Turk J Math
36 (2012) , 245 – 251.
© TÜBİTAK
doi:10.3906/mat-1004-309

On some new inequalities for convex functions

Mevlüt Tunç

Abstract

In the present paper we establish some new integral inequalities analogous to the well known Hadamard's inequality by using a fairly elementary analysis.

Key Words: Hadamard's inequality, convex function, means

1. Introduction

The inequality (see [1])

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which holds for all convex functions $f:[a,b]\to\mathbb{R}$, is known in the literature as Hadamard's inequality. Since its discovery in 1893, Hadamard's inequality [2] has been proven to be one of the most useful inequalities in mathematical analysis. A number of papers have been written on this inequality providing new proofs, noteworthy extensions, generalizations and numerous applications; see [1, 8] and the references cited therein. The main purpose of this paper is to establish some new integral inequalities analogous to Hadamard's inequality given in (1.1) involving two convex functions. The analysis used in the proof is elementary and we believe that the inequalities established here are of independent interest.

2. Main results

We need the following Lemma proved in [6] which deals with the simple characterization of convex functions.

Lemma 1 *The following statements are equivalent for a mapping:* $f:[a,b]\to\mathbb{R}$;

i) f *is convex on* $[a,b]$,

ii) for all x, y *in* $[a,b]$ *the mapping* $g:[0,1]\to\mathbb{R}$, *defined by* $g(t)=f(tx+(1-t)y)$ *is convex on* $[0,1]$.

For the proof of this Lemma, see [6].

Our main result is given in the following theorem.

2000 *AMS Mathematics Subject Classification:* 26A51, 26D15.

Theorem 2 *Let* $f,g:[a,b]\to\mathbb{R}$ *be two convex functions and* $fg\in L^1([a,b])$. *Then,*

$$\begin{aligned}&\frac{1}{(b-a)^2}\int_a^b (b-x)(f(a)g(x)+g(a)f(x))\,dx\\&+\frac{1}{(b-a)^2}\int_a^b (x-a)(f(b)g(x)+g(b)f(x))\,dx\\\leq\ &\frac{M(a,b)}{3}+\frac{N(a,b)}{6}+\frac{1}{b-a}\int_a^b f(x)g(x)\,dx,\end{aligned}\tag{2.2}$$

where $M(a,b)=f(a)g(a)+f(b)g(b), N(a,b)=f(a)g(b)+f(b)g(a)$.

Proof. Since f and g are convex functions, we have that

$$\begin{aligned}f(ta+(1-t)b)&\leq tf(a)+(1-t)f(b)\\g(ta+(1-t)b)&\leq tg(a)+(1-t)g(b)\end{aligned}$$

for $t\in[0,1]$. Using the elementary inequality $e\leq f$ and $p\leq r$ then $er+fp\leq ep+fr$ for $e,f,p,r\in\mathbb{R}$, we get that

$$\begin{aligned}&f(ta+(1-t)b)[tg(a)+(1-t)g(b)]\\&+g(ta+(1-t)b)[tf(a)+(1-t)f(b)]\\\leq\ &[tf(a)+(1-t)f(b)][tg(a)+(1-t)g(b)]\\&+f(ta+(1-t)b)g(ta+(1-t)b)\end{aligned}$$

and we obtain

$$\begin{aligned}&g(a)tf(ta+(1-t)b)+g(b)(1-t)f(ta+(1-t)b)\\&+f(a)tg(ta+(1-t)b)+f(b)(1-t)g(ta+(1-t)b)\\\leq\ &t^2f(a)g(a)+(1-t)^2f(b)g(b)+t(1-t)f(a)g(b)\\&+t(1-t)f(b)g(a)+f(ta+(1-t)b)g(ta+(1-t)b).\end{aligned}$$

