xi = -\nabla_{\phi e} e, \nabla_e \xi = -\phi e = \nabla_\xi e$$
$$\nabla_{\phi e} \xi = e = \nabla_\xi \phi e, \nabla_e e = \nabla_{\phi e} \phi e = \nabla_\xi \xi = 0.$$

Then we have

$$\nabla_X Y = X[Y_1]e + X[Y_2]\phi e + X[Y_3]\xi - \eta(X)\phi Y - \eta(Y)\phi X - g(X, \phi Y)\xi,$$

where $X = X_1 e + X_2 \phi e + X_3 \xi$, $Y = Y_1 e + Y_2 \phi e + Y_3 \xi$. For all s, let $\kappa > 0$. Suppose that $\gamma(s) = (x(s), y(s), z(s))$ is a regular curve with respect to the standard basis defined by

$$x'(s) = -2\sin \int \kappa ds$$

$$y'(s) = 2\cos \int \kappa ds$$

$$z'(s) = -4 \left(\int \left(\cos \int \kappa ds \right) ds \right) \sin \int \kappa ds.$$

Since $t = \gamma'(s)$, we have

$$t = \left(\cos \int \kappa ds \right) e - \left(\sin \int \kappa ds \right) \phi e,$$

which gives us $\eta(t) = 0$ (i.e. the curve is a Legendre curve). If we calculate $t' = \nabla_t t$ and $t'' = \nabla_t t'$, then we have

$$t' = -\left(\kappa \sin \int \kappa ds \right) e - \left(\kappa \cos \int \kappa ds \right) \phi e = -\kappa \phi t.$$

So we obtain

$$t = \left(\cos \int \kappa ds \right) e - \left(\sin \int \kappa ds \right) \phi e$$

$$n = -\left(\sin \int \kappa ds \right) e - \left(\cos \int \kappa ds \right) \phi e.$$

The result appears to be $n = -\phi t$, $b = t \wedge n = -\xi$ and $\tau = g(\nabla_t n, b) = 1$.

The following example is a counterexample of Theorem 1.1 and necessary condition for Theorem 1.3.

Example 4.2 *In $R^3(-3)$ Sasakian space, we define $\gamma(s) = (x(s), y(s), z(s))$ by*

$$
\begin{aligned}
x'(s) &= -2\sqrt{1-\sigma^2}\sin\theta \\
y'(s) &= 2\sqrt{1-\sigma^2}\cos\theta \\
z'(s) &= 2\sigma + yx'(s),
\end{aligned}
$$

where

$$
\begin{aligned}
\theta' &= -2\sigma + \frac{2}{1+\sigma} \\
\sigma(s) &= \frac{1}{2}\left(1 - \cos\left(2\sqrt{2}s\right)\right),
\end{aligned}
$$

Then the tangent vector becomes

$$
t = (\sqrt{1-\sigma^2}\cos\theta)e + (-\sqrt{1-\sigma^2}\sin\theta)\phi(e) + \sigma\xi
$$

and

$$
\begin{aligned}
t' &= (\frac{-\sigma\sigma'}{\sqrt{1-\sigma^2}}\cos\theta - (\theta'+2\sigma)\sqrt{1-\sigma^2}\sin\theta)e \\
&\quad + (\frac{\sigma\sigma'}{\sqrt{1-\sigma^2}}\sin\theta - (\theta'+2\sigma)\sqrt{1-\sigma^2}\cos\theta)\phi(e) + \sigma'\xi.
\end{aligned}
$$

Since $\kappa^2 = \| \nabla_t t \|^2$, we have

$$
\kappa^2 = (\frac{-\sigma\sigma'}{\sqrt{1-\sigma^2}})^2 + ((\theta'+2\sigma)\sqrt{1-\sigma^2})^2 + (\sigma')^2 = 4.
$$

Thus we have $\kappa = 2$ and $n = \frac{1}{2}\nabla_t t$. From equation (2.5) and (3.19), we get

$$
\sigma_b = -\sqrt{1 - \sigma^2 - (\frac{\sigma'}{2})^2}.
$$

From equations (4.27) and (4.28), we find

$$
(\frac{\sigma'}{2})' + 2\sigma = -(\tau - 1)\sqrt{1 - \sigma^2 - (\frac{\sigma'}{2})^2}.
$$

