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Warped product submanifolds of a Kenmotsu manifold

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Abstract

In the present paper, we study warped product semi-slant submanifolds of a Kenmotsu manifold. We obtain some results on the existence of such type warped product submanifolds of a Kenmotsu manifold with an example.

Key Words: Warped product, slant submanifold, semi-slant submanifold, Kenmotsu manifold, canonical structure

1. Introduction

In [13] S. Tanno classified the connected almost contact metric manifold whose automorphism group has maximum dimension; there are three classes:

(a) Homogeneous normal contact Riemannian manifolds with constant ϕ holomorphic sectional curvature if the sectional curvature of the plane section containing ξ, say $C(X,\xi) > 0$.

(b) Global Riemannian product of a line or a circle and a Kaehlerian manifold with constant holomorphic sectional curvature, $C(X,\xi) = 0$.

(c) A warped product space $\mathbb{R} \times_\lambda \mathbb{C}^n$, if $C(X,\xi) < 0$.

Manifolds of class (a) are characterized by some tensor equations, it has a Sasakian structure and manifolds of class (b) are characterized by a tensorial relation admitting a cosymplectic structure. Kenmotsu [7] obtained some tensorial equations to characterize manifolds of class (c). As Kenmotsu manifolds are themselves warped product spaces, it is interesting to study warped product submanifolds in Kenmotsu manifolds.

The notion of semi-slant submanifolds of almost Hermitian manifolds were introduced by N. Papaghuic [11]. In the setting of almost contact metric manifolds, semi-slant submanifolds are defined and investigated by J. L. Cabrerizo et al. [4].

In [2] R. L. Bishop and B. O'Neill introduced the notion of warped product manifolds. These manifolds appear in differential geometric studies in a natural way. The study of warped product submanifolds of Kaehler manifolds was introduced by B. Y. Chen [6]. After that, B. Sahin extended the results of Chen's for warped

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product semi-slant submanifolds of Kaehler manifolds [12]. Later on, K. A. Khan et al. studied warped product semi-slant submanifolds in cosymplectic manifolds and showed that there exist no proper warped product semi-slant submanifolds in the form $N_T \times {}_\lambda N_\theta$ and reversing the two factors in cosymplectic manifolds [8].

Recently, M. Atçeken proved that the warped product submanifolds of the types $M = N_\theta \times {}_\lambda N_T$ and $M = N_\theta \times {}_\lambda N_\perp$ of a Kenmotsu manifold $\bar{M}$ do not exist where the manifolds N_θ and N_T (resp., $N_\perp$) are proper slant and ϕ-invariant (resp. anti-invariant) submanifolds of a Kenmotsu manifold $\bar{M}$, respectively [1]. In this paper, we study the warped product of the types $M = N_T \times {}_\lambda N_\theta$ and $M = N_\perp \times {}_\lambda N_\theta$ which has not been attempted in [1] and obtain some new results for the existence of warped product semi-slant submanifolds of a Kenmotsu manifold $\bar{M}$.

2. Preliminaries

Let $\bar{M}$ be an almost contact metric manifold with structure (ϕ, ξ, η, g) where ϕ is a $(1,1)$ tensor field, ξ a vector field, η is a 1-form and g is a Riemannian metric on $\bar{M}$ satisfying the following properties [3]:

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1. \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.2}$$

These conditions also imply that

$$g(\phi X, Y) + g(X, \phi Y) = 0 \tag{2.3}$$

for all vector fields X, Y on $\bar{M}$. If in addition to the above relations,

$$(\bar{\nabla}_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X \tag{2.4}$$

holds, then $\bar{M}$ is said to be *Kenmotsu* manifold, where $\bar{\nabla}$ is the Levi-Civita connection of g. From (2.4), it follows that

$$\bar{\nabla}_X \xi = X - \eta(X)\xi. \tag{2.5}$$

Let M be submanifold of an almost contact metric manifold $\bar{M}$ with induced metric g and if ∇ and $\nabla^\perp$ are the induced connections on the tangent bundle TM and the normal bundle $T^\perp M$ of M, respectively then Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.6}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^\perp_X N, \tag{2.7}$$

for each X, $Y \in TM$ and $N \in T^\perp M$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N) respectively for the immersion of M into $\bar{M}$. They are related as

$$g(h(X, Y), N) = g(A_N X, Y), \tag{2.8}$$

where g denotes the Riemannian metric on $\bar{M}$ as well as the one induced on M.

