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On α-skew McCoy modules*

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Abstract

Let α be a ring endomorphism. Extending the notions of McCoy modules and α-skew McCoy rings, we introduce the notion of α-skew McCoy modules, which can also be regarded as a generalization of α-skew Armendariz modules. A number of illustrative examples are given. Various properties of these modules are developed, and equivalent conditions for α-skew McCoy modules are established. Furthermore, we study the relationship between a module and its polynomial module.

Key Words: α-skew McCoy module; α-skew Armendariz module; α-skew McCoy ring; polynomial module; zip module

1. Introduction

Throughout this paper all rings considered are associative with unity and all modules are unitary right modules. $R[x]$ denotes the polynomial ring over a ring R and $M[x]$ denotes the polynomial module over a module M. Let $\mathbb{Z}_n$ be the ring of integers modulo n. The symbol I_n stands for the $n \times n$ identity matrix. For a set $X \subseteq M$, $r_R(X)$ stands for the right annihilator of X in R.

Rege and Chhawchharia [23] and Nielsen [22] independently called a ring R *right McCoy* if whenever $f(x)g(x) = 0$ for $f(x) \in R[x]$ and $g(x) \in R[x]\backslash\{0\}$, there exists a nonzero $r \in R$ with $f(x)r = 0$. Left McCoy rings are defined similarly. A ring is said to be *McCoy* if it is both right and left McCoy. The term "McCoy ring" was coined because McCoy [21] had shown that every commutative ring satisfies the above mentioned condition. The class of McCoy rings properly contains the class of Armendariz rings. (These rings are defined through the condition: whenever polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for every i and j. See [23] for basic results on Armendariz rings). Recall that a ring R is *semicommutative* provided $ab = 0$ implies $aRb = 0$ for $a, b \in R$. In [13] it was claimed that all semicommutative rings were McCoy. However, Hirano's claim assumed that $R[x]$ is semicommutative if R is semicommutative, and this was shown to be false in [16]. In 2006, Nielsen [22] gave an example of semicommutative ring which is not right McCoy. Some other properties on McCoy rings have appeared in [5], [11], [18], [20], [23, 24, 25], etc. As a generalization of McCoy rings (resp., Armendariz rings), McCoy

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modules [7] (resp., Armendariz modules [4]) were introduced (Maybe the first result, without a naming, McCoy module, obtained in [2]). A module M_R is said to be *McCoy* (resp., *Armendariz*) if whenever polynomials $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x]$ and $g(x) = \sum_{j=0}^{q} b_j x^j \in R[x]\backslash\{0\}$ satisfy $m(x)g(x) = 0$, there exists $r \in R\backslash\{0\}$ such that $m(x)r = 0$ (resp., $m_i b_j = 0$ for every i and j). Armendariz modules are clearly McCoy.

Given an endomorphism α of a ring R, the *skew polynomial* ring $R[x;\alpha]$ consists of the polynomials in x with coefficients in R written on the left, subject to the relation $xr = \alpha(r)x$ for all $r \in R$. Recently, Başer, Kwak and Lee [3] called a ring R *α-skew McCoy* with respect to an endomorphism α of R if for any nonzero polynomials $f(x)$ and $g(x) \in R[x;\alpha]$, $f(x)g(x) = 0$ implies $f(x)r = 0$ for some nonzero $r \in R$. This notion generalized both concepts of McCoy rings and α-skew Armendariz rings (see [14]).

In this paper, we introduce the notion of α-skew McCoy modules as a straightforward extensions to modules. Many examples of α-skew McCoy modules are given, and properties of this class of modules are investigated. Various results of α-skew McCoy rings are extended to α-skew McCoy modules. We also study the relationship between a module and its polynomial module.

2. α-skew McCoy modules

Let α be an endomorphism of a ring R and M be a right R-module. $M[x;\alpha] = \{\sum_{i=0}^{s} m_i x^i; s \geq 0, m_i \in M\}$ is an abelian group under an obvious addition operation. Moreover, $M[x;\alpha]$ becomes a module over $R[x;\alpha]$ under the following scalar product operation: For $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x;\alpha]$ and $f(x) = \sum_{j=0}^{q} a_j x^j \in R[x;\alpha]$, $m(x)f(x) = \sum_k (\sum_{i+j=k} m_i \alpha^i(a_j))x^k$. According to Zhang and Chen [27], M is *α-skew Armendariz* if $m(x)f(x) = 0$ where $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x;\alpha]$ and $f(x) = \sum_{j=0}^{q} a_j x^j \in R[x;\alpha]$ implies $m_i \alpha^i(a_j) = 0$ for all i and j.

Definition 2.1 *Let α be an endomorphism of a ring R and M be an R-module. M is called α-skew McCoy if whenever $m(x)g(x) = 0$ where $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x;\alpha]$ and $g(x) = \sum_{j=0}^{q} b_j x^j \in R[x;\alpha]\backslash\{0\}$, there exists a nonzero element $r \in R$ such that $m(x)r = 0$ (i.e., $m_i \alpha^i(r) = 0$ for all i).*

Remark 2.2 (1) *M is a McCoy R-module if and only if M is 1_R-skew McCoy, where 1_R is the identity endomorphism of R.*
(2) *A ring R is α-skew McCoy if and only if R_R is an α-skew McCoy module.*
(3) *An R-module M is α-skew McCoy if and only if, for all $m(x) \in M[x;\alpha]$, $r_{R[x;\alpha]}(m(x)) \neq 0$ implies that $r_{R[x;\alpha]}(m(x)) \bigcap R \neq 0$.*

Any α-skew Armendariz module is obviously α-skew McCoy, the falsity of the converse can be inferred from [17, Example 3] or [23, Remark 4.3].

Example 2.3 (1) *Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and $\alpha : R \to R$ be defined by $\alpha((a,b)) = (b,a)$. Then R_R is McCoy but not α-skew McCoy by [3, Example 4] and Remark 2.2(2).*
(2) *For any given ring S, let $R = \mathbb{T}_2(S)$ be the ring of all 2×2 upper triangular matrices over S. Then R_R is not McCoy by [5, Proposition 10.2]. Define $\alpha : R \to R$ by $\alpha(\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)) = \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$. We conclude that R_R is*

α-*skew McCoy. Indeed, suppose that* $F(x)G(x)=0$ *for* $F(x)=\sum_{i=0}^{p}A_ix^i\in R[x;\alpha]$ *and* $G(x)=\sum_{j=0}^{q}B_jx^j\in R[x;\alpha]\backslash\{0\}$. *We may assume that* $B_0\neq 0$ *and write* $B_0=\left(\begin{smallmatrix} b_1 & b_2\\ 0 & b_3\end{smallmatrix}\right)$. *If* $b_1=0$ *then let* $C=\left(\begin{smallmatrix} 0 & b_2\\ 0 & b_3\end{smallmatrix}\right)$, *otherwise, let* $C=\left(\begin{smallmatrix} 0 & b_1\\ 0 & 0\end{smallmatrix}\right)$. *It easily checks that* $A_i\alpha^i(C)=0$ *for both cases, where* $i=0,\dots,p$.