Evaluating $(\frac{\sigma'}{2})' + 2\sigma$ and $1 - \sigma^2 - (\frac{\sigma'}{2})^2$, we have

$$
(\frac{\sigma'}{2})' + 2\sigma = 1 - \sin\alpha
$$

and

$$
1 - \sigma^2 - (\frac{\sigma'}{2})^2 = \frac{(1 - \sin\alpha)^2}{4},
$$

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where $\alpha = 2\sqrt{2}s - \dfrac{\pi}{2}$. Thus we obtain

$$1 - sin\alpha = -\frac{(\tau - 1)}{2}(1 - sin\alpha)$$

and $\tau = -1$. We see that absolute value of the torsion is equal to 1 and at one point $\sigma = \sigma' = 0$. However, although the curve γ in a Sasakian space, it is not a Legendre curve. Furthermore, we see that ratio of $\tau - 1$ and κ is equal to -1 but the curve is not a slant curve.

For the curves in a Sasakian 3-manifold with $\tau = -1$, we state the following theorem.

Theorem 4.2 *Let γ be a regular curve in 3-dimensional Sasakian manifold parameterized by arc length with $\frac{\tau - 1}{\kappa} = c$ and at one point $\sigma = \sigma' = 0$. Then the torsion of the curve is equal to -1 if and only if*

$$\sigma(s) = \pm \left(\frac{c}{1 + c^2} - \frac{c}{1 + c^2} \cos\left(\frac{2\sqrt{1 + c^2}}{c}s \right) \right), \qquad (4.30)$$

where $\mp c\sigma + 1 \geq 0$.

Proof.   From equation (4.27) and (4.28), we have

$$\sigma_n(s) = \frac{\sigma'(s)}{\kappa} \text{ and } \left(\frac{\sigma'(s)}{\kappa} \right)' + \kappa\sigma(s) = (\tau - 1)\sigma_b.$$

Since $\sigma^2 + \sigma_n^2 + \sigma_b^2 = 1$, we obtain

$$\sigma_b(s) = \pm\sqrt{1 - \sigma^2(s) - \left(\frac{\sigma'(s)}{\kappa} \right)^2}.$$

Thus we have

$$\left(\frac{\sigma'(s)}{\kappa} \right)' + \kappa\sigma(s) = \pm(\tau - 1)\sqrt{1 - \sigma^2(s) - \left(\frac{\sigma'(s)}{\kappa} \right)^2}. \qquad (4.31)$$

If we assume that $r(s) = \int_0^s \kappa dh$, from equation (4.31) we have

$$\ddot{\sigma}(r) + \sigma(r) = \pm\frac{(\tau - 1)}{\kappa}\sqrt{1 - \sigma^2(r) - (\dot{\sigma}(r))^2}, \qquad (4.32)$$

where $\dot{\sigma} = \dfrac{d\sigma}{dr}$. Setting $\lambda = 1 - \sigma^2(s) - (\dot{\sigma})^2$, the equation (4.32) becomes

$$\frac{\dot{\lambda}}{\sqrt{\lambda}} = \mp\frac{\tau - 1}{\kappa}\,\dot{\sigma}.$$

If we integrate the equation this equation, we have

$$\sqrt{1 - \sigma^2 - (\dot{\sigma})^2} = \mp \int_0^t \frac{\tau - 1}{\kappa}d\sigma + C_1, \qquad (4.33)$$

where C_1 is a constant. Since $r(0) = 0$, we see that $\sigma(0) = \dot{\sigma}(0) = 0$. If at one point $\sigma = \dot{\sigma} = 0$, then we have $C_1 = 1$ and

$$\sqrt{1 - \sigma^2 - (\dot{\sigma})^2} = \mp \int_0^t \frac{\tau - 1}{\kappa} d\sigma + 1. \tag{4.34}$$

Considering the fact $\frac{\tau-1}{\kappa} = c$ and from equation (4.33), we get

$$\sqrt{1 - \sigma^2 - (\dot{\sigma})^2} = \mp c\sigma + 1,$$

where $\mp c\sigma + 1 \geq 0$. From the above equation, we have

$$(\dot{\sigma})^2 + (1 + c^2)\sigma^2 \mp 2c\sigma = 0.$$

Integration of this equation gives

$$\sigma(t) = \pm \left(\frac{c}{1 + c^2} - \frac{c}{1 + c^2} \sin\left(\sqrt{1 + c^2}t + C_2\right) \right)$$

and

$$\sigma(t) = \pm \left(\frac{c}{1 + c^2} + \frac{c}{1 + c^2} \sin\left(\sqrt{1 + c^2}t + C_3\right) \right),$$

or it can be written in terms of s as follows

$$\sigma(s) = \pm \left(\frac{c}{1+c^2} - \frac{c}{1+c^2} \sin\left(\sqrt{1 + c^2} \int_0^s \kappa \, dh + C_2\right) \right)$$
$$\sigma(s) = \pm \left(\frac{c}{1+c^2} + \frac{c}{1+c^2} \sin\left(\sqrt{1 + c^2} \int_0^s \kappa \, dh + C_3\right) \right). \tag{4.35}$$

