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On $CISE$-normal subgroups of finite groups*

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Abstract

A generalized normality $CISE$ of subgroups of a finite group is introduced. Let $\mathcal{F}$ be a saturated formation containing the class of all supersolvable groups. We give a characterization of $\mathcal{F}$ by using $CISE$-normality of subgroups.

Key Words: Formation, supersolvable subgroup, $CISE$-normal subgroup

1. Introduction

In this paper, all groups considered are finite and G stands for a finite group. We write N $Char$ G to mean that N is a characteristic subgroup of G. Let $\pi(G)$ stand for the set of all prime divisors of $|G|$. Let $\mathcal{F}$ denote a saturated formation and $\mathcal{U}$ the class of supersolvable groups. Let $G^{\mathcal{F}} = \cap\{N \trianglelefteq G \mid G/N \in \mathcal{F}\}$, say the $\mathcal{F}$-residual of G. Write $Z_{\mathcal{F}}(G)$ for the $\mathcal{F}$-hypercentre of G(see[9]). The other notations and terminologies are standard (see[10]).

It is always a question of particular interest in the theory of groups to study the structure of a group G by using a certain generalized normality of some subgroups of G. Kegel in [12] introduced the concept of s-quasinormal subgroups. A subgroup H of a group G is said to be s-permutable, s-quasinormal, or π-quasinormal in G if $PH = HP$ for all Sylow subgroups P of G. Ballester-Bolinches and Pedraza-Aguilera in [4] introduced the notion of s-quasinormal embedding. A subgroup H of a group G is said to be S-quasinormally embedded or π-quasinormally embedded in G if for each prime number p in $\pi(H)$, A Sylow p-subgroup of H is also a Sylow p-subgroup of a certain s-quasinormal subgroup of G. In [23], Wang Yanming introduced the concept of c-normal subgroups. A subgroup H of a group G is said to be c-normal in G if G has a normal subgroup T such that $HT = G$ and $H \cap T \leq H_G$, where $H_G = \cap_{x\in G} H^x$ is the core of H in G. In [25], Wei and Wang integrated these terminologies into the concept of c^*-normal subgroups. A subgroup H of a group G is said to be c^*-normal in G if G has a normal subgroup T such that $HT = G$ and $H \cap T$ is π-quasinormally embedded in G. In [21], Skiba introduced that: a subgroup H of G is said to be weakly s-permutable in G if G has a subnormal subgroup T such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the maximal s-permutable subgroup of G contained in H. Many authors have investigated the structure of a finite

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group G under the assumption that some subgroups of prime power order of G or some maximal subgroups of Sylow subgroups of G have those generalized normality in G and have obtained many results (see [3], [5], [6], [15], [17], [18], [21], [23], [24] and [25]). Now, we introduce another generalized normality of subgroups of a finite group and generalize further the concept of c^*-normal. The aim is to improve and extend these works mentioned before.

Definition 1.1 *Let H be a subgroup of a group G.*

(1) If there exists a subnormal subgroup K of G such that $G = HK$ and $H \cap K$ is π-quasinormally embedded in G, H is called an ISE-normal subgroup of G.

(2) If H has a supersolvable supplement in G or H is ISE-normal in G, H is called a $CISE$-normal subgroup of G.

It is clear that the concept of ISE-normal is obtained by replacing the word "normal subgroup" by "subnormal subgroup" in the definition of c^*-normal and the concept of $CISE$-normal is a generalization of ISE-normal. The following example shows that the generalization is proper.

Let $N = L_2(8)$, $G = Aut(N)$, then $G \cong N : 3$. We identify G with $N : 3$. Let $P \in Syl_2(N)$ and $Q \in Syl_3(N)$. Let $H = N_G(P)$ and $K = N_G(Q)$. By [7], $N_N(P) \cong 2^3 : 7$, $H \cong 2^3 : 7 : 3$ and $K \cong 9 : 6$. So K is supersolvable and $G = HK$, which shows that H is $CISE$-normal in G. But it is clear that the subnormal subgroup containing the Sylow 3-subgroup of H must be G. Hence H is not an ISE-normal subgroup. This shows that H is not ISE-normal in G.

2. Preliminary results

Lemma 2.1 *([4, Lemma 1]) Suppose that U is π-quasinormally embedded in a group G, $H \leq G$ and K a normal subgroup of G.*

(a) If $U \leq H$, then U is π-quasinormally embedded in H.

(b) UK is π-quasinormally embedded in G and UK/K is π-quasinormally embedded in G/K.

(c) Let $K \leq H$ such that H/K is π-quasinormally embedded in G/K, then H is π-quasinormally embedded in G.

Lemma 2.2 *Let H be a subgroup of a group G.*

(1) If H is ISE-normal ($CISE$-normal) in G and $H \leq M \leq G$, then H is ISE-normal (respectively, $CISE$-normal) in M.

(2) Let $N \lhd G$ and $N \leq H$. Then H is ISE-normal in G if and only if H/N is ISE-normal in G/N; if H is $CISE$-normal in G, then H/N is $CISE$-normal in G/N.

(3) Let π be a set of primes, H a π-subgroup of G, and N a normal π'-subgroup of G. If H is ISE-normal (respectively, $CISE$-normal) in G, then HN/N is ISE-normal (respectively, $CISE$-normal) in G/N.

(4) Suppose that H is a p-subgroup for some prime p and H is not π-quasinormally embedded in G but ISE-normal in G, then G has a normal subgroup M such that $|G:M|=p$ and $G=HM$.