For any $X \in TM$, we write

$$\phi X = PX + FX, \tag{2.9}$$

where PX is the tangential component and FX is the normal component of ϕX.

Similarly for any $N \in T^{\perp}M$, we write

$$\phi N = tN + fN, \tag{2.10}$$

where tN is the tangential component and fN is the normal component of ϕN. From (2.3) and (2.9) we have

$$g(PX, Y) + g(X, PY) = 0 \tag{2.11}$$

for all $X,\ Y \in TM$. The covariant derivatives of the tensor fields P and F are defined as

$$(\bar{\nabla}_X P)Y = \nabla_X PY - P\nabla_X Y, \tag{2.12}$$

$$(\bar{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y. \tag{2.13}$$

The canonical structures P and F on a submanifold M are said to be *parallel* if $\bar{\nabla} P = 0$ and $\bar{\nabla} F = 0$, respectively. On a submanifold of a Kenmotsu manifold by equations (2.5) and (2.6), we get

$$\nabla_X \xi = X - \eta(X)\xi \tag{2.14}$$

and

$$h(X, \xi) = 0 \tag{2.15}$$

for each $X \in TM$. Furthermore, from equation (2.15)

$$A_N \xi = 0, \quad \eta(A_N X) = 0 \tag{2.16}$$

for each $N \in T^{\perp}M$ and $X \in TM$. Also, from equations (2.4), (2.6), (2.7) (2.9), (2.10), (2.12) and (2.13), we obtain

$$(\bar{\nabla}_X P)Y = A_{FY}X + th(X, Y) - g(X, PY)\xi - \eta(Y)PX, \tag{2.17}$$

$$(\bar{\nabla}_X F)Y = fh(X, Y) - h(X, PY) - \eta(Y)FX. \tag{2.18}$$

We shall always consider ξ to be tangent to M. The submanifold M is said to be *invariant* if F is identically zero, that is, $\phi X \in TM$ for any $X \in TM$. On the other hand M is said to be *anti-invariant* if P is identically zero, that is, $\phi X \in T^{\perp}M$, for any $X \in TM$.

For each non zero vector X tangent to M at x, such that X is not proportional to ξ, we denotes by $\theta(X)$, the angle between ϕX and PX.

M is said to be *slant* [5] if the angle $\theta(X)$ is constant for all $X \in TM - \{\xi\}$ and $x \in M$. The angle θ is called a *slant angle* or *Wirtinger angle*. Obviously, if $\theta = 0$, M is invariant and if $\theta = \pi/2$, M is an anti-invariant submanifold. If the slant angle of M is different from 0 and $\pi/2$ then it is called *proper slant*.

A characterization of slant submanifolds is given by the following theorem.

Theorem 2.1 [5] *Let M be a submanifold of an almost contact metric manifold $\bar{M}$ such that $\xi \in TM$. Then M is slant if and only if there exists a constant $\delta \in [0, 1]$ such that*

$$P^2 = \delta(-I + \eta \otimes \xi). \tag{2.19}$$

Furthermore, in such case, if θ is slant angle, then $\delta = \cos^2\theta$.

The following relations are straight forward consequence of equation (2.19):

$$g(PX, PY) = \cos^2\theta[g(X, Y) - \eta(X)\eta(Y)] \tag{2.20}$$

$$g(FX, FY) = \sin^2\theta[g(X, Y) - \eta(X)\eta(Y)] \tag{2.21}$$

for any X, Y tangent to M.

3. Warped product submanifolds

Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds and λ, a positive differentiable function on N_1. The warped product of N_1 and N_2 is the Riemannian manifold $N_1 \times {}_\lambda N_2 = (N_1 \times N_2, g)$, where

$$g = g_1 + \lambda^2 g_2. \tag{3.1}$$

A warped product manifold $N_1 \times {}_\lambda N_2$ is said to be *trivial* if the warping function λ is constant.We recall the following general formula on a warped product [2].

$$\nabla_X V = \nabla_V X = (X \ln\lambda)V, \tag{3.2}$$

where X is tangent to N_1 and V is tangent to N_2.