An ideal I of a ring R is an α-*ideal* if $\alpha(I)\subseteq I$, where α is an endomorphism of R.

Proposition 2.4 (1) *Every submodule of an* α-*skew McCoy module is* α-*skew McCoy. In particular, if* I *is a right ideal of an* α-*skew McCoy ring* R, *then* I *is* α-*skew McCoy.*
(2) M *is an* α-*skew McCoy module if and only if every finitely generated submodule of* M *is* α-*skew McCoy.*
(3) *For any index set* Γ, *if* M_i *is an* α_i-*skew McCoy* R_i-*module for each* $i\in\Gamma$, *then* $\prod_{i\in\Gamma}M_i$ *is an* α-*skew McCoy* $\prod_{i\in\Gamma}R_i$-*module, where* $\alpha=(\alpha_i)_{i\in\Gamma}$.
(4) *Let* I *be any nonzero* α-*ideal of a ring* R, *then* R/I *is an* α-*skew McCoy* R-*module.*

Proof. (1) - (3) are obvious. (4) For each $\overline{f(x)}\in(R/I)[x;\alpha]$, take any nonzero $r\in I\ (\subseteq R)$. Since $\alpha(I)\subseteq I$, $f(x)r\in I[x;\alpha]$, i.e., $\overline{f(x)}r=0$. □

Remark 2.5 *The condition "*I *is an* α-*ideal" in Proposition 2.4(4) is necessary. Take the ring and the ring endomorphism in Example 2.3(1). Let* $I=0\oplus\mathbb{Z}_2\subseteq R$. *Then* I *is an ideal but* $\alpha(I)\subsetneq I$. *Note that* $R/I\cong\mathbb{Z}_2\oplus 0$. *We show that* R/I *is not* α-*skew McCoy as a right* R-*module. For* $f(x)=(1,0)+(1,0)x\in(\mathbb{Z}_2\oplus 0)[x;\alpha]$ *and* $g(x)=(0,1)+(1,0)x\in R[x;\alpha]$, $f(x)g(x)=0$. *However,* $f(x)r=0$ *implies* $r=0$ *for* $r\in R$.

A module M_R is *semicommutative* [4] if for any $m\in M$ and $a\in R$, $ma=0$ implies $mRa=0$. In [27], a module M_R with a ring endomorphism α of R is called α-*semicommutative* if whenever $ma=0$ for $m\in M$ and $a\in R$, $mR\alpha(a)=0$; a ring R is α-*semicommutative* if R_R is α-semicommutative. We can infer that 1_R-semicommutative modules need not be 1_R-McCoy from Section 3 of [22].

Proposition 2.6 *Let* α *be an endomorphism of a ring* R. *Then a semicommutative module* M_R *with* $m\alpha(a)=0$ *whenever* $m\alpha(a)a=0$ *for* $m\in M$ *and* $a\in R$ *is* α-*skew Armendariz.*

Proof. Firstly, we show that M_R is α-semicommutative. Let $ma=0$ for $m\in M$ and $a\in R$. Then $mRa=0$. Clearly, $m\alpha(a)a=0$. Thus $m\alpha(a)=0$ and $mR\alpha(a)=0$ by the hypotheses.

Let $m(x)=\sum_{i=0}^{p}m_ix^i\in M[x;\alpha]$ and $f(x)=\sum_{j=0}^{q}a_jx^j\in R[x;\alpha]\backslash\{0\}$ with $m(x)f(x)=0$. Then $\sum_{i+j=k}m_i\alpha^i(a_j)=0$ for $k=0,\dots,p+q$. So $m_0a_0=0$ and $m_0a_1+m_1\alpha(a_0)=0$, and then $m_0a_1a_0+m_1\alpha(a_0)a_0=0$. Since M_R is semicommutative, $m_0a_1a_0=0$. So we have $m_1\alpha(a_0)a_0=0$, and $m_1\alpha(a_0)=0$ by the hypothesis. Hence $m_0a_1=m_1\alpha(a_0)=0$. Assume that $s\geq 1$ and $m_i\alpha^i(a_j)=0$ for all i,j with $i+j\leq s$. Note that

$$m_0a_{s+1}+m_1\alpha(a_s)+\cdots+m_s\alpha^s(a_1)+m_{s+1}\alpha^{s+1}(a_0)=0, \tag{2.1}$$

where m_i and a_j are 0 if $i>p$ and $j>q$. Multiplying (2.1) by $\alpha^s(a_0)$ on the right yields

$$m_0a_{s+1}\alpha^s(a_0)+m_1\alpha(a_s)\alpha^s(a_0)+\cdots+m_s\alpha^s(a_1)\alpha^s(a_0)+m_{s+1}\alpha^{s+1}(a_0)\alpha^s(a_0)=0. \tag{2.2}$$

Since M_R is α-semicommutative and $m_i\alpha^i(a_0) = 0$ for $i \leq s$, it follows that $m_iR\alpha^s(a_0) = 0$. Thus (2.2) becomes $m_{s+1}\alpha^{s+1}(a_0)\alpha^s(a_0) = m_{s+1}\alpha(\alpha^s(a_0))\alpha^s(a_0) = 0$, which implies $m_{s+1}\alpha^{s+1}(a_0) = 0$ by the assumption. So (2.1) becomes

$$m_0a_{s+1} + m_1\alpha(a_s) + \cdots + m_{s-1}\alpha^{s-1}(a_2) + m_s\alpha^s(a_1) = 0. \tag{2.3}$$

Analogously, multiplying (2.3) by $\alpha^{s-1}(a_1)$ on the right, one obtains

$$m_0a_{s+1}\alpha^{s-1}(a_1) + m_1\alpha(a_s)\alpha^{s-1}(a_1) + \cdots + m_{s-1}\alpha^{s-1}(a_2)\alpha^{s-1}(a_1) + m_s\alpha^s(a_1)\alpha^{s-1}(a_1) = 0.$$

The similar argument as the above reveals that $m_s\alpha^s(a_1)\alpha^{s-1}(a_1) = 0$. Thus $m_s\alpha^s(a_1) = 0$. Continuing this process, we have $m_s\alpha^s(a_1) = \cdots = m_1\alpha(a_s) = m_0a_{s+1} = 0$. So we prove that $m_i\alpha^i(a_j) = 0$ for all i, j with $i + j \leq s + 1$. By the induction principle, $m_i\alpha^i(a_j) = 0$ for every i and j. □

The converse of Proposition 2.6 is not true. We use the ring given in [14].