By the Lemma 1, $f(ta+(1-t)b)$ and $g(ta+(1-t)b)$ are convex on $[0,1]$, they are integrable on $[0,1]$ and consequently $f(ta+(1-t)b)g(ta+(1-t)b)$ is also integrable on $[0,1]$. Similarly, since f and g are convex on $[a,b]$, they are integrable on $[a,b]$ and hence fg is also integrable on $[a,b]$. Integrating both sides of the above inequality over $[0,1]$, we get

$$\begin{aligned}&g(a)\int_0^1 tf(ta+(1-t)b)\,dt+g(b)\int_0^1(1-t)f(ta+(1-t)b)\,dt\\&+f(a)\int_0^1 tg(ta+(1-t)b)\,dt+f(b)\int_0^1(1-t)g(ta+(1-t)b)\,dt\\\leq\ &f(a)g(a)\int_0^1 t^2dt+f(b)g(b)\int_0^1(1-t)^2dt+f(a)g(b)\int_0^1 t(1-t)\,dt\\&+f(b)g(a)\int_0^1 t(1-t)\,dt+\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)\,dt.\end{aligned}$$

By substituting $ta+(1-t)b=x$, $(a-b)\,dt=dx$ it is easy to observe that

$$\begin{aligned}&\int_0^1 tg\left(ta+(1-t)\,b\right)dt\\ =\ &\frac{1}{b-a}\int_a^b \frac{x-b}{a-b}g\left(x\right)dx=\frac{1}{\left(b-a\right)^2}\int_a^b \left(b-x\right)g\left(x\right)dx,\\ &\int_0^1 \left(1-t\right)g\left(ta+(1-t)\,b\right)dt\\ =\ &\frac{1}{b-a}\int_a^b \frac{a-x}{a-b}g\left(x\right)dx=\frac{1}{\left(b-a\right)^2}\int_a^b \left(x-a\right)g\left(x\right)dx,\end{aligned}$$

and

$$\begin{aligned}\int_0^1 tf\left(ta+(1-t)\,b\right)dt &= \frac{1}{\left(b-a\right)^2}\int_a^b \left(b-x\right)f\left(x\right)dx\\ \int_0^1 \left(1-t\right)f\left(ta+(1-t)\,b\right)dt &= \frac{1}{\left(b-a\right)^2}\int_a^b \left(x-a\right)f\left(x\right)dx.\end{aligned}$$

It can be easily checked that

$$\int_0^1 t^2dt=\int_0^1 \left(1-t\right)^2 dt=\frac{1}{3},\qquad \int_0^1 t\left(1-t\right)dt=\frac{1}{6}$$

$$\int_0^1 f\left(ta+(1-t)\,b\right)g\left(ta+(1-t)\,b\right)dt=\frac{1}{b-a}\int_a^b f\left(x\right)g\left(x\right)dx.$$

When the above expressions are taken into account, the proof is complete. □

Theorem 3 *Let* $f,g:[a,b]\to\mathbb{R}$ *be two convex functions and* $fg\in L^1\left([a,b]\right)$*. Then*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b \left(f\left(\frac{a+b}{2}\right)g\left(x\right)+g\left(\frac{a+b}{2}\right)f\left(x\right)\right)dx \qquad (2.3)\\ \le\ &\frac{1}{2\left(b-a\right)}\int_a^b f\left(x\right)g\left(x\right)dx\\ &+\frac{1}{12}M\left(a,b\right)+\frac{1}{6}N\left(a,b\right)+f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right),\end{aligned}$$

where $M(a,b)=f(a)g(a)+f(b)g(b), N(a,b)=f(a)g(b)+f(b)g(a)$.

Proof. Since f and g are convex on $[a,b]$, we have that

$$f\left(\frac{a+b}{2}\right)=f\left(\frac{ta+(1-t)\,b}{2}+\frac{(1-t)\,a+tb}{2}\right)\le\frac{f\left(ta+(1-t)\,b\right)+f\left((1-t)\,a+tb\right)}{2}$$

$$g\left(\frac{a+b}{2}\right)=g\left(\frac{ta+(1-t)\,b}{2}+\frac{(1-t)\,a+tb}{2}\right)\leq\frac{g\,(ta+(1-t)\,b)+g\,((1-t)\,a+tb)}{2}$$

for $t\in[0,1]$. Again as explained, in the proof of inequality (2.2) given above, we multiply by one under the other and by one across the other of the above inequality and then we add these inequalities, so we obtain