Let $M = N_1 \times {}_\lambda N_2$ be a warped product manifold, this means that N_1 is totally geodesic and N_2 is totally umbilical submanifold of M, respectively.

The following corollary shows that the warped product of the type $M = N_1 \times {}_\lambda N_2$ is trivial if $\xi \in TN_2$.

Corollary 3.1 [9] *Let $\bar{M}$ be a Kenmotsu manifold and N_1 and N_2 be any Riemannian submanifolds of $\bar{M}$.Then there does not exist a warped product submanifold $M = N_1 \times {}_\lambda N_2$ of $\bar{M}$ such that ξ is tangential to N_2.*

Thus, throughout we assume that the structure vector filed ξ is tangential to the first factor N_1 of a warped product submanifold $N_1 \times {}_\lambda N_2$ of $\bar{M}$. In this case, first we obtain some useful formulae for later use.

Lemma 3.1 *Let $M = N_1 \times {}_\lambda N_2$ be warped product submanifold of a Kenmotsu manifold $\bar{M}$ such that N_1 tangent to ξ, where N_1 and N_2 are any Riemannian submanifolds of $\bar{M}$. Then*

(i) $\xi \ln\lambda = 1$,

(ii) $g(h(X, Y), FZ) = g(h(X, Z), FY)$,

(iii) $g(h(X, Z), FW) = g(h(X, W), FZ)$

for any X, $Y \in TN_1$ and Z, $W \in TN_2$.

Proof. Consider the warped product $M = N_1 \times {}_\lambda N_2$ of a Kenmotsu manifold $\bar{M}$ and $X \in TN_1$ and $Z \in TN_2$, then

$$\nabla_X Z = \nabla_Z X = (X \ln\lambda)Z.$$

In particular $\xi \in TN_1$, then the above formula becomes

$$\nabla_Z \xi = (\xi \ln \lambda) Z. \tag{3.3}$$

Also, by (2.14) and the fact that ξ is tangent to N_1, we get

$$\nabla_Z \xi = Z. \tag{3.4}$$

Equations (3.3) and (3.4) follow the first part of the Lemma.

Now, for any $X \in TN_1$ and $Z \in TN_2$ we have

$$(\bar{\nabla}_X \phi) Z = \bar{\nabla}_X \phi Z - \phi \bar{\nabla}_X Z.$$

Using (2.4) and the fact that $\xi \in TN_1$, left hand side of the above equation is zero by orthogonality of two distributions, then

$$\bar{\nabla}_X \phi Z = \phi \bar{\nabla}_X Z.$$

By (2.6), (2.7), (2.9) and (2.10) we obtain

$$\nabla_X PZ + h(X, PZ) - A_{FZ} X + \nabla^{\perp}_X FZ = P \nabla_X Z + F \nabla_X Z + th(X, Z) + fh(X, Z).$$

Equating the tangential components and using (2.12), we get

$$(\bar{\nabla}_X P) Z = A_{FZ} X + th(X, Z). \tag{3.5}$$

As $\xi \in TN_1$ then using formulae (2.12) and (3.2), the left hand side of the above equation is zero, then

$$A_{FZ} X = -th(X, Z). \tag{3.6}$$

Parts (ii) and (iii) follow by taking the product in (3.6) with $Y \in TN_1$ and $W \in TN_2$, respectively. □

In the following section we shall investigate warped product semi-slant submanifolds of a Kenmotsu manifold.

4. Warped product semi-slant submanifolds

The study of semi-slant submanifolds of almost Hermitian manifolds was introduced by N. Papaghuic [11], which were latter extended to almost contact metric manifold by J. L. Cabrerizo et al. [4]. A semi-slant submanifold M of an almost contact metric manifold $\bar{M}$ is a submanifold which admits two orthogonal complementary distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ such that $\mathcal{D}_1$ is invariant under ϕ and $\mathcal{D}_2$ is slant with slant angle $\theta \neq 0$ i.e., $\phi \mathcal{D}_1 = \mathcal{D}_1$ and ϕZ makes a constant angle θ with TM for each $Z \in \mathcal{D}_2$. In particular, if $\theta = \frac{\pi}{2}$, then a semi-slant submanifold reduces to a contact CR-submanifold. For a semi-slant submanifold M of an almost contact metric manifold, we have