Example 2.7 *Let* $R = \{\left(\begin{smallmatrix} a & \overline{b} \\ 0 & a \end{smallmatrix}\right) | a \in \mathbb{Z}, \overline{b} \in \mathbb{Z}_4\}$. *Clearly,* R *is commutative. Let* $\alpha : R \to R$ *be an endomorphism defined by*

$$\alpha(\left(\begin{smallmatrix} a & \overline{b} \\ 0 & a \end{smallmatrix}\right)) = \left(\begin{smallmatrix} a & -\overline{b} \\ 0 & a \end{smallmatrix}\right).$$

Then R_R *is* α*-skew Armendariz by [14, Example 7]. However,* $I_2\alpha(\left(\begin{smallmatrix} 0 & \overline{b} \\ 0 & 0 \end{smallmatrix}\right))\left(\begin{smallmatrix} 0 & \overline{b} \\ 0 & 0 \end{smallmatrix}\right) = 0$, *but* $I_2\alpha(\left(\begin{smallmatrix} 0 & \overline{b} \\ 0 & 0 \end{smallmatrix}\right)) \neq 0$ *in case* $\overline{b} \neq 0$.

Let α be an endomorphism of a ring R and M be an R-module. According to Lee and Zhou [19], M is called *α-reduced* if the following conditions hold: For any $m \in M$ and $a \in R$, (1) $ma = 0$ implies $mRa = mR\alpha(a) = 0$; (2) $ma\alpha(a) = 0$ implies $ma = 0$; (3) $ma^2 = 0$ implies $ma = 0$. A ring is *reduced* if R_R is 1_R-reduced.

Remark 2.8 *Assume that* M *is an* α*-reduced* R*-module. For some* $m \in M$ *and* $a \in R$ *with* $m\alpha(a)a = 0$, *by* (1) *we have* $m\alpha(a)\alpha(a) = m[\alpha(a)]^2 = 0$, *and so* $m\alpha(a) = 0$ *by applying condition* (3). *In view of Proposition 2.6, it is clear that any* α*-reduced module is* α*-skew Armendariz and is therefore* α*-skew McCoy.*

Proposition 2.9 *Let* α *be an endomorphism of a reduced ring* R. *Then every* α*-semicommutative module* M_R *is* α*-skew McCoy.*

Proof. Since M_R is α-semicommutative, it is easy to obtain that for $m \in M$ and $a \in R$,

$$ma = 0 \ \Rightarrow \ mR\alpha^s(a^t) = 0 \text{ for any } s, t \geq 1. \tag{2.4}$$

Suppose that $m(x) = m_0 + m_1x + \cdots + m_px^p \in M[x;\alpha]$, $f(x) = a_0 + a_1x + \cdots + a_qx^q \in R[x;\alpha]\backslash\{0\}$ satisfy $m(x)f(x) = 0$. We may assume that $m(x) \neq 0$ and k is minimal such that $m_k \neq 0$, and let l be minimal such that $a_l \neq 0$. Since $m(x)f(x) = 0$, we have the following equations:

$$\begin{array}{ll} (0) & m_k\alpha^k(a_l)=0, \\ (1) & m_{k+1}\alpha^{k+1}(a_l)+m_k\alpha^k(a_{l+1})=0, \\ & \dots \\ (p+q-k-l) & m_p\alpha^p(a_q)=0. \end{array}$$

If $\alpha^k(a_l)=0$ then $m(x)a_l=0$, and we are done. Next assume that $\alpha^k(a_l)\neq 0$. Multiplying Eq. (1) by $\alpha^{k+1}(a_l)$ from the right, we obtain $m_{k+1}\alpha^{k+1}(a_l)\alpha^{k+1}(a_l)+m_k\alpha^k(a_{l+1})\alpha^{k+1}(a_l)=0$. Combining Eq. (0) with (2.4), one has $m_k\alpha^k(a_{l+1})\alpha^{k+1}(a_l)=0$. Thus $m_{k+1}\alpha^{k+1}(a_l)\alpha^{k+1}(a_l)=m_{k+1}\alpha^{k+1}(a_l^2)=0$. Continuing this procedure, multiplying Eq. (i) on the right by $\alpha^{k+i}(a_l^i)$ yields $m_{k+i}\alpha^{k+i}(a_l^{i+1})=0$, where $i=1,2,\dots,p-k$. Let $r=a_l^p$. Then $r\neq 0$ since R is reduced. So, by (2.4) we get $m_i\alpha^i(r)=0$ for each i, proving that M_R is α-skew McCoy. □

In view of [7, Example 2.5], the converse of Proposition 2.9 does not hold generally.

Following [12], an endomorphism α of a ring R is called *compatible* if for each $a,\ b\in R$, $ab=0\Leftrightarrow a\alpha(b)=0$. A ring R is said to be α-*compatible* if there exists a compatible endomorphism α of R. We define the following:

Definition 2.10 (1) *An endomorphism α of a ring R is called weakly compatible (or W-compatible for short) if whenever $ab=0$ for $a,\ b\in R$, $a\alpha(b)=0$.*
(2) *An endomorphism α of a ring R is called weakly finitely compatible (or WF-compatible for short) if for a finite number of elements $a_i,b_i\in R$, $\sum_i a_ib_i=0$ implies $\sum_i a_i\alpha(b_i)=0$.*

The following examples reveal the relationships among the above endomorphisms (for a given ring).