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right)\frac{g\,(ta+(1-t)\,b)+g\,((1-t)\,a+tb)}{2} \\
& +g\left(\frac{a+b}{2}\right)\frac{f\,(ta+(1-t)\,b)+f\,((1-t)\,a+tb)}{2} \\
\leq\; & \frac{f\,(ta+(1-t)\,b)+f\,((1-t)\,a+tb)}{2}\,\frac{g\,(ta+(1-t)\,b)+g\,((1-t)\,a+tb)}{2} \\
& +f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)
\end{aligned}$$

then

$$\begin{aligned}
& \frac{1}{2}f\left(\frac{a+b}{2}\right)[g\,(ta+(1-t)\,b)+g\,((1-t)\,a+tb)] \\
& +\frac{1}{2}g\left(\frac{a+b}{2}\right)[f\,(ta+(1-t)\,b)+f\,((1-t)\,a+tb)] \\
\leq\; & \frac{1}{4}f\,(ta+(1-t)\,b)\,g\,(ta+(1-t)\,b) \\
& +\frac{1}{4}f\,((1-t)\,a+tb)\,g\,((1-t)\,a+tb) \\
& +\frac{1}{4}f\,(ta+(1-t)\,b)\,g\,((1-t)\,a+tb) \\
& +\frac{1}{4}f\,((1-t)\,a+tb)\,g\,(ta+(1-t)\,b)+f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\
\leq\; & f\,(ta+(1-t)\,b)\,g\,(ta+(1-t)\,b) \\
& +\frac{1}{4}f\,((1-t)\,a+tb)\,g\,((1-t)\,a+tb) \\
& +\frac{1}{4}\left[tf\,(a)+(1-t)\,f\,(b)\right]\left[(1-t)\,g\,(a)+tg\,(b)\right] \\
& +\frac{1}{4}\left[(1-t)\,f\,(a)+tf\,(b)\right]\left[tg\,(a)+(1-t)\,g\,(b)\right]+f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\
=\; & \frac{1}{4}f\,(ta+(1-t)\,b)\,g\,(ta+(1-t)\,b) \\
& +\frac{1}{4}f\,((1-t)\,a+tb)\,g\,((1-t)\,a+tb) \\
& +\frac{1}{4}\left[2t\,(1-t)\right]\left[f\,(a)\,g\,(a)+f\,(b)\,g\,(b)\right] \\
& +\frac{1}{4}\left[t^2+(1-t)^2\right]\left[f\,(a)\,g\,(b)+f\,(b)\,g\,(a)\right]+f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right).
\end{aligned}$$

Again as explained, in the proof of inequality (2.2) given above we integrate both sides of the above inequality over $[0,1]$ and obtain

$$\begin{aligned}
& \frac{1}{2}f\left(\frac{a+b}{2}\right)\int_0^1 \left[g\left(ta+(1-t)b\right)+g\left((1-t)a+tb\right)\right]dt \\
& +\frac{1}{2}g\left(\frac{a+b}{2}\right)\int_0^1 \left[f\left(ta+(1-t)b\right)+f\left((1-t)a+tb\right)\right]dt \\
\leq\ & \frac{1}{4}\int_0^1 f\left(ta+(1-t)b\right)g\left(ta+(1-t)b\right)dt \\
& +\frac{1}{4}\int_0^1 f\left((1-t)a+tb\right)g\left((1-t)a+tb\right)dt \\
& +\frac{1}{4}\left[f(a)g(a)+f(b)g(b)\right]\int_0^1 \left[2t(1-t)\right]dt \\
& +\frac{1}{4}\left[f(a)g(b)+f(b)g(a)\right]\int_0^1 \left[t^2+(1-t)^2\right]dt \\
& +f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\int_0^1 dt.
\end{aligned}$$

By substituting $ta+(1-t)b=x$, it is easy to observe that

$$\int_0^1 f\left(ta+(1-t)b\right)g\left(ta+(1-t)b\right)dt=\frac{1}{b-a}\int_a^b f(x)g(x)\,dx$$

$$\int_0^1 f\left(ta+(1-t)b\right)dt=\int_0^1 f\left((1-t)a+tb\right)dt=\frac{1}{b-a}\int_a^b f(x)\,dx.$$