Since $\sigma(0) = \sigma'(0) = 0$, we have $C_2 = \frac{\pi}{2}$ and $C_3 = -\frac{\pi}{2}$. If $\sigma_b \geq 0$, then we have

$$\sigma(s) = \left(\frac{c}{1 + c^2} - \frac{c}{1 + c^2} \cos\left(\sqrt{1 + c^2} \int_0^s \kappa \, dh\right) \right). \tag{4.36}$$

If $\sigma_b \leq 0$, then we have

$$\sigma(s) = -\left(\frac{c}{1 + c^2} - \frac{c}{1 + c^2} \cos\left(\sqrt{1 + c^2} \int_0^s \kappa \, dh\right) \right).$$

If torsion of the curve is equal to -1, we have $\kappa = \frac{-2}{c}$ and $\int_0^s \kappa \, dh = \frac{-2}{c}s$. Then we obtain the equation (4.30).

Conversely, suppose that we have one of the two condition as

$$\sigma_b = \sqrt{1 - \sigma^2(s) - \left(\frac{\sigma'(s)}{\kappa}\right)^2} \geq 0$$

and

$$\sigma(s) = \left(\frac{c}{1+c^2} - \frac{c}{1+c^2} \cos\left(\frac{2\sqrt{1+c^2}}{c} s \right) \right), \qquad (4.37)$$

where $-c\sigma + 1 \geq 0$. So from equation (4.36) and (4.37), we obtain

$$\sqrt{1+c^2} \int\limits_0^s \kappa dh = (2k+1)\pi - \frac{2\sqrt{1+c^2}}{c} s,$$

which gives us $\kappa = \frac{-2}{c}$ and $\tau = -1$. $\qquad\qquad\qquad\square$

A new proof of Theorem 1.1:

If $\tau = 1$, from equation (4.33) we have

$$\ddot{\sigma} + \sigma = 0.$$

Integrating the above equation, we have

$$\sigma(s) = A \cos\left(\int\limits_a^s \kappa dh \right) + B \sin\left(\int\limits_a^s \kappa dh \right).$$

If at one point $\sigma(0) = \sigma'(0) = 0$, then we find

$$A \cos\left(\int\limits_a^0 \kappa dh \right) + B \sin\left(\int\limits_a^0 \kappa dh \right) = 0,$$

$$-A \sin\left(\int\limits_a^0 \kappa dh \right) + B \cos\left(\int\limits_a^0 \kappa dh \right) = 0.$$

So we have $A = B = 0$ and $\sigma = 0$. Thus the curve is a Legendre curve.

A new proof of Theorem 1.2:

Let γ be a Legendre curve in a 3-dimensional contact metric manifold parametrized by arc length. Since $h(\xi) = 0$, Then $\lambda_1 = 0$ is eigenvalue and ξ is an eigenvector. Since h is a self-adjoint operator, its eigenvalues are real and the eigenvectors (corresponding to the nonzero eigenvalues) are perpendicular to ξ [2]. Thus an eigencurve of h in a contact metric manifold is a Legendre curve. If γ is an eigencurve of h, we have $\sigma_t(s) = \sigma'_t(s) = 0$, where $h(t) = \lambda t$. From (3.23) , (3.24) and (3.25), we see that

$$\sigma_n(s) = 0 \text{ and } (\tau - 1 - \lambda)\sigma_b = 0.$$

If we assume that $\tau = 1$, we have

$$-\lambda \sigma_b = 0.$$

By equation (3.21), we have $\sigma_b = \pm 1 \neq 0$. From the above equation since σ_b is not equal to zero, λ must be zero. It is well known that if λ is eigenvalue of h, then $-\lambda$ is eigenvalue of h [2]. Thus we have $h = 0$. Using equation (1.4) and (1.3), as a result, we see that the manifold is a Sasakian manifold.

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