Example 2.11 (1) *Both compatible and WF-compatible endomorphisms of given rings are W-compatible, but the converse is not true. Let $R=\mathbb{Z}_2[x]$, and $\alpha:R\to R$ be defined by $\alpha(f(x))=f(0)$ for $f(x)\in R$. Since R is an integral domain, $g(x)h(x)=0$ implies that either $g(x)=0$ or $h(x)=0$, so $g(x)\alpha(h(x))=0$. Hence α is W-compatible. But α is neither compatible nor WF-compatible. Indeed, let $f_1(x)=x,\ g_1(x)=1,\ f_2(x)=1$ and $g_2(x)=x$. Then $f_2(x)\alpha(g_2(x))=0$ and $f_1(x)g_1(x)+f_2(x)g_2(x)=0$, but both $f_2(x)g_2(x)$ and $f_1(x)\alpha(g_1(x))+f_2(x)\alpha(g_2(x))$ do not equal* 0.
(2) *WF-compatible endomorphisms need not be compatible. Given the ring and the ring endomorphism in Example 2.3(2), it is easy to check that α is WF-compatible. However, since $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\alpha(\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right))=0$ and $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)\neq 0$, α is not compatible.*
(3) *Let $R=\mathbb{Z}_2[x_1,x_2,\dots]$ be a ring of polynomials in infinitely countably many indeterminates. Define $\alpha:R\to R$ by $x_i\mapsto x_{i+1}$ for $i=1,2,\dots$. Notice that R is an integral domain and α is monic. So α is compatible. Nevertheless, since $x_2x_3+x_3x_2=0$ and $x_2\alpha(x_3)+x_3\alpha(x_2)=x_2x_4+(x_3)^2\neq 0$, α is not WF-compatible.*

For an endomorphism α of a ring R, write $\alpha(h(x))=\sum_{i=0}^n\alpha(c_i)x^i$, where $h(x)=\sum_{i=0}^n c_ix^i\in R[x;\alpha]$.

Lemma 2.12 *Let α be a WF-compatible endomorphism of a ring R. If $f(x)g(x)=0$ in $R[x;\alpha]$, then $f(x)\alpha(g(x))=0$.*

Proof. Suppose that $f(x)=\sum_{i=0}^{m}a_ix^i$ and $g(x)=\sum_{j=0}^{n}b_jx^j$ are elements of $R[x;\alpha]$ with $f(x)g(x)=0$. Then looking at the degree k part of the equation $f(x)g(x)=0$ we have $\sum_{i+j=k}a_i\alpha^i(b_j)=0$. Because α is WF-compatible, $0=\sum_{i+j=k}a_i\alpha^{i+1}(b_j)=\sum_{i+j=k}a_i\alpha^i(\alpha(b_j))$ for each k. Thus $f(x)\alpha(g(x))=0$. □

Lemma 2.13 *Let R be a semicommutative ring and α be a W-compatible endomorphism of R. Suppose that $f(x)g(x)=0$ for nonzero $f(x)=\sum_{i=0}^{m}a_ix^i$ and $g(x)=\sum_{j=0}^{n}b_jx^j$ of $R[x;\alpha]$. If $a_lg(x)\neq 0$ for some minimal index l then $a_l^{n+1}\alpha^l(g(x))=0$.*

Proof. A direct check shows that R is α-semicommutative. By hypothesis, $a_kg(x)=0$ for every $k<l$. So $a_kb_j=0$ for $j=0,\dots,n$. Since α is W-compatible, we have $a_k\alpha^k(b_j)=0$. It follows that $0=f(x)g(x)=(\sum_{i=l}^{m}a_ix^i)(\sum_{j=0}^{n}b_jx^j)$. One easily obtains the following system of equations:

$$\begin{array}{ll}(l) & a_l\alpha^l(b_0)=0,\\ (l+1) & a_l\alpha^l(b_1)+a_{l+1}\alpha^{l+1}(b_0)=0,\\ (l+2) & a_l\alpha^l(b_2)+a_{l+1}\alpha^{l+1}(b_1)+a_{l+2}\alpha^{l+2}(b_0)=0,\\ & \dots\\ (m+n) & a_m\alpha^m(b_n)=0.\end{array}$$

Since R is α-semicommutative, by Eq. (l) one has $a_la_{l+1}\alpha^{l+1}(b_0)=0$. Now multiplying Eq. $(l+1)$ by a_l on the left yields $a_l^2\alpha^l(b_1)=0$. Similarly, multiplying Eq. $(l+2)$ by a_l^2 from the left, we have $a_l^3\alpha^l(b_2)=0$ by using the α-semicommutativity of R. Repeating this process finite times, we obtain $a_l^{j+1}\alpha^l(b_j)=0$ for $j=0,\dots,n$. Thus $a_l^{n+1}\alpha^l(b_j)=0$, and so $a_l^{n+1}\alpha^l(g(x))=0$. □

A ring is said to be *right duo* (resp., *left duo*) if all its right (resp., left) ideals are two-sided ideals. It is not difficult to show that one-sided duo rings are semicommutative.

Lemma 2.14 *Suppose that R is a right duo ring and α is a WF-compatible automorphism of R. If $f(x)g(x)=0$ for nonzero $f(x)=\sum_{i=0}^{m}a_ix^i$, $g(x)=\sum_{j=0}^{n}b_jx^j$ of $R[x;\alpha]$ and $a_kg(x)\neq 0$ for minimal $k\geq 0$, then there exists $h(x)\in R[x;\alpha]\backslash\{0\}$ such that $f(x)h(x)=0$ and $a_ih(x)=0$ for all $i\leq k$.*

Proof. Since $a_kg(x)\neq 0$, there exists a minimal index l such that $a_kb_l\neq 0$. If $a_k\alpha^k(b_l)=0$ then let $h_1(x)=\alpha^k(g(x))$. As α is monic, $h_1(x)\neq 0$. Next assume that $a_k\alpha^k(b_l)\neq 0$. Note that k is minimal such that $a_kg(x)\neq 0$. By Lemma 2.13, there exists an integer $p\geq 1$ such that $a_k^{p+1}\alpha^k(b_l)=0\neq a_k^p\alpha^k(b_l)$. Since R is right duo, there exists $s\in R$ with $a_k^p\alpha^k(b_l)=\alpha^k(b_l)s$. As α is an automorphism, we may let $s=\alpha^l(r)$ for some $r\in R$. Write $h_1(x)=\alpha^k(g(x))r$. Then $h_1(x)\neq 0$ since $\alpha^k(b_l)\alpha^l(r)=\alpha^k(b_l)s\neq 0$. By Lemma 2.12, $f(x)h_1(x)=0$ for both cases. In addition, since $a_ib_j=0$ for all j and $i<k$, it follows that $a_0h_1(x)=a_1h_1(x)=a_{k-1}h_1(x)=0$ by using the W-compatibility of α, and a_k annihilates the first l coefficients of $h_1(x)$.

