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Note on Hilbert-type inequalities

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Abstract

The main objective of this paper is to prove Hilbert-type and Hardy-Hilbert-type inequalities with a general homogeneous kernel, thus generalizing a result obtained in [Namita Das and Srinibas Sahoo, A generalization of multiple Hardy-Hilbert's integral inequality, Journal of Mathematical Inequalities, 3(1), (2009), 139–154].

1. Introduction

Bicheng Yang in [4] proved a Hilbert-type inequality for conjugate parameters and with the kernel $K(x,y) = (u(x)+u(y))^{-s}$, $s > 0$. His result is contained in the following theorem.

Theorem A *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\phi_r > 0$ $(r = p, q)$, $\phi_p + \phi_q = s$, $u(t)$ *is a differentiable strict increasing function in* (a,b) $(-\infty \le a < b \le \infty)$, *such that* $u(a+) = 0$ *and* $u(b-) = \infty$, *and* $f, g \ge 0$ *satisfy* $0 < \int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f^p(x)dx < \infty$ *and* $0 < \int_a^b \frac{(u(x))^{q(1-\phi_p)-1}}{(u'(x))^{q-1}} g^q(x)dx < \infty$, *then*

$$\int_a^b \int_a^b \frac{f(x)g(y)}{(u(x)+u(y))^s} dxdy \tag{1.1}$$

$$< B(\phi_p, \phi_q) \left(\int_a^b \frac{(u(x))^{p(1-\phi_q)-1}}{(u'(x))^{p-1}} f^p(x)dx \right)^{\frac{1}{p}} \left(\int_a^b \frac{(u(x))^{q(1-\phi_p)-1}}{(u'(x))^{q-1}} g^q(x)dx \right)^{\frac{1}{q}},$$

where the constant factor $B(\phi_p, \phi_q)$ *is the best possible. If* $p < 1$ $(p \neq 0)$, $\{s; \phi_r > 0\ (r = p,q), \phi_p + \phi_q = s\} \neq \emptyset$, *with the above assumption, one has the reverse of (1.1), and the constant is still the best possible.*

Recently, Namita Das et al. [1] gave a generalization of Yang's result:

Theorem B *Let* $n \in \mathbb{N}\backslash\{1\}$, $p_i > 1$, $(i = 1, 2, \ldots, n)$, $\sum_{i=1}^n \frac{1}{p_i} = 1$, $s > 0$, $\lambda_i > 0$ $(i = 1, 2, \ldots, n)$ *with* $\sum_{i=1}^n \lambda_i = s$. *Suppose for every* $= 1, \ldots, n$; $u_i : (a_i, b_i) \to (0, \infty)$, *is a strictly increasing differentiable function such that* $u_i(a_i) = 0$ *and* $u_i(b_i) = \infty$. *If* $f_i \ge 0$ $(j = 1, 2, \ldots, n)$, *satisfy*

$$0 < \int_{a_j}^{b_j} (u_j(x))^{p_j(1-\lambda_j)-1} (u_j'(x))^{1-p_j} f_j^{p_j}(x)dx < \infty,$$

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then

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \frac{1}{(\sum_{i=1}^n u_i(x_i))^s} \prod_{j=1}^n f_j(x_j) dx_1 \dots dx_n$$
$$< \frac{1}{\Gamma(s)} \prod_{j=1}^n \Gamma(\lambda_j) \left(\int_{a_j}^{b_j} (u_j(x_j))^{p_j(1-\lambda_j)-1} (u_j'(x_j))^{1-p_j} f_j^{p_j}(x_j) dx_j \right)^{\frac{1}{p_j}},$$

where the constant factors $\frac{1}{\Gamma(s)} \prod_{j=1}^n \Gamma(\lambda_j)$ *is the best possible.*

Our main objective is to emphasize the previous theorem. Our generalization will include a general homogeneous kernel. In what follows we suppose that $K(x_1, \dots, x_n)$ is non-negative measurable homogeneous function of degree $-s$, $s > 0$. To obtain the main results we define the function $k(\beta_1, \dots, \beta_{n-1})$ by

$$k(\beta_1, \dots, \beta_{n-1}) := \int_{(0,\infty)^{n-1}} K(1, t_1 \dots, t_{n-1}) t_1^{\beta_1} \cdots t_{n-1}^{\beta_{n-1}} dt_1 \cdots dt_{n-1}, \tag{1.2}$$

where we suppose that $k(\beta_1, \dots, \beta_{n-1}) < \infty$ for $\beta_1, \dots, \beta_{n-1} > -1$ and $\beta_1 + \dots + \beta_{n-1} + n < s + 1$.