Proof. (1) Suppose that H is ISE-normal in G. By the hypothesis, there exists a subnormal subgroup T of G such that $G=HT$ and $H\cap T$ is π-quasinormally embedded in G. Then $M=M\cap HT=H(M\cap T)$, $M\cap T\lhd\lhd M$ by [8, Chap A, Lemma 14.1(a)], and $H\cap(M\cap T)=H\cap T$ is π-quasinormally embedded in M by Lemma 2.1(a). So H is ISE-normal in M. If H has a supersolvable supplement L in G, then $G=HL$, $M=M\cap HL=H(M\cap L)$. Obviously, $M\cap L$ is also a supersolvable subgroup of M. Hence if H is $CISE$-normal in G, then H is $CISE$-normal in M. □

(2) Suppose that H is ISE-normal in G. Then there exists a subnormal subgroup T of G such that $G=HT$ and $H\cap T$ is π-quasinormally embedded in G. Thus $G/N=HT/N=(H/N)\cdot(TN/N)$, $TN/N\lhd\lhd G/N$ by [8, Chap A, Lemma 14.1(b)], and $(H/N)\cap(TN/N)=(H\cap TN)/N=(H\cap T)N/N$ is π-quasinormally embedded in G/N by Lemma 2.1(b). So H/N is ISE-normal in G/N. Conversely, if H/N is ISE-normal in G/N, then there exists a subnormal subgroup T/N of G/N such that $G/N=(H/N)\cdot(T/N)$ and $(H/N)\cap(T/N)=(H\cap T)/N$ is π-quasinormally embedded in G/N. Then $G=HT$, T is subnormal G, and $H\cap T$ is π-quasinormally embedded in G by Lemma 2.1(c). So H is ISE-normal in G. The second part is clear. □

(3) Suppose that H is ISE-normal in G, then there exists a subnormal subgroup T of G such that $G=HT$ and $H\cap T$ is π-quasinormally embedded in G. Then $G/N=HT/N=(HN/N)\cdot(TN/N)$ and $HN/N\cap TN/N=(HN\cap TN)/N$. By [8, Chap A, Lemma 14.1(b)], TN/N is subnormal in G/N. Since H is a π-subgroup of G and $G=HT$, $\pi^{'}(G)\subseteq\pi(T)$. Since T is subnormal in G, we have $O_{\pi'}(G)\leq T$, so $N\leq O_{\pi'}(G)\leq T$, thus by Lemma 2.1(b), $(HN\cap TN)/N=(HN\cap T)/N=(H\cap T)N/N$ is π-quasinormally embedded in G/N, hence HN/N is ISE-normal in G/N. If H has a supersolvable supplement L in G, then $G=HL$, so $G/N=HL/N=(HN/N)\cdot(LN/N)$. Obviously $LN/N\cong L/(L\cap N)$ is supersolvable. Thus HN/N also has a supersolvable supplement LN/N in G/N. Hence we have claim (3).

(4) Since H is ISE-normal in G, there exists a subnormal subgroup K of G such that $G=HK$ and $H\cap K$ is π-quasinormally embedded in G. Since H is not π-quasinormally embedded in G, $H\cap K\neq H$. Hence K is a proper subnormal subgroup of G, then there exists a proper normal subgroup T of G such that $K\leq T$, so $|G/T|=|H:T\cap H|=p^i$, where i is a natural number, thus G has a normal maximal subgroup M such that $G=HM$ and $|G:M|=p$. □

Lemma 2.3 *([25, Lemma 2.5]) Let G be a group, K a π-quasinormally embedded subgroup of G and P a Sylow p-subgroup of K, where p is a prime. If either $P\leq O_p(G)$ or $K_G=1$, then P is π-quasinormal in G.*

Lemma 2.4 *Let N be an elementary abelian normal p-subgroup of G. Assume that N has a subgroup U with $1<|U|<|N|$ such that every subgroup H of N of order $|U|$ is ISE-normal in G. Then N is not a minimal normal subgroup of G.*

Proof. Suppose that this lemma is false, then N is a minimal normal subgroup of G. If some subgroup H of N satisfying $|H|=|U|$ is not π-quasinormally embedded in G, then by Lemma 2.2 (4), there exists a normal subgroup of G, M such that $|G:M|=p$ and $G=HM$. It follows that $NM=G$, $p=|NM:M|=|N:N\cap M|$. So $M\cap N$ is a maximal subgroup of N and $N\cap M\lhd G$. By the minimality of

N, we get $N \cap M = 1$. By $HM = NM$, we get $|N| = |H| < |N|$, a contradiction. Hence every subgroup H of N satisfying $|H| = |U|$ is π-quasinormally embedded in G, and $H \leq N \leq O_p(G)$. By Lemma 2.3, these subgroups H are π-quasinormal in G. By [16, Lemma 2.2], we have $O^p(G) \leq N_G(H)$, hence $|G : N_G(H)| = p^{j_H}$, where $j_H \geq 0$. Since $|H| < |N|$ and N is a minimal normal subgroup of G, we have $N_G(H) \neq G$. So $j_H > 0$ and j_H are natural numbers. Let Ω be the set of all subgroups with order $|U|$ of N. Then G acts on Ω by conjugation and we can obtain a partition of Ω into orbits. For $K \in \Omega$, let the G-orbit of K be $\{K_1, K_2, \cdots, K_s\}$, then $s = |G : N_G(K)| = p^{j_K}$. Hence p divides $|\Omega|$. On the other hand, by ([10, III 8.5(d)]), we know $|\Omega| \equiv 1(mod\ p)$, a contradiction. □

Lemma 2.5 *Let $\mathcal{F}$ be a saturated formation containing the class of all nilpotent groups $\mathcal{N}$, G be a group. Suppose that $G^{\mathcal{F}}$ is soluble and every maximal subgroup of G not containing $G^{\mathcal{F}}$ belongs to $\mathcal{F}$. If every cyclic subgroup of $G^{\mathcal{F}}$ with prime order or order 4 (if $G^{\mathcal{F}}$ is a non-abelian 2-group) is $CISE$-normal in G, then $|G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})| = p$.*