$$TM = D_1 \oplus D_2 \oplus \{\xi\}.$$

Similarly we say M is *anti-slant submanifold* of $\bar{M}$ if $\mathcal{D}_1$ is an anti-invariant distribution of M i.e., $\phi\mathcal{D}_1 \subseteq T^{\perp}M$ and $\mathcal{D}_2$ is slant with slant angle $\theta \neq 0$. For a slant submanifold the orthogonal complement of $F\mathcal{D}_2$ in the normal bundle $T^{\perp}M$, is an invariant subbundle of $T^{\perp}M$ and is denoted by μ. Thus, we have

$$T^{\perp}M = F\mathcal{D}_2 \oplus \mu.$$

Let M be a semi-slant submanifold and for any $X \in TM$. Then as $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \{\xi\}$, we can write

$$X = T_1X + T_2X + \eta(X)\xi \tag{4.1}$$

where $T_1X \in \mathcal{D}_1$ and $T_2X \in \mathcal{D}_2$. Now by using equations (2.9) and (4.1), we have

$$\phi X = \phi T_1X + PT_2X + FT_2X. \tag{4.2}$$

As $\mathcal{D}_1$ is invariant under ϕ, we obtain

$$\phi T_1X = PT_1X, \quad FT_1X = 0, \quad PT_2X \in \mathcal{D}_2. \tag{4.3}$$

Thus

$$PX = \phi T_1X + PT_2X \tag{4.4}$$

and

$$FX = FT_2X. \tag{4.5}$$

From Corollary 3.1 we have seen that the warped product submanifolds of the type $M = N_1 \times {}_\lambda N_2$ of a Kenmotsu manifold $\bar{M}$ do not exist if the structure vector field ξ is tangent to N_2. Thus, in this section we study warped product semi-slant submanifolds $M = N_1 \times {}_\lambda N_2$ of $\bar{M}$, when $\xi \in TN_1$. If the manifolds N_θ and N_T (resp. $N_\perp$) are slant and invariant (resp. anti-invariant) submanifolds of a Kenmotsu manifold $\bar{M}$, then their warped product semi-slant submanifolds may given by one of the following forms:

(i) $N_T \times {}_\lambda N_\theta$, *(ii)* $N_\perp \times {}_\lambda N_\theta$,

(iii) $N_\theta \times {}_\lambda N_T$, *(iv)* $N_\theta \times {}_\lambda N_\perp$.

In this paper we are concerned with cases (i) and (ii) and are left with the last two cases as these are already studied by M. Atcȩken [1]. For the warped products of the type *(i)*, we have the following lemma.

Lemma 4.1 *Let $M = N_T \times {}_\lambda N_\theta$ be warped product semi-slant submanifold of a Kenmotsu manifold $\bar{M}$ such that ξ is tangent to N_T where N_T and N_θ are invariant and proper slant submanifolds of $\bar{M}$. Then*

(i) $g(h(X,Z), FPZ) = g(h(X,PZ), FZ) = \{X \ln\lambda - \eta(X)\}\cos^2\theta\|Z\|^2$,

(ii) $g(h(X,Z), FZ) = -(PX \ln\lambda)\|Z\|^2$,

for any $X \in TN_T$ and $Z \in TN_\theta$.

Proof. The equality first and second of *(i)* follows directly by Lemma 3.1 *(iii)*. Now, for any $X \in TN_T$ and $Z \in TN_\theta$ we have

$$(\bar{\nabla}_X\phi)Z = \bar{\nabla}_X\phi Z - \phi\bar{\nabla}_XZ.$$

On using (2.4) and the fact that ξ is tangent to N_T, the left hand side of the above equation is zero by orthogonality of two distributions, then

$$\bar{\nabla}_X \phi Z = \phi \bar{\nabla}_X Z.$$

Thus, from (2.6), (2.7), (2.9) and (2.10) we obtain

$$\nabla_X PZ + h(X, PZ) - A_{FZ}X + \nabla^{\perp}_X FZ = P\nabla_X Z + F\nabla_X Z + th(X, Z) + fh(X, Z).$$