Let A_{ij}, $i, j = 1, \dots, n$, be the real numbers satisfying

$$\sum_{i=1}^n A_{ij} = 0, \qquad j = 1, 2, \dots, n. \tag{1.3}$$

We also define

$$\alpha_i = \sum_{j=1}^n A_{ij}, \qquad i = 1, 2, \dots, n. \tag{1.4}$$

Our results will be based on the following result of Perić and Vuković from [2].

Theorem C *Let* $p_1, \dots, p_n$ *be conjugate parameters such that* $p_i > 1$, $i = 1, \dots, n$, *and let* $\frac{1}{q} = \sum_{i=1}^{n-1} \frac{1}{p_i}$. *Let* $K : (0,\infty)^n \to \mathbb{R}$ *be non-negative measurable homogeneous function of degree* $-s$, $s > 0$, *and let* A_{ij}, $i, j = 1, \dots, n$, *and* α_i, $i = 1, \dots, n$ *be real parameters satisfying (1.3) and (1.4). If* $f_i : (0, \infty) \to \mathbf{R}$, $f_i \neq 0$, $i = 1, \dots, n$ *are non-negative measurable functions, then the following inequalities hold and are equivalent:*

$$\int_{(0,\infty)^n} K(x_1, \dots, x_n) \prod_{i=1}^n f_i(x_i) dx_1 \cdots dx_n$$
$$< L \prod_{i=1}^n \left(\int_0^\infty x_i^{n-s-1+p_i\alpha_i} f_i^{p_i}(x_i) dx_i \right)^{\frac{1}{p_i}} \tag{1.5}$$

and

$$\int_0^\infty x_n^{(1-q)(n-1-s)-q\alpha_n}\left(\int_{(0,\infty)^{n-1}} K(x_1,\ldots,x_n)\prod_{i=1}^{n-1} f_i(x_i)dx_1\cdots dx_{n-1}\right)^q dx_n$$

$$< L^q\prod_{i=1}^{n-1}\left(\int_0^\infty x_i^{n-1-s+p_i\alpha_i} f_i^{p_i}(x_i)dx_i\right)^{\frac{q}{p_i}}, \tag{1.6}$$

where

$$L = k(p_1A_{12},\ldots,p_1A_{1n})^{\frac{1}{p_1}}\cdot k(s-n-p_2(\alpha_2-A_{22}),p_2A_{23},\ldots,p_2A_{2n})^{\frac{1}{p_2}}$$

$$\cdots k(p_nA_{n2},\ldots,p_nA_{n,n-1},s-n-p_n(\alpha_n-A_{nn}))^{\frac{1}{p_n}}, \tag{1.7}$$

and $p_iA_{ij} > -1,\ i\neq j,\ p_i(A_{ii}-\alpha_i) > n-s-1.$

In what follows, without further explanation, we assume that all integrals exist on the respective domains of their definitions.

2. Main results

By applying Theorem C we get the following theorem.

Theorem 1 *Let* $K:(0,\infty)^n\to\mathbb{R}$ *and* $A_{ij},\ i,j=1,\ldots,n,$ *be as in Theorem C. Suppose for every* $i=1,\ldots,n$; $u_i:(a_i,b_i)\to(0,\infty)$, *is a strictly increasing differentiable function such that* $u_i(a_i)=0$ *and* $u_i(b_i)=\infty$. *If* $f_i:(0,\infty)\to\mathbb{R},\ f_i\neq 0,\ i=1,\ldots,n$ *are non-negative measurable functions, then the following inequalities hold and are equivalent*

$$\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} K(u_1(t_1),\ldots,u_n(t_n))\prod_{i=1}^{n} f_i(t_i)dt_1\ldots dt_n \tag{2.1}$$

$$< L\prod_{i=1}^{n}\left(\int_{a_i}^{b_i}(u_i(t_i))^{n-s-1+p_i\alpha_i}(u_i'(t_i))^{1-p_i}f_i^{p_i}(t_i)dt_i\right)^{\frac{1}{p_i}}$$

and

$$\int_{a_n}^{b_n}(u_n(t_n))^{(1-q)(n-1-s)-q\alpha_n}\left(\int_{a_1}^{b_1}\cdots\int_{a_{n-1}}^{b_{n-1}} K(u_1(t_1),\ldots,u_n(t_n))\prod_{i=1}^{n-1} f_i(t_i)dt_1\ldots dt_{n-1}\right)^q dt_n$$