If a_k annihilates all coefficients of $h_1(x)$ then we are done by letting $h(x)=h_1(x)$. Otherwise, repeating the above procedure, and after finite times we can construct $h(x)\in R[x;\alpha]\backslash\{0\}$ satisfying $f(x)h(x)=0$ and for each $i\leq k$, $a_ih(x)=0$. □

Theorem 2.15 *Let R be a right duo ring and α be a WF-compatible automorphism of R. Then R is an α-skew McCoy ring.*

Proof. Let $f(x) = \sum_{i=0}^{m} a_i x^i,\ g(x) = \sum_{j=0}^{n} b_j x^j \in R[x;\alpha]\backslash\{0\}$ satisfy $f(x)g(x) = 0$. It suffices to show the following.

There exists $g'(x) \in R[x;\alpha]\backslash\{0\}$ such that $f(x)g'(x) = 0$ and for all i, $a_i g'(x) = 0$.

Note that α is WF-compatible. If $a_i g(x) = 0$ for all i then let $g'(x) = g(x)$. So $a_i\alpha^i(b_j) = 0$, which implies that $f(x)b_j = 0$ for each j, and we are done. Next we assume that $a_i g(x) \neq 0$ for some i. Let k be minimal such that $a_k g(x) \neq 0$. Then by Lemma 2.14, there exists a nonzero $h(x) \in R[x;\alpha]$ such that $f(x)h(x) = 0$ and $a_0 h(x) = a_1 h(x) = \cdots = a_k h(x) = 0$. Now, if $a_i h(x) = 0$ for all i, then the proof is finished by letting $g'(x) = h(x)$. If not, there must exist an integer i_0 $(> k)$ satisfying $a_{i_0} h(x) \neq 0$, and apply Lemma 2.14 again. So after finite times check, we can produce a nonzero polynomial $g'(x) \in R[x;\alpha]$ such that $f(x)g'(x) = 0$ and $a_i g'(x) = 0$ for all i. The proof is complete. □

Remark 2.16 *Notice that in Example 2.3(1), the ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is commutative, and thus duo. But R is not α-skew McCoy. So we conclude that the condition "α is a WF-compatible automorphism" in Theorem 2.15 is not superfluous.*

Corollary 2.17 *[5, Theorem 8.2] Right duo rings are necessarily right McCoy.*

Let α be an endomorphism of a ring R and M be an R-module. M is said to be *α-compatible* if for any $m \in M$ and $r \in R$, $mr = 0 \Leftrightarrow m\alpha(r) = 0$ (see [1]). Based on this, we call M *weakly α-compatible* (or *W-α-compatible* for short) if $m\alpha(r) = 0$ whenever $mr = 0$; and call M *weakly finitely α-compatible* (or *WF-α-compatible* for short) if for a finite number of elements $m_i \in M$ and $r_i \in R$, $\sum_i m_i r_i = 0$ implies $\sum_i m_i\alpha(r_i) = 0$.

Proposition 2.18 *Let R be a right duo ring and α be an automorphism of R. Then every WF-α-compatible cyclic R-module is α-skew McCoy.*

Proof. In view of Theorem 2.15, R_R is α-skew McCoy. Let N be a cyclic R-module. Then $N \cong R/I$ with $I = r_R(n)$ for some $n \in N$. By hypothesis, N is WF-α-compatible. Then for any $s \in I$, we have $ns = 0$, implying $n\alpha(s) = 0$. Thus $\alpha(I) \subseteq I$. Therefore, the result follows from Proposition 2.4(4). □

Recall that a module is called a *Bezout module* if each of its finitely generated submodules is cyclic.

Corollary 2.19 *Let R be a right duo ring with an automorphism α. Then WF-α-compatible Bezout R-modules are α-skew McCoy.*

Proof. By Proposition 2.18, every WF-α-compatible cyclic R-module is α-skew McCoy. Hence Bezout R-modules are α-skew McCoy by Proposition 2.4(2). □

In what follows R_n denotes (for a positive integer n) the following subring of the upper triangular matrix ring $T_n(R)$ over a ring R:

$$R_n = \{(a_{ij}) \in T_n(R) : a_{ij} \in R, a_{11} = a_{22} = \cdots = a_{nn}\};$$

we also consider the following subgroup of the additive group of all formal upper triangular matrices over M, namely,

$$M_n = \{(m_{ij}) \in T_n(M) : m_{ij} \in M, m_{11} = m_{22} = \cdots = m_{nn}\}.$$

Then M_n is an R_n-module under the usual matrix addition operation and the following scalar product operation. For $W = (w_{ij}) \in M_n$ and $A = (a_{ij}) \in R_n$, $WA = (m_{ij})$ with $m_{ij} = \sum_{k=1}^{n} w_{ik}a_{kj}$, for $i, j = 1, 2, \ldots, n$. An endomorphism α of R can be extended to an endomorphism $\overline{\alpha}$ of R_n defined by $\overline{\alpha}((a_{ij})) = (\alpha(a_{ij}))$.

Proposition 2.20 *A module M_R is α-skew McCoy if and only if M_n is $\overline{\alpha}$-skew McCoy as an R_n-module.*

Proof. The result for modules can be proved in exactly the same manner as that results for rings in [3, Theorem 14]. □

For a commutative domain R and a module M_R, the *torsion submodule* of M is defined by $T(M) = \{x \in M | r_R(x) \neq 0\}$; M is called *torsion free* if $T(M) = 0$.

Proposition 2.21 *Let α be a monomorphism of a commutative domain D and M be a D-module. Then M is α-skew McCoy if and only if its torsion submodule $T(M)$ is α-skew McCoy.*

Proof. Let $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x; \alpha]$ and $d(x) = \sum_{j=0}^{q} d_j x^j \in D[x; \alpha] \backslash \{0\}$ satisfy $m(x)d(x) = 0$. We have

(0) $m_0 d_0 = 0$,
(1) $m_0 d_1 + m_1 \alpha(d_0) = 0$,
(2) $m_0 d_2 + m_1 \alpha(d_1) + m_2 \alpha^2(d_0) = 0$,
$\cdots$
$(p+q)$ $m_p \alpha^p(d_q) = 0$.