Equating the tangential and normal components and using (2.12), (2.13) we get

$$(\bar{\nabla}_X P)Z = A_{FZ}X + th(X, Z) \tag{4.6}$$

and

$$(\bar{\nabla}_X F)Z = fh(X, Z) - h(X, PZ). \tag{4.7}$$

On the Other hand for any $X \in TN_T$ and $Z \in TN_\theta$, we have

$$(\bar{\nabla}_Z \phi)X = \bar{\nabla}_Z \phi X - \phi \bar{\nabla}_Z X.$$

Using the structure equation of Kenmotsu manifold and the fact that ξ is tangent to N_T, we get

$$g(PZ, X)\xi - \eta(X)\phi Z = \nabla_Z \phi X + h(Z, \phi X) - P\nabla_Z X - F\nabla_Z X$$

$$- th(X, Z) - fh(X, Z).$$

Thus, by (4.3) and orthogonality of two distributions we obtain

$$-\eta(X)PZ - \eta(X)FZ = \nabla_Z PX + h(Z, PX) - P\nabla_Z X - F\nabla_Z X$$

$$- th(X, Z) - fh(X, Z).$$

Equating tangential and normal components, we get

$$(\bar{\nabla}_Z P)X = th(X, Z) - \eta(X)PZ \tag{4.8}$$

and

$$F\nabla_Z X = h(PX, Z) - fh(X, Z) + \eta(X)FZ. \tag{4.9}$$

Then, from (4.6) and (4.8) we have

$$(\bar{\nabla}_X P)Z + (\bar{\nabla}_Z P)X = A_{FZ}X + 2th(X, Z) - \eta(X)PZ. \tag{4.10}$$

Using (2.12) and (3.2), we obtain

$$(PX \ln \lambda)Z - (X \ln \lambda)PZ = A_{FZ}X + 2th(X, Z) - \eta(X)PZ. \tag{4.11}$$

Taking product with PZ and then using (2.8), we get

$$-(X \ln \lambda)g(PZ, PZ) = g(h(X, PZ), FZ) + 2g(th(X, Z), PZ) - \eta(X)g(PZ, PZ).$$

Then on applying (2.20) we obtain

$$\{X \ln \lambda - \eta(X)\} \cos^2 \theta \|Z\|^2 = -2g(\phi h(X,Z), PZ) - g(h(X,PZ), FZ),$$

$$\text{or,} \ \ \{X \ln \lambda - \eta(X)\} \cos^2 \theta \|Z\|^2 = 2g(h(X,Z), FPZ) - g(h(X,PZ), FZ).$$

Thus by Lemma 3.1 (iii), we obtain

$$g(h(X,Z), FPZ) = \{X \ln \lambda - \eta(X)\} \cos^2 \theta \|Z\|^2. \tag{4.12}$$

This is the first and third equality of (i).

Now, for part (ii), taking product in (4.11) with $Z \in TN_\theta$ we obtain

$$(PX \ln \lambda)\|Z\|^2 = g(h(X,Z), FZ) + 2g(th(X,Z), Z)$$

$$\text{or,} \quad (PX \ln \lambda)\|Z\|^2 = g(h(X,Z), FZ) - 2g(h(X,Z), FZ),$$

that is,

$$g(h(X,Z), FZ) = -(PX \ln \lambda)\|Z\|^2.$$

Thus, the proof is complete. □

The following theorems provide an explicit mechanism of warped product semi-slant submanifold $M = N_T \times {}_\lambda N_\theta$ of Kenmotsu manifold.

Theorem 4.1 *Let $M = N_T \times {}_\lambda N_\theta$ be warped product semi-slant submanifold of a Kenmotsu manifold $\bar{M}$ such that $h(X,Z) \in \mu$. Then at least one of the following statements is true:*

(i) $X \ln \lambda = \eta(X)$*,*

(ii) M *is a CR-warped product,*

(iii) M *is an invariant submanifold*

for each $X \in TN_T$ and $Z \in TN_\theta$.