$$< L^q\prod_{i=1}^{n-1}\left(\int_{a_i}^{b_i}(u_i(t_i))^{n-s-1+p_i\alpha_i}(u_i'(t_i))^{1-p_i}f_i^{p_i}(t_i)dt_i\right)^{\frac{q}{p_i}}, \tag{2.2}$$

where the constant L *is defined by (1.7).*

Proof. The proof follows directly from Theorem C setting the functions $g_i : [0,\infty) \to \mathbb{R}$, $i = 1,\dots,n$, such that $f_i(t_i) = g_i(u_i(t_i))u_i'(t_i)$. Namely, the inequality (1.5) with the functions g_i defined above, becomes

$$\int_{(0,\infty)^n} K(x_1,\dots,x_n)\prod_{i=1}^{n} g_i(x_i)dx_1\dots dx_n \tag{2.3}$$
$$< L\prod_{i=1}^{n}\left(x_i^{n-s-1+p_i\alpha_i} g_i^{p_i}(x_i)dx_i\right)^{\frac{1}{p_i}},$$

where the constant L is defined by (1.7). Now, let I and J denote the left-hand and right-hand side of the inequalities (2.3) respectively. By using the substitution $x_i = u_i(t_i)$, $i = 1,\dots,n$, we obtain

$$I = \int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} K(u_1(t_1),\dots,u_n(t_n))\prod_{i=1}^{n}\left(g_i(u_i(t_i))u_i'(t_i)\right)dt_1\dots dt_n, \tag{2.4}$$

where we used the facts $u_i(a_i) = 0$ and $u_i(b_i) = \infty$.
Similarly, we get

$$J = L\prod_{i=1}^{n}\left(\int_{a_i}^{b_i}(u_i(t_i))^{n-s-1+p_i\alpha_i} g_i^{p_i}(u_i(t_i))u_i'(t_i)dt_i\right)^{\frac{1}{p_i}}$$
$$= L\prod_{i=1}^{n}\left(\int_{a_i}^{b_i}(u_i(t_i))^{n-s-1+p_i\alpha_i}(u_i'(t_i))^{1-p_i} g_i^{p_i}(u_i(t_i))(u_i'(t_i))^{p_i}dt_i\right)^{\frac{1}{p_i}}. \tag{2.5}$$

Now, from (2.3), (2.4), (2.5) and the fact $f_i(t_i) = g_i(u_i(t_i))u_i'(t_i)$ follows the inequality (2.1). The second inequality (2.2) can be proved by applying (1.6) from Theorem C. □

To obtain a case of the best possible inequality it is natural to impose the following conditions on the parameters A_{ij}:

$$p_jA_{ji} = s - n - p_i(\alpha_i - A_{ii}),\ i,j = 1,2,\dots,n,\ i \neq j. \tag{2.6}$$

In that case the constant L from Theorem 1 is simplified to the form:

$$L^* = k(\widetilde{A}_2,\dots,\widetilde{A}_n), \tag{2.7}$$

where

$$\widetilde{A}_i = p_jA_{ji},\ \ i,j = 1,2,\dots,n,\ i \neq j. \tag{2.8}$$

It is easy to see that the parameters $\widetilde{A}_i$ satisfy the relation

$$\sum_{i=1}^{n}\widetilde{A}_i = s - n. \tag{2.9}$$

By using (1.3) and (2.8) we have

$$\begin{aligned} A_{ii} &= -A_{1i} - A_{2i} - \cdots - A_{i-1,i} - A_{i+1,i} - \cdots - A_{ni} \\ &= -\frac{\widetilde{A}_i}{p_1} - \frac{\widetilde{A}_i}{p_2} - \cdots - \frac{\widetilde{A}_i}{p_{i-1}} - \frac{\widetilde{A}_i}{p_{i+1}} - \cdots - \frac{\widetilde{A}_i}{p_n} \\ &= \widetilde{A}_i \left(\frac{1}{p_i} - 1 \right). \end{aligned} \tag{2.10}$$

Further, by using (2.7) and (2.8) and (2.10), the inequalities (2.1) and (2.2) with the parameters A_{ij}, satisfying the relation (2.6), become