Proof. Let $R = G^{\mathcal{F}}$. By [19, VI Theorem 24.2], R is a p-subgroup, $expR = p$ or 4 (if R is a non-abelian 2-group) and $R/\Phi(R)$ is a minimal normal subgroup of $G/\Phi(R)$. Suppose that there exists $x \in R$ such that $\langle x\rangle$ has a supersolvable supplement M in G. If $G = M$, then $G \in \mathcal{F}$ and $G^{\mathcal{F}} = 1$, a contradiction. So we may assume that $M \neq G$, then $M\Phi(R) \neq G$ by $\Phi(R) \leq \Phi(G)$. And $G = \langle x\rangle M = RM$, so $(M\Phi(R)/\Phi(R))(\langle x\rangle\Phi(R)/\Phi(R)) = G/\Phi(R)$. Hence $|G/\Phi(R) : M\Phi(R)/\Phi(R)| = p$ and $G/\Phi(R) = (M\Phi(R)/\Phi(R))(R/\Phi(R))$. Since $R/\Phi(R)$ is a minimal normal subgroup of $G/\Phi(R)$, $R/\Phi(R) \cap M\Phi(R)/\Phi(R) = \overline{1}$, so $|(G/\Phi(R)) : (M\Phi(R)/\Phi(R))| = |R/\Phi(R)| = p$. If there exists $y \in R$ with $y \notin \Phi(R)$ such that $y \in M_G$, since $R/\Phi(R)$ is a minimal normal subgroup of $G/\Phi(R)$, we have $R \leq M_G$, contrary to $G = RM$. So for any $y \in R$ with $y \notin \Phi(R)$, there exists $g \in G$ such that $y \notin M^g$. Thus $G = \langle y\rangle M^g$ and $\langle y\rangle$ has a supersoluble supplement. Hence $|R/\Phi(R)| = p$ and the result is true. If for any $y \in R$ with $y \notin \Phi(R)$, $\langle y\rangle$ has no supersoluble supplement, then $\langle y\rangle$ is ISE-normal in G by the hypothesis. By Lemma 2.2 (2), every cyclic subgroup of $R/\Phi(R)$ of order prime is ISE-normal in $G/\Phi(R)$. Since $R/\Phi(R)$ is a minimal normal subgroup of $G/\Phi(R)$, by Lemma 2.4, $R/\Phi(R)$ has no a proper cyclic subgroup of prime order, hence we conclude that $|R/\Phi(R)| = p$. Thus we have Lemma 2.5. □

Lemma 2.6 *([22, Lemma 1.6]) Let P be a nilpotent normal subgroup of a group G. If $P \cap \Phi(G) = 1$, then P is the direct product of some minimal normal subgroups of G.*

Lemma 2.7 *Let G be a group and let P be a Sylow 2-subgroup of G. Suppose that there exists a maximal subgroup P_1 of P such that P_1 is $CISE$-normal in G. Then G is not a nonabelian simple group.*

Proof. Suppose that G is a nonabelian simple group. By the hypothesis, P_1 has a supersolvable supplement M in G or it is ISE-normal in G. If the former case is true, then $P_1M = G$ and $M \neq G$ where M is supersolvable. So $|G : M| = 2^r$ and $2^r < |P|$. By [1, Theorem 5.8], we get either M is a Hall r'-subgroup of G or G is isomorphic to A_n with $5 \leq n = 2^r$, $r \geq 2$ and $M \cong A_{n-1}$. Since $2^r < |P|$ and M is supersolvable, we have G is not a nonabelian simple group. If the later case holds, then there exists a subnormal subgroup T such that $G = P_1T$ and $P_1 \cap T$ is π-quasinormally embedded in G. Since G is simple, we have $T = G$, then

$P_1 \cap T = P_1$ is π-quasinormally embedded in G. So there exists a π-quasinormal subgroup K of G such that $P_1 \in Syl_2(K)$. Since G is simple, we get $K = G$. So $P_1 = P$, a contradiction. Then G is not a nonabelian simple group. □

Lemma 2.8 *Let G be a group and M a subgroup of G. Then we have:*

(1) If M is normal in G, then $F^*(M) \leq F^*(G)$.

(2) $F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = Soc(F(G)C_G(F(G))/F(G))$.

(3) $F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is soluble, then $F^*(G) = F(G)$.

(4) $C_G(F^*(G)) \leq F(G)$.

Proof. (1) $\sim$ (4) can be found in [11, Chap. X, §13]. □

3. Main results

Theorem 3.1 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a normal subgroup N such that $G/N \in \mathcal{F}$. Assume that every Sylow subgroup of $F^*(N)$ is cyclic, then $G \in \mathcal{F}$.*