Proof. The given statement is $h(X,Z) \in \mu$ for each $X \in TN_T$ and $Z \in TN_\theta$, then by (4.12) we have

$$\{X \ln \lambda - \eta(X)\} \cos^2 \theta \|Z\|^2 = 0.$$

This means that either $X \ln \lambda - \eta(X) = 0$ or $\theta = \frac{\pi}{2}$ i.e., $M = N_T \times {}_\lambda N_\perp$ is a CR-warped product submanifold or $N_\theta = \{0\}$. This proves the theorem. □

Theorem 4.2 *Let $M = N_T \times {}_\lambda N_\theta$ be warped product semi-slant submanifold of a Kenmotsu manifold $\bar{M}$ such that $\xi \in TN_T$. Then $(\bar{\nabla}_X F)Z \in \mu$ for each $X \in TN_T$ and $Z \in TN_\theta$, where μ is an invariant normal subbundle of TM.*

Proof. As ξ is tangent to TN_T, then by (2.2) we have

$$g(\phi\bar{\nabla}_X Z, \phi Z) = g(\bar{\nabla}_X Z, Z),$$

for any $X \in TN_T$ and $Z \in TN_\theta$. Using (2.6) and (3.2), the above equation takes the form

$$g(\phi\bar{\nabla}_X Z, \phi Z) = g(\nabla_X Z, Z) = (X \ln \lambda)\|Z\|^2. \tag{4.13}$$

On the other hand, we have

$$(\bar{\nabla}_X \phi)Z = \bar{\nabla}_X \phi Z - \phi\bar{\nabla}_X Z,$$

for any $X \in TN_T$ and $Z \in TN_\theta$. On using (2.4) and the fact that $\xi \in TN_T$, then by orthogonality of two distributions the left hand side of the above equation is zero, that is

$$\phi\bar{\nabla}_X Z = \bar{\nabla}_X \phi Z.$$

Then by (2.9), we have

$$\phi\bar{\nabla}_X Z = \bar{\nabla}_X PZ + \bar{\nabla}_X FZ.$$

On using (2.6) and (2.7), we obtain

$$\phi\bar{\nabla}_X Z = \nabla_X PZ + h(X, PZ) - A_{FZ}X + \nabla^{\perp}_X FZ.$$

Taking product with ϕZ and using (2.8), (2.9), we get

$$g(\phi\bar{\nabla}_X Z, \phi Z) = g(\nabla_X PZ, PZ) + g(\nabla^{\perp}_X FZ, FZ).$$

Thus by (2.13) and (3.2) we obtain

$$g(\phi\bar{\nabla}_X Z, \phi Z) = (X \ln \lambda)g(PZ, PZ) + g((\bar{\nabla}_X F)Z, FZ) + g(F\nabla_X Z, FZ).$$

Which, on using (2.20) and (2.21), implies

$$g(\phi\bar{\nabla}_X Z, \phi Z) = (X \ln \lambda)\cos^2\theta\|Z\|^2 + g((\bar{\nabla}_X F)Z, FZ) + \sin^2\theta g(\nabla_X Z, Z).$$

By (3.2) and (4.13), we get

$$(X \ln \lambda)\|Z\|^2 = (X \ln \lambda)\cos^2\theta\|Z\|^2 + g((\bar{\nabla}_X F)Z, FZ) + (X \ln \lambda)\sin^2\theta\|Z\|^2.$$

Therefore,

$$g((\bar{\nabla}_X F)Z, FZ) = 0. \tag{4.14}$$

As $Z \in TN_\perp$, then $FZ \in FTM$ then by orthogonality of normal space, we obtain $(\bar{\nabla}_X F)Z \in \mu$. This proves the theorem completely. □

The other case is dealt with by the following theorem.

Theorem 4.3 *Let $M = N_\perp \times_\lambda N_\theta$ be warped product semi-slant submanifold of a Kenmotsu manifold $\bar{M}$ such that $\xi \in TN_\perp$. Then at least one of the following statements is true:*

(i) $\eta(Z) = Z\ln\lambda$,

(ii) M *is an anti-invariant submanifold*

for each $Z \in TN_\perp$.