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} K(u_1(t_1), \ldots, u_n(t_n)) \prod_{i=1}^{n} f_i(t_i) dt_1 \ldots dt_n \tag{2.11}$$

$$< L^* \prod_{i=1}^{n} \left(\int_{a_i}^{b_i} (u_i(t_i))^{-1-p_i \widetilde{A}_i} (u_i'(t_i))^{1-p_i} f_i^{p_i}(t_i) dt_i \right)^{\frac{1}{p_i}}$$

and

$$\int_{a_n}^{b_n} (u_n(t_n))^{(1-q)(-1-p_n \widetilde{A}_n)} \left(\int_{a_1}^{b_1} \cdots \int_{a_{n-1}}^{b_{n-1}} K(u_1(t_1), \ldots, u_n(t_n)) \prod_{i=1}^{n-1} f_i(t_i) dt_1 \ldots dt_{n-1} \right)^q dt_n$$

$$< (L^*)^q \prod_{i=1}^{n-1} \left(\int_{a_i}^{b_i} (u_i(t_i))^{-1-p_i \widetilde{A}_i} (u_i'(t_i))^{1-p_i} f_i^{p_i}(t_i) dt_i \right)^{\frac{q}{p_i}}. \tag{2.12}$$

In the following theorem we show that, if the parameters A_{ij} satisfy condition (2.6), then one obtains the best possible constant.

Theorem 2 *If the parameters A_{ij}, $i, j = 1, \ldots, n$, satisfy conditions (1.3) and (2.6), then the constants L^* and $(L^*)^q$ are the best possible in inequalities (2.11) and (2.12).*

Proof. As in the proof of Theorem 1, let $g_i : [0, \infty) \to \mathbb{R}$, $i = 1, \ldots, n$ be the functions such that $f_i(t_i) = g_i(u_i(t_i)) u_i'(t_i)$. The inequality (1.5) with the functions g_i defined above, becomes

$$\int_{(0,\infty)^n} K(x_1, \ldots, x_n) \prod_{i=1}^{n} g_i(x_i) dx_1 \ldots dx_n \tag{2.13}$$

$$< L^* \prod_{i=1}^{n} \left(\int_0^\infty x_i^{-1-p_i \widetilde{A}_i} g_i^{p_i}(x_i) dx_i \right)^{\frac{1}{p_i}},$$

where the constant L^* is defined by (2.7).

Now, let's suppose that the constant factor L^* is not the best possible. Then, there exists a positive constant L_1, smaller than L^* such that the inequality (2.13) is still valid if we replace L^* by L_1. For this purpose, set

$$\widetilde{g}_i(x_i) = \begin{cases} 0 & x \in (0,1) \\ x_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i}} & x \in [1, \infty) \end{cases}, \quad i = 1, \ldots, n,$$

where $0<\varepsilon<\min_{1\le i\le n}\{p_i+p_i\widetilde{A}_i\}$. If we put these functions in the inequality (2.13), then the right-hand side of the inequality becomes $\frac{L_1}{\varepsilon}$, since

$$\prod_{i=1}^{n}\left[\int_0^\infty {x_i}^{-1-p_i\widetilde{A}_i}\widetilde{g}_i^{p_i}(x_i)dx_i\right]^{\frac{1}{p_i}}=\frac{1}{\varepsilon}. \tag{2.14}$$

Further, let J denotes the left-hand side of the inequality (2.13), for above choice of the functions $\widetilde{g}_i$. By using substitution $u_i=\frac{x_i}{x_1}$, $i=2,\dots,n$ in J, we find that

$$J=\int_1^\infty {x_1}^{-1-\varepsilon}\left[\int_{\frac{1}{x_1}}^\infty\cdots\int_{\frac{1}{x_1}}^\infty K(1,u_2,\dots,u_n)\prod_{i=2}^{n}{u_i}^{\widetilde{A}_i-\frac{\varepsilon}{p_i}}du_2\dots du_n\right]dx_1. \tag{2.15}$$

It is easy to see that the following inequality holds

$$\begin{aligned}
J &\ge \int_1^\infty {x_1}^{-1-\varepsilon}\left[\int_{(0,\infty)^{n-1}}K(1,u_2,\dots,u_n)\prod_{i=2}^{n}{u_i}^{\widetilde{A}_i-\frac{\varepsilon}{p_i}}du_2\dots du_n\right]dx_1\\
&\quad-\int_1^\infty {x_1}^{-1-\varepsilon}\sum_{j=2}^{n}I_j(x_1)dx_1\\
&= \frac{1}{\varepsilon}k\left(\widetilde{A}_2-\frac{\varepsilon}{p_2},\dots,\widetilde{A}_n-\frac{\varepsilon}{p_n}\right)-\int_1^\infty x_1^{-1-\varepsilon}\sum_{j=2}^{n}I_j(x_1)dx_1,
\end{aligned} \tag{2.16}$$