Proof. Assume that the result is false and let G be a counterexample with $|G| + |N|$ minimal. Suppose that all Sylow subgroups of $F^*(N)$ are cyclic. Clearly, $F^*(N)$ is supersolvable, then $F^*(N) = F(N)$ by Lemma 2.8 (3). Since $F(N)/\Phi(N) = F(N/\Phi(N))$, all Sylow subgroups of $F(N/\Phi(N))$ are cyclic. So $(G/\Phi(N), N/\Phi(N))$ satisfies the hypothesis of Theorem 3.1. Thus if $\Phi(N) \neq 1$, then $G \in \mathcal{F}$. Hence we may assume that $\Phi(N) = 1$. Since $\Phi(F(N)) \subseteq \Phi(N)$, $\Phi(F(N)) = 1$. Let $\pi(F(N)) = \{p_i \mid 1 \leq i \leq t\}$ and $K_i \in Syl_{p_i}(F(N))$, where $p_1 < p_2 < \cdots < p_t$, then $K_i \lhd G$ and $|K_i| = p_i$. Then $|Aut(K_i)| = p_i - 1$ and $G/C_G(K_i)$ is isomorphic to a subgroup of $Aut(K_i)$. From $Aut(K_i)$ cyclic, we get that $G/C_G(K_i)$ is cyclic. Let $U = \cap_{i=1}^{t} C_G(K_i)$, then $G/U \in \mathcal{U}$ and so $G/U \cap N \in \mathcal{F}$. It is easy to see that $F^*(U \cap N) = F^*(N)$. Hence if $U \cap N < N$, then $G \in \mathcal{F}$, a contradiction. So we may assume that $N \leq U$. It is clear that $U = C_G(F(N))$. So $N \leq C_G(F(N))$, $F(N) \leq Z(N)$. By Lemma 2.8 (4), $N = C_N(F(N)) = C_N(F^*(N)) \leq F(N)$, so $N = F(N)$ and N is cyclic. Since $(G/K_1)/(N/K_1) \cong G/N \in \mathcal{F}$, and $F(N/K_1) = N/K_1$ is cyclic, by the minimality of $|G| + |N|$, we get $G/K_1 \in \mathcal{F}$. So we may assume that $N = K_1$ and $K_1 \nleq \Phi(G)$. Let M be a maximal subgroup of G such that $G = K_1 M$, then $M \cap K_1 = 1$. Since K_1 centralizes $C_G(K_1) \cap M$ and M normalizes $C_G(K_1) \cap M$, we get $C_G(K_1) \cap M \lhd G$. Let $T = C_G(K_1) \cap M$. Then $K_1 \nleq T$ and so $K_1 \cap T = 1$. By $G = C_G(K_1)M$, we have that $M/T \cong C_G(K_1)M/C_G(K_1) = G/C_G(K_1)$. Since $G/C_G(K_1)$ is cyclic and $G/T = K_1T/T \rtimes M/T$, we have $G/T \in \mathcal{U}$. Because $\mathcal{F}$ is a formation, we have obtained that $G \cong G/K_1 \cap T \in \mathcal{F}$, a contradiction. This contradiction completes the proof of Theorem 3.1. □

Theorem 3.2 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a normal subgroup N such that $G/N \in \mathcal{F}$. Assume that every non-cyclic Sylow subgroup P of N has a subgroup U with $1 < |U| < |P|$*

such that every subgroup H of P of order $|U|$ and every cyclic subgroup of P of order 4 (if $|U|=2$ and P is a non-abelian 2-group) is $CISE$-normal in G, then $G\in\mathcal{F}$.

Proof. Assume that the result is false and let (G,N) be a counterexample with $|G|+|N|$ minimal.

If all Sylow subgroups of N are cyclic, then all Sylow subgroups of $F(N)$ are cyclic. By Theorem 3.1, $G\in\mathcal{F}$. Hence when we want to prove $\overline{G}\in\mathcal{F}$ in the following arguments, we assume always that $\overline{N}$ has a non-cyclic Sylow subgroup if $(\overline{G},\overline{N})$ satisfy the hypothesis of (G,N) in Theorem 3.2.

Step 1. If T is a Hall subgroup of N, the hypothesis is true for (T,T). In addition, if $T\lhd N$, then the hypothesis is also true for $(G/T,N/T)$. Especially, if T is a non-identity normal Hall subgroup of N, we may assume that $N=T$.

Let T be a Hall subgroup of N, P be a non-cyclic Sylow subgroup of T, of course, P is also a non-cyclic Sylow subgroup of N. By the hypothesis, P has a subgroup U with $1<|U|<|P|$ such that every subgroup H of P of order $|U|$ and every cyclic subgroup of P of order 4 (if $|U|=2$ and P is a non-abelian 2-group) is $CISE$-normal in G. By Lemma 2.2 (1), the hypothesis is true for (T,T). In addition, if $T\lhd N$, then by T is a characteristic subgroup of N and $N\lhd G$, we get that $T\lhd G$. Let P^*/T be a non-cyclic Sylow subgroup of N/T, then by Shur-Zassenhaus Theorem, it is easy to prove that $P^*=T\rtimes P$, where $P\in Syl_p(P^*)$. Obviously, P is also a non-cyclic Sylow subgroup of N. By the hypothesis, P has a subgroup U such that $1<|U|<|P|$ and every subgroup H of P of order $|U|$ and every cyclic subgroup of P of order 4 (if $|U|=2$ and P is a non-abelian 2-group) is $CISE$-normal in G. By Lemma 2.2 (3), H^*/T is $CISE$-normal in G/T. Especially, if T is a non-identity normal Hall subgroup of N, then the hypothesis is true for $(G/T,N/T)$, so $G/T\in\mathcal{F}$. Thus the hypothesis is still true for (G,T). By the minimality of $|G|+|N|$, we may assume that $T=N$.

Step 2. Let $p=min\pi(N)$, then Sylow p-subgroups P of N are not cyclic. Thus by the hypothesis, P has a subgroup U such that $1<|U|<|P|$ and every subgroup H of P of order $|U|$ and every cyclic subgroup of P of order 4 (if $|U|=2$ and P is a non-abelian 2-group) is $CISE$-normal in G.

If P is cyclic, then N is p-nilpotent by [10, V, 2.8], so N has a normal p'-Hall subgroup $N_{p'}$. If $N_{p'}\neq 1$, by **Step** 1, $(G/N_{p'},N/N_{p'})$ satisfies the hypothesis, then $G\in\mathcal{F}$ by the minimality of $|G|+|N|$, a contradiction. So $N_{p'}=1$, $N=P$. Since $G/N\in\mathcal{F}$ and P is cyclic, we get $G\in\mathcal{F}$ by Theorem 3.1, a contradiction.

Step 3. If either $N=G$ or $N=P$, then $|U|>p$.