Proof. For any $X \in TN_\theta$ and $Z \in TN_\perp$, we have

$$(\bar{\nabla}_X\phi)Z = \bar{\nabla}_X\phi Z - \phi\bar{\nabla}_X Z.$$

Using (2.4), (2.6), (2.7) and (2.9) we obtain

$$g(PX,Z)\xi - \eta(Z)PX - \eta(Z)FX = -A_{FZ}X + \nabla^\perp_X FZ - P\nabla_X Z - F\nabla_X Z$$
$$- th(X,Z) - fh(X,Z).$$

By (4.3) and the orthogonality of two distributions first term of the left hand side is zero, then the tangential components are

$$\eta(Z)PX = A_{FZ}X + P\nabla_X Z + th(X,Z).$$

Thus by (3.2), we have

$$\eta(Z)PX = A_{FZ}X + (Z\ln\lambda)PX + th(X,Z). \tag{4.15}$$

Taking product with PX in equation (4.15) and making use of formulae (2.8) and (2.20) we obtain

$$\eta(Z)\cos^2\theta\|X\|^2 = g(h(X,PX),FZ) + (Z\ln\lambda)\cos^2\theta\|X\|^2 + g(th(X,Z),PX).$$

That is,

$$\{\eta(Z) - Z\ln\lambda\}\cos^2\theta\|X\|^2 = g(h(X,PX),FZ) - g(h(X,Z),FPX). \tag{4.16}$$

As $\theta \neq \pi/2$, interchanging X by PX in (4.16) and taking account of equation (2.19), we deduce that

$$\{\eta(Z) - Z\ln\lambda\}\cos^4\theta\|X\|^2 = -\cos^2\theta g(h(X,PX),FZ) + \cos^2\theta g(h(PX,Z),FX),$$

i.e.,

$$\{\eta(Z) - Z\ln\lambda\}\cos^2\theta\|X\|^2 = -g(h(X,PX),FZ) + g(h(PX,Z),FX). \tag{4.17}$$

Adding equations (4.16) and (4.17), we get

$$2\{\eta(Z) - Z\ln\lambda\}\cos^2\theta\|X\|^2 = g(h(PX,Z),FX) - g(h(X,Z),FPX).$$

The right hand side of the above equation is zero by Lemma 3.1 (iii), then

$$\{Z\ln\lambda - \eta(Z)\}\cos^2\theta\|X\|^2 = 0. \tag{4.18}$$

Showing that either $Z\ln\lambda = \eta(Z)$ or $\theta = \frac{\pi}{2}$ or $N_\theta = \{0\}$. This completes the proof. □

In the following we construct an example of warped product submanifold $M = N_\perp \times_f N_\theta$, which is not trivial.

Example 4.1 Consider the complex space $\mathbb{C}^5$ with the usual Kaehler structure and real global coordinates $(x^1, y^1, x^2, y^2, x^3, y^3, x^4, y^4, x^5, y^5)$. Let $\bar{M} = \mathbb{R} \times_f \mathbb{C}^5$ be the warped product between the real line $\mathbb{R}$ and $\mathbb{C}^5$, where the warping function is $f = e^t$, t being the global coordinate on $\mathbb{R}$. Then $\bar{M}$ is Kenmotsu manifold. Now, define the orthonormal frame of the tangent space TM by $\{e_1, e_2, \cdots, e_7\}$ and for any $\theta \in (0, \pi/2)$, the tangent bundle TM is spanned by

$$e_1 = \cos\theta \frac{\partial}{\partial x^1} + \sin\theta \frac{\partial}{\partial y^3}, \quad e_2 = -\sin\theta \frac{\partial}{\partial x^3} + \cos\theta \frac{\partial}{\partial y^1},$$

$$e_3 = \cos\theta \frac{\partial}{\partial x^2} - \sin\theta \frac{\partial}{\partial y^4}, \quad e_4 = \sin\theta \frac{\partial}{\partial x^4} + \cos\theta \frac{\partial}{\partial y^2},$$

$$e_5 = \frac{\partial}{\partial x^3}, \quad e_6 = \frac{\partial}{\partial x^4}, \quad e_7 = \frac{\partial}{\partial t}.$$

Then the distributions $D_\theta = span\{e_1, e_2, e_3, e_4\}$ and $D^\perp = span\{e_5, e_6, e_7\}$ which obviously are integrable. Let us denote by N_θ and $N_\perp$ their, integral submanifolds, respectively. Then their metrics are $g_{N_\theta} = \sum_{i=1}^{2}((dx^i)^2 + (dy^i)^2)$ and $g_{N_\perp} = dt^2 + e^{2t} \sum_{a=3}^{4}(dx^a)^2$. Then, $M = N_\perp \times_f N_\theta$ is a warped product submanifold, isometrically immersed in $\bar{M}$. The warping function is given by $f = e^t$.

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