We may assume that $d_0 \neq 0$. Then by Eq. (0), $m_0 \in T(M)$. Multiplying Eq. (1) by d_0 on the right, one obtains $m_1 \alpha(d_0) d_0 = 0$. Since α is monic and D is a domain, $m_1 \in T(M)$. Multiplying Eq. (2) by $\alpha(d_0)d_0$ from the right yields $m_2 \alpha^2(d_0)\alpha(d_0)d_0 = 0$, so $m_2 \in T(M)$. Repeating this process, we have $m(x) \in T(M)[x]$. Since $T(M)$ is α-skew McCoy, there exists $r \in R \backslash \{0\}$ satisfying $m_i \alpha^i(r) = 0$. This proves that M is an α-skew McCoy module. The other implication is trivial. □

By a similar proof as above, we have the following result.

Proposition 2.22 *Let α be an endomorphism of a commutative domain D and M be a torsion free D-module. Then M is an α-skew McCoy module.*

A module is *uniform* [8] if any two nonzero submodules have a nonzero intersection.

Lemma 2.23 *Let $\{M_i\}_{i \in \Lambda}$ be a family of α-skew McCoy R-modules with Λ an index set. If R_R is uniform, then a direct sum $M = \coprod_{i \in \Lambda} M_i$ is α-skew McCoy.*

Proof. Let $m(x) = \sum_{k=0}^{p} (m_{ik})_{i \in \Lambda} x^k \in M[x; \alpha]$, $g(x) \in R[x; \alpha] \backslash \{0\}$ satisfy $m(x)g(x) = 0$. Let $m_i(x) = \sum_{k=0}^{p} m_{ik} x^k \in M_i[x]$. Since $m_i(x)g(x) = 0$ and M_i is α-skew McCoy, there exists $r_i \in R \backslash \{0\}$ such that $m_i(x) r_i = 0$. Note that the set $\Lambda' = \{i \in \Lambda \mid m_i(x) \neq 0\}$ is finite. Put $U = \bigcap_{i \in \Lambda'} r_i R$. Then $U \neq 0$ since R_R

is uniform. Take any $r \in U\backslash\{0\}$. Then $m_i(x)r = 0$ for each i, whence $m(x)r = 0$. Thus, $M = \coprod_{i\in\Lambda} M_i$ is α-skew McCoy. □

Theorem 2.24 *Let α be an endomorphism of a ring R and R_R be uniform. Then R is α-skew McCoy if and only if every flat R-module is α-skew McCoy.*

Proof. Let M be a flat module. Let $0 \to K \to F \to M \to 0$ be an exact sequence with F free. (In what follows, for any $y \in F$, we denote $\overline{y} = y + K$ in M). Let $m(x) = \sum_{i=0}^{p} \overline{y}_i x^i \in M[x;\alpha]$ and $g(x) = \sum_{j=0}^{q} b_j x^j \in R[x;\alpha]\backslash\{0\}$ satisfy $m(x)g(x) = 0$, then we have

$$\sum_{i+j=k} \overline{y_i}\alpha^i(b_j) = 0 \text{ for } k = 0, \ldots, p+q.$$

Therefore $y_0b_0, y_0b_1 + y_1\alpha(b_0), \ldots, y_p\alpha^p(b_q)$ all belong to K. Since M is a flat R-module, there exists an R-homomorphism $\nu : F \to K$ such that $\nu(y_0b_0) = y_0b_0, \nu(y_0b_1 + y_1\alpha(b_0)) = y_0b_1 + y_1\alpha(b_0), \ldots, \nu(y_p\alpha^p(b_q)) = y_p\alpha^p(b_q)$. Write $w_i := \nu(y_i) - y_i$ for $i = 0, \ldots, p$. Each w_i is an element of F and therefore the polynomial $n(x) = \sum_{i=0}^{p} w_i x^i \in F[x;\alpha]$ and $n(x)g(x) = 0$. Since R is α-skew McCoy and F_R is free, by Lemma 2.23 F is α-skew McCoy. Thus, there exists a nonzero $r \in R$ such that $w_i\alpha^i(r) = 0$ for all i. It follows that $y_i\alpha^i(r) \in K$, and so $\overline{y}_i\alpha^i(r) = 0$ in M, proving that M is α-skew McCoy. The other implication is obvious. □

Question: Can the words "R_R is uniform" be removed in Theorem 2.24?

Recall that if α is an endomorphism of a ring R, then the map $R[x] \to R[x]$ defined by $\sum_{i=0}^{m} a_i x^i \mapsto \sum_{i=0}^{m} \alpha(a_i)x^i$ is an endomorphism of the polynomial ring $R[x]$. We also denote the extended map by α. In [27, Theorem 3.3], Zhang and Chen proved that, if the endomorphism α of a ring R satisfies $\alpha^l = 1_R$ for some integer $l \geq 1$, then a module M_R is α-skew Armendariz iff $M[x]$ is α-skew Armendariz over $R[x]$. We have a similar result.

Theorem 2.25 *Let α be an endomorphism of a ring R and $\alpha^l = 1_R$ for some integer $l \geq 1$. Then a module M_R is α-skew McCoy if and only if $M[x]$ is α-skew McCoy over $R[x]$.*

Proof. Assume that M is α-skew McCoy. Let $n(y) = \sum_{i=0}^{p} n_i(x)y^i \in M[x][y;\alpha]$ and $g(y) = \sum_{j=0}^{q} g_j(x)y^j \in R[x][y;\alpha]$ with $n(y)g(y) = 0$, where $n_i(x) = \sum_{k=0}^{p_i} n_{ik}x^k \in M[x]$ and $g_j(x) = \sum_{l=0}^{q_j} b_{jl}x^l \in R[x]$. Take an integer u such that $u \geq deg(n_0(x)) + deg(n_1(x)) + \cdots + deg(n_p(x)) + deg(g_0(x)) + deg(g_1(x)) + \cdots + deg(g_q(x))$, where the degree of $n_i(x)$ is as polynomial in $M[x]$, the degree of $g_j(x)$ is as polynomial in $R[x]$ and the degree of the zero polynomial is taken to be 0. Put

$$m(x) = n_0(x^l) + n_1(x^l)x^{lu+1} + n_2(x^l)x^{2lu+2} + \cdots + n_p(x^l)x^{plu+p} \in M[x;\alpha],$$
$$h(x) = g_0(x^l) + g_1(x^l)x^{lu+1} + g_2(x^l)x^{2lu+2} + \cdots + g_q(x^l)x^{qlu+q} \in R[x;\alpha].$$

Then $h(x) \neq 0$, and the set of coefficients of $n_i(x)$'s (resp., $g_j(x)$'s) equals the set of coefficients of $m(x)$ (resp., $h(x)$). Since $\alpha^l = 1_R$, x^l commutes with elements of R in $R[x;\alpha]$. By $n(y)g(y) = 0$, we have

$m(x)h(x) = 0 \in M[x;\alpha]$. Since M is α-skew McCoy, there exists $r \in R\backslash\{0\}$ such that $m(x)r = 0 \in M[x;\alpha]$. That is, $n_i(x^l)x^{ilu+i}r = 0$ for $i = 0, 1, \ldots, p$. Again, since $\alpha^l = 1_R$, we have $n_{ik}\alpha^i(r) = 0$ for all i and k. Hence $n(y)r = 0$ in $M[x][y;\alpha]$. Thus, $M[x]$ is α-skew McCoy over $R[x]$.