Suppose that $N=G$, by the hypothesis, G is not a nilpotent group and so it has a p-closed Schmidt subgroup E [10, IV, 5.4]. If $|U|=p$, then E satisfies the condition of Lemma 2.5, so E is p-nilpotent, a contradiction. Hence $|U|>p$. Suppose that $N=P$, then $G/P\in\mathcal{F}$. Assume that $|U|=p$ and $R=G^{\mathcal{F}}$. Let T be an arbitrary maximal subgroup of G not containing P, then $G/P=TP/P\cong T/T\cap P$, so the hypothesis is still true for $(T,T\cap P)$, thus $T\in\mathcal{F}$ by the choice of G. So every maximal subgroup of G not containing P belongs to $\mathcal{F}$. Since $R\leq P$, every maximal subgroup of G not containing R dose not contain P, thus every maximal subgroup of G not containing R belongs to $\mathcal{F}$, so $|R/\Phi(R)|=p$ by Lemma 2.5. Since $(G/\Phi(R))/(R/\Phi(R))\cong G/R\in\mathcal{F}$, we have $G/\Phi(R)\in\mathcal{F}$ by Theorem 3.1, which implies $G\in\mathcal{F}$, a contradiction. Hence $|U|>p$.

Step 4. If $|P:U|>p$, then every subgroup H of P of order $|U|$ not having a supersoluble supplement in G is π-quasinormally embedded in G. If P is a non-abelian 2-group and $|U|=2$, then every subgroup H of P of order 4 not having a supersoluble supplement in G is also π-quasinormally embedded in G.

Assume that P has a subgroup H of order $|U|$ and H neither has a supersoluble supplement in G nor is π-quasinormally embedded in G. Then G has a normal subgroup M such that $G=HM$ and $|G:M|=p$ by Lemma 2.2 (4), so $G/N\cap M\in\mathcal{F}$. Hence the hypothesis is still true for $(G,N\cap M)$ by $|P:U|>p$ and Lemma 2.2. If $M\cap N=N$, then $N\le M$, so $G=HM=NM=M$, a contradiction. Thus $|N\cap M|<|N|$, so $|G|+|N\cap M|<|G|+|N|$, contrary to the minimality of $|G|+|N|$. Similarly, we can prove the second statement of **Step** 4.

Step 5. If L is a minimal normal subgroup of G and $L\le P$, then $|L|\le|U|$.

Suppose that $|L|>|U|$, then every subgroup H of L of order $|U|$ is $CISE$-normal in G. If H has a supersoluble supplement M in G, then $MH=G$. Since $L=L\cap HM=H(L\cap M)$ and $|L|>|U|$, $L\cap M\ne 1$. Obviously, $L\cap M\lhd ML=MH=G$. By the minimality of L, we get $L\cap M=L$, then $L\le M$. So $G=MH=ML=M$ is supersoluble, a contradiction. Thus every subgroup H of L of order $|U|$ is ISE-normal in G. But by Lemma 2.4, L is not a minimal normal subgroup of G, a contradiction.

Step 6. If either $N=G$ or $N=P$ and L is an abelian minimal normal subgroup of G contained in N, then the hypothesis is still true for $(G/L,N/L)$ and so $G/L\in\mathcal{F}$.

Assume that $|P:U|=p$. By the hypothesis, every maximal subgroup of P is $CISE$-normal in G. Let T/L be a maximal subgroup of PL/L, then $p=|(PL/L):(T/L)|$, and $T=PL\cap T=(P\cap T)L$. Let $P_1=P\cap T$, then $P_1\cap L=P\cap T\cap L=P\cap L$, so $p=|PL:T|=|PL:(P\cap T)L|=|P:P\cap T|=|P:P_1|$, thus P_1 is $CISE$-normal in G. If P_1 has a supersoluble supplement M in G, then $G=P_1M$, and $G/L=P_1M/L=TM/L=(T/L)(ML/L)$. Since $ML/L\cong M/M\cap L$ is supersoluble, T/L has a supersoluble supplement ML/L in G/L. Suppose that P_1 is ISE-normal in G, then there is a subnormal subgroup B of G such that $G=P_1B$ and $P_1\cap B$ is π-quasinormally embedded in G. We have $P_1L\cap BL=(P_1L\cap B)L$. Now let $\pi(G)=\{p_1,p_2,\cdots,p_n\}$ where $p_1=p$, and B_{p_i} denotes a Sylow p_i-subgroup of B $(i=2,\cdots,n)$. Then B_{p_i} is a Sylow p_i-subgroup of G, hence $B_{p_i}\cap N$ is a Sylow p_i-subgroup of N $(i=2,\cdots,n)$. Write $V=\langle L\cap B_{p_2},\cdots,L\cap B_{p_n}\rangle$, then $V\le B$ and $L=(P_1\cap L)V$, thus $P_1L\cap BL=(P_1L\cap B)L=(P_1V\cap B)L=(P_1\cap B)VL=(P_1\cap B)L$. It follows from Lemma 2.1(b) that $(P_1L/L)\cap(BL/L)=(P_1\cap B)L/L$ is π-quasinormally embedded in G/L. Therefore T/L is $CISE$-normal in G/L, then the hypothesis is true for $(G/L,N/L)$.

Assume that $|L|<|U|$. If $N=G$, then L is a p-subgroup, then for $L<H$ with $|H|=|U|$, we have H/L is $CISE$-normal in G/L by Lemma 2.2 (2) and $1<|H/L|<|P/L|$. If $N=P$, then by Lemma 2.2 (2), we have H/L is $CISE$-normal in G/L. Hence the hypothesis is still true for $(G/L,N/L)$.