From the above inequality it is easy to observe that

$$\begin{aligned}
& \frac{f\left(\frac{a+b}{2}\right)}{b-a}\int_a^b g(x)\,dx+\frac{g\left(\frac{a+b}{2}\right)}{b-a}\int_a^b f(x)\,dx \\
\leq\ & \frac{1}{2(b-a)}\int_a^b f(x)g(x)\,dx \\
& +\frac{1}{12}M(a,b)+\frac{1}{6}N(a,b)+f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)
\end{aligned}$$

The proof is complete. □

3. Applications for special means

As in [8], we shall consider the means as arbitrary real numbers $a,b,a\neq b$. In the sources there includes $A(a,b)=\frac{a+b}{2}$, $a,b\in\mathbb{R}$, (arithmetic mean);

$K(a,b)=\sqrt{\frac{a^2+b^2}{2}}, \quad a,b\in\mathbb{R}, \quad$ (quadratic mean);

$L(a,b)=\frac{b-a}{\ln|b|-\ln|a|}, \quad |a|\neq|b|, ab\neq 0, \quad$ (logarithmic mean);

$G(a,b)=\sqrt{ab}, \quad$ (geometric mean);

$L_n(a,b)=\left[\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}\right]^{1/n}, \quad n\in\mathbb{Z}\backslash\{-1,0\}, a,b\in\mathbb{R}, a\neq b,$ (generalized log-mean).

Now, using the results of Section 2, we illustrate some applications of special means of real numbers.

Proposition 4 *Let* $0<a<b$*. Then,*

$$\frac{4A(a,b)-2L(a,b)}{L(a,b)G^2(a,b)}\leq\frac{2K^2(a,b)+4G^2(a,b)}{3G^4(a,b)} \tag{3.4}$$

or

$$\frac{4A(a,b)}{L(a,b)}\leq\frac{2K^2(a,b)}{3G^2(a,b)}+\frac{10}{3}.$$

Proof. The assertion from Theorem 2 is applied to the convex mapping $f(x)=g(x)=\frac{1}{x}, x\in[a,b]$. □

Proposition 5 *Let* $a,b\in I^o\subset\mathbb{R}$ *(*I^o *is the interior of* I*),* $a<b, 0\notin[a,b]$ *and* $n\in\mathbb{Z}\backslash\{-1,0\}, a\neq b$*. Then,*

$$\begin{aligned}&2A^n(a,b)L_n^n(a,b) \\ \leq\;&\frac{1}{2}L_{2n}^{2n}(a,b)+\frac{K^2(a^n,b^n)}{6}+\frac{G^{2n}(a,b)}{3}+A^{2n}(a,b).\end{aligned} \tag{3.5}$$

Proof. The assertion from Theorem 3 is applied to the convex mapping $f(x)=g(x)=x^n, x\in[a,b]$ and $n\in\mathbb{Z}\backslash\{-1,0\}, a\neq b$. □

References

[1] Dragomir, S.S. *Two mappings in connection to Hadamard's inequalities,* J.Math.Anal.Appl. 167, 49-56, (1992).

[2] Hadamard, J. *Etude sur les properties des fonctions entières et en particulier d'une fonction considerée par Riemann,* J.Math.Pures appl. 58, 171-215, (1893).

[3] Heing, H.P., Maligranda, L. *Chebyshev inequality in function spaces,* Real Analysis Exchange 17, 211-47, (1991/92).

[4] Maligranda, L., Pecaric, J.E. and Persson, L.E. *On some inequalities of the Gruss-Barnes and Borell type,* J.Math.Ana.Appl. 187, 306-323, (1994).

[5] Mitrinovic, D.S. *Analytic Inequalities,* Springer-Verlag, Berlin, New York 1970.

[6] Pecaric, J.E., Dragomir, S.S. *A generalization of Hadamard's inequality for isotonic linear functionals,* Radovi Matematicki 7, 103-107, (1991).

[7] Pachpatte, B.G. *On some inequalities for convex functions,* RGMIA Res. Rep. Coll., 6 (E), (2003).

[8] Kırmacı, U.S. *Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula,* Appl. Math.Comput. 147, 137–146, (2004).

Mevlüt TUNÇ
Department of Mathematics,
Faculty of Arts and Sciences,
Kilis 7 Aralik University,
Kilis, 79000, TURKEY
e-mail: mevluttunc@kilis.edu.tr

Received: 26.05.2010