Conversely, assume that $M[x]$ is α-skew McCoy. Let $m(x)g(x) = 0$ with $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x;\alpha]$ and $g(x) = \sum_{j=0}^{q} b_j x^j \in R[x;\alpha]\backslash\{0\}$. Set $n(y) = \sum_{i=0}^{p} m_i y^i$ and $h(y) = \sum_{j=0}^{q} b_j y^j$. Then $h(y) \neq 0$ and $n(y)h(y) = 0 \in M[x][y;\alpha]$. By hypothesis, there exists a nonzero element $c(x) = \sum_{i=0}^{m} c_i x^i \in R[x]$ satisfying $n(y)c(x) = 0$. It follows that $m_i\alpha^i(c(x)) = 0$, and so $m_i\alpha^i(c_j) = 0$, implying $m(x)c_j = 0$ in $M[x;\alpha]$, where $0 \leq i \leq p$ and $0 \leq j \leq m$. Thus M_R is α-skew McCoy. □

Corollary 2.26 *[3, Theorem 20] Let α be an endomorphism of a ring R and $\alpha^l = 1_R$ for some positive integer l. Then R is α-skew McCoy if and only if $R[x]$ is α-skew McCoy.*

We write $M_n(R)$ for the $n \times n$ matrix ring over R. For a module M_R and $A = (a_{ij}) \in M_n(R)$, let $MA = \{(ma_{ij}) : m \in M\}$. For $n \geq 2$, let $V = \sum_{i=1}^{n-1} E_{i(i+1)}$ where $\{E_{ij} : 1 \leq i, j \leq n\}$ are the matrix units, and set $V_n(R) = RI_n + RV + \cdots + RV^{n-1}$ and $V_n(M) = MI_n + MV + \cdots + MV^{n-1}$. Then $V_n(R)$ is a ring and $V_n(M)$ becomes a right module over $V_n(R)$ under usual addition and multiplication of matrices. There is a ring isomorphism $\theta : V_n(R) \to R[x]/(x^n)$ given by $\theta(r_0I_n + r_1V + \cdots + r_{n-1}V^{n-1}) = r_0 + r_1x + \cdots + r_{n-1}x^{n-1} + (x^n)$, and an abelian group isomorphism $\phi : V_n(M) \to M[x]/(M[x](x^n))$ given by $\phi(m_0I_n + m_1V + \cdots + m_{n-1}V^{n-1}) = m_0 + m_1x + \cdots + m_{n-1}x^{n-1} + M[x](x^n)$ such that $\phi(WA) = \phi(W)\theta(A)$ for all $W \in V_n(M)$ and $A \in V_n(R)$.

Let α be an endomorphism of a ring R, the map $V_n(R) \to V_n(R)$ defined by $a_0I_n + a_1V + \cdots + a_{n-1}V^{n-1} \mapsto \alpha(a_0)I_n + \alpha(a_1)V + \cdots + \alpha(a_{n-1})V^{n-1}$ is an endomorphism of $V_n(R)$. Similarly the map $R[x]/(x^n) \to R[x]/(x^n)$ defined by $a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + (x^n) \mapsto \alpha(a_0) + \alpha(a_1)x + \cdots + \alpha(a_{n-1})x^{n-1} + (x^n)$ is an endomorphism of $R[x]/(x^n)$. We shall denote the two maps above by $\overline{\alpha}$.

Proposition 2.27 *Let α be an endomorphism of a ring R. Then a module M_R is α-skew McCoy if and only if $M[x]/M[x](x^n)$ is $\overline{\alpha}$-skew McCoy over $R[x]/R[x](x^n)$ for any $n \geq 2$.*

Proof. By the remark above, it suffices to show that M_R is α-skew McCoy iff $V_n(M)_{V_n(R)}$ is $\overline{\alpha}$-skew McCoy.

"⇒". Suppose that $W(x)A(x) = 0$ where $W(x) = \sum_{i=0}^{p} W_i x^i \in V_n(M)[x;\overline{\alpha}]$ and $A(x) = \sum_{j=0}^{q} A_j x^j \in V_n(R)[x;\overline{\alpha}]\backslash\{0\}$. Let $W_i = m_{i0}I_n + m_{i1}V + \cdots + m_{i(n-1)}V^{n-1}$ and $A_j = a_{j0}I_n + a_{j1}V + \cdots + a_{j(n-1)}V^{n-1}$ for $0 \leq i \leq p$ and $0 \leq j \leq q$. It follows that $[m_0(x)I_n + m_1(x)V + \cdots + m_{n-1}(x)V^{n-1}][a_0(x)I_n + a_1(x)V + \cdots + a_{n-1}(x)V^{n-1}] = 0$ in $V_n(M)[x;\overline{\alpha}]$, where $m_k(x) = m_{0k} + m_{1k}x + \cdots + m_{pk}x^p \in M[x;\alpha]$ and $a_l(x) = a_{0l} + a_{1l}x + \cdots + a_{ql}x^q \in R[x;\alpha]$ for $0 \leq k, l \leq n-1$, and hence $\sum_{k+l=t} m_k(x)a_l(x) = 0$ in $M[x;\alpha]$ for $t = 0, 1, \ldots, n-1$. In particular, we have

$$m_0(x)a_{l_0}(x) = 0$$

with a minimal index l_0 (l_0 exists since $A(x) \neq 0$) such that $a_{l_0}(x) \neq 0$. Since M_R is α-skew McCoy, there exists a nonzero $r \in R$ such that $m_0(x)r = 0$. Let $A = rE_{1n}$. Then $A \in V_n(R)\backslash\{0\}$ and $W(x)A = 0$. So $V_n(M)_{V_n(R)}$ is $\overline{\alpha}$-skew McCoy.