So let $|L|=|U|$ and $|P:U|>p$. Then by **Step** 4 every subgroup H of P with order $|H|=|U|$ not having a supersoluble supplement in G is π-quasinormally embedded in G, and if P is a non-abelian 2-group and $|U|=2$, then every cyclic subgroup H of P with order 4 not having a supersoluble supplement in G is also π-quasinormally embedded in G. By **Step** 3, L is non-cyclic, hence every subgroup of G containing L is not cyclic. Let $L<K\le P$, where $|K:L|=p$. Since K is non-cyclic, it has a maximal subgroup $M\ne L$. If M has a supersoluble supplement in G, then K has a supersoluble supplement in G. If M is π-quasinormally embedded in G, so $K=LM$ does by Lemma 2.1(b). Thus if P/L is p-group$(p>2)$ or an abelian 2-group or a non-abelian 2-group with $|U|>2$, the hypothesis is true for $(G/L,N/L)$ by Lemma 2.2 (2) and **Step** 4. If P/L is a non-abelian 2-group and $|U|=2$, then P is a non-abelian 2-group and so every cyclic subgroup of P with order 4 not having a supersoluble supplement in G is π-quasinormally embedded in G. In this case, using the same method as above, one can show that every subgroup X of P containing L such that X/L is a cyclic subgroup of order 4 either has a supersoluble supplement in G or is π-quasinormally embedded in G. Thus again the hypothesis is still true for $(G/L, N/L)$.

Step 7. N is solvable.

By **Step** 1 and the choice of G we only need consider the case $N = G$. Let $2 = \min\pi(N)$. Then the Sylow 2-subgroup P of N are not cyclic by **Step** 2. Assume that $|P : U| = 2$. Let G be a counterexample of minimal order. By **Step** 6 and Lemma 2.7, we get G has the unique minimal normal subgroup L of G such that G/L is solvable and $L \neq 1$. Suppose that $L \cap P \leq \Phi(P)$, then L is 2-nilpotent by J. Tate Theorem([10, Theorem 4.4.7]), so G is solvable, a contradiction. Then $L \cap P \nleq \Phi(P)$, so there exists a maximal subgroup P_1 of P such that $(L \cap P)P_1 = P$. By the hypothesis, P_1 is $CISE$-normal in G. If P_1 is ISE-normal in G, then there exists a subnormal subgroup T of G such that $G = P_1T$ and $P_1 \cap T$ is π-quasinormally embedded in G. So there exists a π-quasinormal subgroup K of G such that $P_1 \cap T \in \mathrm{Syl}_2(K)$. Assume that $K_G \neq 1$, then $L \leq K_G \leq K$, so $P_1 \cap T \cap L \in \mathrm{Syl}_2(L)$ and $P_1 \cap T \cap L \leq P_1 \cap L \leq P \cap L$. Thus $P_1 \cap T \cap L = P \cap L = P_1 \cap L$, hence $P = (P \cap L)P_1 = P_1$, a contradiction. So $K_G = 1$. By Lemma 2.3, $P_1 \cap T$ is π-quasinormal in G. If $P_1 \cap T \neq 1$, then $P_1 \cap T \leq O_2(G)$, $O_2(G) \neq 1$, so $L \leq O_2(G)$, thus G is solvable, a contradiction. If $P_1 \cap T = 1$, then $2 \mid |T|$, but $4 \nmid |T|$, so T is solvable, thus G is solvable by the subnormality of T, a contradiction. Thus every maximal of P has a supersolvable supplement in G, then G is q-closed by [21, Lemma 2.2] and **Step** 2, where $q = \max\pi(G)$, so G is solvable, a contradiction.

Assume that $|P : U| > 2$. By **Step** 4, every subgroup H of P of order $|U|$ not having a supersoluble supplement in G is π-quasinormally embedded in G. If P is a non-abelian 2-group and $|U| = 2$, then every subgroup H of P of order 4 not having a supersoluble supplement in G is also π-quasinormally embedded in G. By **Step** 1 and **Step** 6, we may assume that $O_{2'}(G) = 1$ and $O_2(G) = 1$. Suppose that H is π-quasinormally embedded in G, then there exists a π-quasinormal subgroup K such that $H \in \mathrm{Syl}_2(K)$. If $K_G = 1$, then H is π-quasinormal subgroup of G by Lemma 2.3, so $H \leq O_2(G)$, thus $O_2(G) \neq 1$, a contradiction. If $K_G \neq 1$, we choose $H \lessdot H_1 \leq P$, then H_1K_G satisfies the hypothesis. By the first paragraph discussion, we have H_1K_G is solvable, so is K_G. Thus $O_2(K_G) \neq 1$ or $O_{2'}(K_G) \neq 1$, hence $O_2(G) \neq 1$ or $O_{2'}(G) \neq 1$, a contradiction. Therefore, every subgroup H of P of order $|U|$ has a supersoluble supplement in G, that is, every maximal subgroup of P has a supersoluble supplement in G, then G is q-closed by [21, Lemma 2.2] and **Step** 2, where $q = \max\pi(G)$, so G is solvable, the final contradiction. This contradiction implies that G is solvable.

Step 8. Let $q = \max\pi(N)$, then N is q-closed.

Assume that N_q is not normal in N and let N be a counterexample with $|N| + |G|$ of minimal order for q-closed.

By **Step** 7, we can assume that $\{N_r \mid r \in \pi(N)\}$ is a Sylow system of N. Let $K = N_qN_r$ for any $r \in \pi(N)$ with $r \neq q$. By Step 1, the hypothesis is still true for (K, K). If $|\pi(N)| > 3$ or $G \neq N$, then $N_q \lhd K$, which implies that $N_q \lhd N$, a contradiction. Thus we may assume that $G = N$ and $|G| = p^aq^b$.

Let L be a minimal normal subgroup of G, then G/L is q-closed by **Step** 6. Since q-closed groups are a saturated formation, we may assume that $L \nleq \Phi(G)$ and L is the only minimal normal subgroup of G. If L is a q-group, then $G_q \lhd G$, where G_q denotes a Sylow q-subgroup of G, a contradiction. Thus $L \leq P$ and so $L \leq O_p(G)$. Now we show that $L = O_p(G)$. Let W be a maximal subgroup of G such that $L \nleq W$, then $G = LW$ and $L \cap W = 1$. Since $W \cong G/L$, W is q-closed. By $L \leq O_p(G)$, $G = LW = O_p(G)W$. From $O_p(G) \leq F(G) \leq C_G(L)$, it is easy to see that L and W normalize $O_p(G) \cap W$, thus $O_p(G) \cap W \lhd G$. So

$O_p(G) \cap W = 1$ or $L \leq O_p(G) \cap W$. If the later case happened, then $L \leq W$, that is, $G = L \rtimes W = W$, a contradiction. So $O_p(G) \cap W = 1$, thus $|O_p(G)| = |G : W| = |L|$, hence $L = O_p(G)$.

Assume that $|P : U| = p$. For every maximal subgroup A of P containing L we have $G = AW$, hence A has a supplement W in G such that W is q-closed. If every maximal subgroup of P not containing L has a supersolvable supplement M in G, then M is q-closed. Thus every maximal subgroup of P has a supplement M such that M is q-closed. By [21, Lemma 2.2], G is q-closed, a contradiction. Thus there exists one maximal subgroup S of P neither containing L nor having a supersoluble supplement in G. By the hypothesis, S is ISE-normal in G. It follows that there exists a subnormal subgroup K of G such that $G = SK$ and $S \cap K$ is π-quasinormally embedded in G. It is easy to prove that all Sylow q-subgroups of G are in every subnormal subgroup of G containing a Sylow q-subgroup of G. Since for any $g \in G$, K^g is subnormal in G, all Sylow q-subgroups of G are in K^g, so $K_G \neq 1$ and $G_q \leq K_G$. By the uniqueness of L, $L \leq K_G \leq K$. If $S \cap K = 1$, then $|K| = pq^b$. Since $q > p$, $K_q \lhd K$, which implies that G is q-closed, a contradiction. If $S \cap K \neq 1$, then $S \cap K$ is a Sylow p-subgroup of some π-quasinormal subgroup K_1 of G. Now we claim that $(K_1)_G = 1$. If $(K_1)_G \neq 1$, by the uniqueness of L, we have $L \leq (K_1)_G$, $O_p((K_1)_G) \neq 1$. Thus $L \leq O_p((K_1)_G) \leq (K_1)_p = S \cap K \leq S$, where $(K_1)_p \in \mathrm{Syl}_p(K_1)$, which contradicts the choice of S. Hence $S \cap K$ is π-quasinormal in G by Lemma 2.3, so $S \cap K$ is a subnormal subgroup of G, thus $S \cap K \leq O_p(G) = L$, then $S \cap K \leq S \cap L \leq S \cap K$, so $S \cap K = S \cap L$. It is clear that $S \cap K = S \cap L$ is normalized by P. Since $S \cap K$ is also a subnormal Sylow subgroup of $(S \cap K)G_q$, $S \cap K$ is normalized by G_q. By $G = PG_q$, $S \cap K \lhd G$, then $L \leq S \cap K \leq S$, which contradicts the choice of S.

Therefore we may assume that $|P : U| > p$, then by **Step** 4, every subgroup H of P satisfying $|H| = |U|$ and not having a supersoluble supplement in G is a π-quasinormally embedded subgroup. If H has not supersoluble supplement in G, then H is a π-quasinormally embedded subgroup of G, so there exists a π-quasinormal subgroup K of G such that $K_p = H$, where $K_p \in \mathrm{Syl}_p(K)$. If $K_G = 1$, then H is a π-quasinormal subgroup of G by Lemma 2.3, so H is subnormal in G. Since every subnormal p-subgroup is contained in $O_p(G)$ and $O_p(G) = L$ by the previous argument, $H \leq L$. On the other hand, by **Step** 5, $|H| \geq |L|$, so $L = H$. If $K_G \neq 1$, then $L \leq K_G$ and $L \leq K_p = H$. Summing up, we have $L \leq H$, so $G = WL = WH$ and $W \cap L = 1$, H has a q-closed supplement W in G. If H has a supersolvable supplement M in G, then M is also q-closed. We have obtained that every subgroup H of order $|U|$ in P has a q-closed supplement in G. Since every maximal subgroup of P contains at least one subgroup H such that $|H| = |U|$, we get that every maximal subgroup of P has a q-closed supplement in G. By [21, Lemma 2.2], G is q-closed, a final contradiction.

Step 9. Final contradiction.

Let $q = \max \pi(N)$ and Q be a Sylow q-subgroup of N. Then by **Step** 8, Q is normal in N and so we may assume that $Q = N = P$ by **Step** 1. Let L be a minimal normal subgroup of G contained in P. Then by **Step** 6, L is the only minimal normal subgroup of G contained in P and so $L = O_p(G) = P$. But by Lemma 2.4, L is not a minimal normal subgroup of G, a contradiction. This contradiction completes the proof of this theorem. □

Proposition 3.3 *(a) Let H be a p-subgroup of $F(G)$. If H is an ISE-normal subgroup of G, then it is also a weakly s-permutable subgroup of G.*

(b) If every ISE-normal subgroup of G is also weakly s-permutable in G, then $G/O_p(G)$ is p-nilpotent for arbitrary $p \in \pi(G)$.

Proof. (a) Let H be an ISE-normal subgroup of G, then there exists a subnormal subgroup K of G such that $G = HK$ and $H \cap K$ is π-quasinormally embedded in G. Since $H \leq F(G)$, we have $H \leq O_p(G)$, so $H \cap K \leq O_p(G)$, thus $H \cap K$ is π-quasinormal in G by Lemma 2.3, so $H \cap K \leq H_{sG}$, consequently, H is weakly s-permutable in G, as desired.