"$\Leftarrow$". Assume that $m(x)g(x)=0$, where $m(x)\in M[x;\alpha]$ and $g(x)\in R[x;\alpha]\backslash\{0\}$. Let $\alpha(x)=m(x)I_n$ and $\beta(x)=g(x)I_n$. Then $\alpha(x)\in V_n(M)[x;\overline{\alpha}]$, $\beta(x)\in V_n(R)[x;\overline{\alpha}]\backslash\{0\}$ and $\alpha(x)\beta(x)=0$. As $V_n(M)$ is an $\overline{\alpha}$-skew McCoy $V_n(R)$-module, there exists a nonzero $A\in R_n$ such that $\alpha(x)A=0$. Obviously, there is an element $r\in R\backslash\{0\}$ such that $m(x)r=0$. Therefore, M_R is α-skew McCoy. □

The following definition is due to Zhang and Chen [28]. A module M_R is a *zip module* if for any subset X of M, $r_R(X)=0$ implies $r_R(Y)=0$ for some finite subset Y of X. By [6, Proposition 1] and [15, Example 10], (in general) the class of α-skew McCoy modules neither contains nor is contained in the class of zip modules. According to [6, Example 2], R_R is a zip module does not imply that $R[x;1_R]_{R[x;1_R]}$ is zip (Some notable results on zip rings have appeared in [9], [10], [26], etc).

Theorem 2.28 *Let α be an endomorphism of a ring R with $\alpha^l=1_R$ for some positive integer l and M_R be a W-α-compatible α-skew McCoy module. Then M is a zip R-module if and only if $M[x;\alpha]$ is a zip $R[x;\alpha]$-module.*

Proof. Suppose that $M[x;\alpha]_{R[x;\alpha]}$ is zip. Let $Y\subseteq M$ with $r_R(Y)=0$. If $f(x)=a_0+a_1x+\cdots+a_nx^n\in r_{R[x;\alpha]}(Y)$, then $mf(x)=0$ for each $m\in Y$. Thus $ma_i=0$, and so $a_i\in r_R(Y)=0$ for $i=1,2,\ldots,n$. Therefore $f(x)=0$, i.e., $r_{R[x;\alpha]}(Y)=0$. Since $M[x;\alpha]$ is zip, there exists a finite subset $Y_0\subseteq Y$ such that $r_{R[x;\alpha]}(Y_0)=0$. Hence, $r_R(Y_0)=r_{R[x;\alpha]}(Y_0)\bigcap R=0$.

Conversely, assume that M is zip. Let $X\subseteq M[x;\alpha]$ with $r_{R[x;\alpha]}(X)=0$. Now let Y be the set of all coefficients of elements in X. Then $Y\subseteq M$. If $a\in r_R(Y)$, then $wa=0$ for each $w\in Y$. Since M_R is W-α-compatible, $w\alpha^i(a)=0$ for all $i\geq 0$. Thus we have $m(x)a=0$ for every $m(x)\in X$, and so $a\in r_{R[x;\alpha]}(X)=0$. That is $r_R(Y)=0$. Since M is zip, there exists a finite subset $Y_0=\{w_1,w_2,\ldots,w_t\}\subseteq Y$ such that $r_R(Y_0)=0$. For each $w_i\in Y_0$ and $i=1,2,\ldots,t$, let $m_{w_i}(x)\in X$ be such that some coefficient of $m_{w_i}(x)$ is w_i. Let $X_0=\{m_{w_1}(x),m_{w_2}(x),\ldots,m_{w_t}(x)\}\subseteq X$ and Y_1 be the set of all coefficients of elements in X_0, where $m_{w_i}(x)=\sum_{q=0}^{p_{w_i}}a_{w_iq}x^q$. Then $Y_0\subseteq Y_1$ and so $r_R(Y_1)\subseteq r_R(Y_0)=0$. If $f(x)=\sum_{j=0}^{n}b_jx^j\in r_{R[x;\alpha]}(X_0)\backslash\{0\}$, then $m_{w_i}(x)f(x)=0$ for $i=1,2,\ldots,t$. Write $u=\sum_{k=1}^{t}p_{w_k}+n$. Let $n(x)=m_{w_1}(x)+m_{w_2}(x)x^{lu}+\cdots+m_{w_t}(x)x^{lu(t-1)}\in M[x;\alpha]$, by $\alpha^l=1_R$ we have $n(x)f(x)=0$. Since M_R is α-skew McCoy, there exists $r\in R\backslash\{0\}$ such that $n(x)r=0$. So $m_{w_i}(x)r=0$ in $M[x;\alpha]$ for each i, i.e., $a_{w_iq}\alpha^q(r)=0$. The condition M_R is W-α-compatible implies that there exists an integer z such that $a_{w_iq}\alpha^z(r)=0$ for all w_i and q. Then $\alpha^z(r)\in r_R(Y_1)=0$, and so $r=0$, a contradiction. Therefore $f(x)=0$, that is, $r_{R[x;\alpha]}(X_0)=0$. □

Corollary 2.29 *[7, Theorem 3.6] Let M be a McCoy R-module. Then M is a zip R-module if and only if $M[x]$ is a zip $R[x]$-module.*

Corollary 2.30 *Let R be a right McCoy ring. Then R is right zip if and only if $R[x]$ is right zip.*

Remark 2.31 *Notice that all R-modules are W-1_R-compatible. We conclude that there exists an α-skew McCoy module which is not W-α-compatible. Consider the ring $R=\{\left(\begin{smallmatrix} a & b\\ 0 & a\end{smallmatrix}\right)\,|a,b\in\mathbb{Z}_4\}$. Let $\alpha:R\to R$ be* *an endomorphism defined by* $\alpha(\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)) = \left(\begin{smallmatrix} a & -b \\ 0 & a \end{smallmatrix}\right)$. *By [3, Example 7],* R_R *is* α*-skew McCoy. Let* R_2 *be a ring and the endomorphism* $\overline{\alpha} : R_2 \to R_2$ *both as defined in Proposition 2.20. Write* $M = R_2$. *Then* M *is* $\overline{\alpha}$*-skew McCoy as an* R_2*-module also by Proposition 2.20. Nevertheless,* M *is not W-*$\overline{\alpha}$*-compatible. Indeed, for* $A = \begin{pmatrix} \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) & \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \\ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \end{pmatrix} \in M$, $B = \begin{pmatrix} \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) & \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \\ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \end{pmatrix} \in R_2$, $AB = 0$ *but* $A\overline{\alpha}(B) \neq 0$.

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