(b) Because every Sylow subgroup of G is always normally embedded in G, it is of course, ISE-normal, it follows that G_p is weakly s-permutable in G. By definition in [21], there exists a subnormal subgroup T such that $G = G_pT$ and $G_p \cap T \leq (G_p)_{sG}$. Since $(G_p)_{sG} \leq O_p(G)$, we have $G_p \cap T = T \cap O_p(G)$, so

$$|(G/O_p(G)) : (TO_p(G)/O_p(G))| = |G : TO_p(G)| = |G_pT : TO_p(G)| = |G_p : O_p(G)|.$$

Hence $TO_p(G)/O_p(G)$ is a Hall p'-subgroup of $G/O_p(G)$. Since $TO_p(G)$ is subnormal in G, we get $TO_p(G)/O_p(G)$ is a normal p-complement of $G/O_p(G)$, so $G/O_p(G)$ is p-nilpotent. □

Theorem 3.4 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a normal subgroup N such that $G/N \in \mathcal{F}$. Assume that every non-cyclic Sylow subgroup P of $F^*(N)$ has a subgroup U with $1 < |U| < |P|$ such that every subgroup H of P of order $|U|$ and every cyclic subgroup of P of order 4 (if $|U| = 2$ and P is a non-abelian 2-group) is CISE-normal in G, then $G \in \mathcal{F}$.*

Proof. Assume that the result is false and consider a counterexample (G, N) with minimal $|G| + |N|$. If all Sylow subgroups of $F^*(N)$ are cyclic, then by Theorem 3.1, $G \in \mathcal{F}$. Next, we assume always that $F^*(N)$ has a non-cyclic Sylow subgroup. We claim that $F^*(N) = F(N) \neq 1$. In fact, $F^*(N)$ is supersolvable by Theorem 3.2. So $F^*(N) = F(N) \neq 1$ by Lemma 2.8 (2), (3). By the hypothesis, every non-cyclic Sylow subgroup P of $F^*(N) = F(N)$ has a subgroup U with $1 < |U| < |P|$ such that every subgroup H of P of order $|U|$ and every cyclic subgroup of P of order 4 (if $|U| = 2$ and P is a non-abelian 2-group) is $CISE$-normal in G. If H has supersolvable supplement M in G, then $G = HM = PM$, so $G/P \cong M/P \cap M \in \mathcal{U} \subseteq \mathcal{F}$, thus (G, P) satisfy the condition of Theorem 3.2, hence $G \in \mathcal{F}$, a contradiction. Thus every subgroup H of P of order $|U|$ and every cyclic subgroup of P of order 4 (if $|U| = 2$ and P is a non-abelian 2-group) is ISE-normal in G, by Proposition 3.3 (a), every subgroup H of P of order $|U|$ and every cyclic subgroup of P of order 4 (if $|U| = 2$ and P is a non-abelian 2-group) is weakly s-permutable in G. Applying [20, Corollary 5.4], $G \in \mathcal{F}$, a contradiction. This contradiction completes the proof of this theorem. □

It is well known that if a subgroup H of G is c-normal, c^*-normal, S-permutable, S-quasinormally embedded in G respectively and has a supersolvable supplement in G, then H is $CISE$-normal in G. Hence [21, Corollary 5.1∼2.24] are corollaries of our Theorem 3.2 and Theorem 3.4. Moreover, we have the following corollaries.

Corollary 3.5 *(See [2, Theorem 3.3]) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, and let G be a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup H such that $G/H \in \mathcal{F}$ and all maximal subgroups of any Sylow subgroup of H are S-quasinormally embedded in G.*

Corollary 3.6 *(See [2, Corollary 3.4]) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a solvable group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup H such that $G/H \in \mathcal{F}$ and all maximal subgroups of the Sylow subgroups of $F(H)$ are S-quasinormally embedded in G.*

Corollary 3.7 *(See [15, Theorem 3.2]) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G be a group with a soluble normal subgroup H such that $G/H \in \mathcal{F}$. If all minimal subgroups and all cyclic subgroups with order 4 of $F(H)$ are π-quasinormally embedded in G, then $G \in \mathcal{F}$.*

Corollary 3.8 *(See [15, Theorem 1.1]) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, and let G be a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup H such that $G/H \in \mathcal{F}$ and all maximal subgroups of any Sylow subgroup of $F^*(H)$ are π-quasinormally embedded in G.*

Corollary 3.9 *(See [15, Theorem 1.2]) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, and let G be a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup H such that $G/H \in \mathcal{F}$ and the cyclic subgroups of prime order or order 4 of $F^*(H)$ are π-quasinormally embedded in G.*

Corollary 3.10 *(See [25, Theorem 4.1]) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, and let G be a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup H such that $G/H \in \mathcal{F}$ and all maximal subgroups of any Sylow subgroup of $F^*(H)$ are c^*-normal in G.*

Corollary 3.11 *(See [13, Theorem 3.5]) Let G be a group and $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Then $G \in \mathcal{F}$ if and only if there is a solvable normal subgroup H such that $G/H \in \mathcal{F}$ and every maximal subgroup of all Sylow subgroups of $F(H)$, the Fitting subgroup of H, is either c-normal or S-quasinormally embedded in G.*

Corollary 3.12 *(See [13, Theorem 3.2]) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. Then G is in $\mathcal{F}$ if and only if there is a normal subgroup H such that $G/H \in \mathcal{F}$ and every maximal subgroup of all Sylow subgroups of H is either c-normal or S-quasinormally embedded in G.*

In [14], the following concept was introduced: Let G be a group. A subgroup H of G is said to be an SS-quasinormal subgroup (Supplement-Sylow-quasinormal subgroup) of G if there is a supplement B to H in G such that H permutes with every Sylow subgroup of B. Let H be a nilpotent subgroup of G and $H \leq F(G)$. We know that H is SS-quasinormal in G if and only if H is S-quasinormally embedded in G. our final corollary:

Corollary 3.13 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a soluble normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of the Sylow subgroups of $F(H)$ (all minimal subgroups and all cyclic subgroups with order 4 of $F(E)$) are SS-quasinormal in G, then $G \in \mathcal{F}$.*

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