where for $j=2,\dots,n$, $I_j(x_1)$ is defined by

$$I_j(x_1)=\int_{D_j}K(1,u_2,\dots,u_n)\prod_{i=2}^{n}{u_i}^{\widetilde{A}_i-\frac{\varepsilon}{p_i}}du_2\dots du_n,$$

where $D_j=\{(u_2,u_3,\dots,u_n);0<u_j\le\frac{1}{x_1},\ 0<u_k<\infty,k\ne j\}$. Without losing generality, we only estimate the integral $I_2(x_1)$. In fact, since $1-u_2^\varepsilon\to 1$ $(u_2\to 0^+)$, there exists $M\ge 0$ such that $1-u_2^\varepsilon\le M$ $(u_2\in(0,1])$, and by Fubini's theorem, it follows that

$$\begin{aligned}
0 &\leq \varepsilon \int_1^\infty x_1^{-1-\varepsilon} I_2(x_1)dx_1 \\
&= \varepsilon \int_1^\infty x_1{}^{-1-\varepsilon} \left[\int_{(0,\infty)^{n-2}} \int_0^{\frac{1}{x_1}} K(1,u_2,\ldots,u_n) \prod_{i=2}^n u_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i}} du_2 \ldots du_n \right] dx_1 \\
&= \varepsilon \int_{(0,\infty)^{n-2}} \int_0^1 K(1,u_2,\ldots,u_n) \prod_{i=2}^n u_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i}} \left(\int_1^{\frac{1}{u_2}} x_1{}^{-1-\varepsilon} dx_1 \right) du_2 \ldots du_n \\
&= \varepsilon \int_{(0,\infty)^{n-2}} \int_0^1 K(1,u_2,\ldots,u_n) \prod_{i=2}^n u_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i}} \left(\frac{1}{\varepsilon}(1-u_2^\varepsilon) \right) du_2 \ldots du_n \\
&\leq M \int_{(0,\infty)^{n-2}} \int_0^1 K(1,u_2,\ldots,u_n) \prod_{i=2}^n u_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i}} du_2 \ldots du_n \\
&\leq M \int_{(0,\infty)^{n-1}} K(1,u_2,\ldots,u_n) \prod_{i=2}^n u_i^{\widetilde{A}_i - \frac{\varepsilon}{p_i}} du_2 \ldots du_n \\
&= M \cdot k \left(\widetilde{A}_2 - \frac{\varepsilon}{p_2}, \ldots, \widetilde{A}_n - \frac{\varepsilon}{p_n} \right) < \infty.
\end{aligned}$$

Hence by (2.16), we have that

$$J \geq \frac{1}{\varepsilon} k \left(\widetilde{A}_2 - \frac{\varepsilon}{p_2}, \ldots, \widetilde{A}_n - \frac{\varepsilon}{p_n} \right) - 0(1). \tag{2.17}$$

We conclude, by using (2.14) and (2.17), that $L^* \leq L_1$ when $\varepsilon \to 0^+$, which is an obvious contradiction. It follows that the constant L^* in (2.11) is the best possible.

Further, since the equivalence keeps the best possible constant, the proof is completed. □

3. Some applications

To obtain the following results we need some lemmas.

Lemma 1 *(see [3]) If $n \in \mathbb{N}$, $r_i > 0$, $i = 1, \ldots, n$, then*

$$\int_{(0,\infty)^{n-1}} \frac{\prod_{i=1}^{n-1} u_i^{r_i - 1}}{\left(1 + \sum_{i=1}^{n-1} u_i\right)^{\sum_{i=1}^n r_i}} du_1 \ldots du_{n-1} = \frac{\prod_{i=1}^n \Gamma(r_i)}{\Gamma\left(\sum_{i=1}^n r_i\right)}. \tag{3.1}$$

By using Lemma 1 we have

Lemma 2 *If $n \in \mathbb{N}$, $s, \lambda > 0$, $\beta_i > -1$, $i = 1, \dots, n-1$, and $\sum_{i=1}^{n-1} \beta_i < \lambda s - n + 1$, then*

$$\int_{(0,\infty)^{n-1}} \frac{\prod_{i=1}^{n-1} t_i^{\beta_i}}{\left(1 + \sum_{i=1}^{n-1} t_i^{\lambda}\right)^s} dt_1 \dots dt_{n-1} \tag{3.2}$$
$$= \frac{1}{\Gamma(s)\lambda^{n-1}} \left(\prod_{i=1}^{n-1} \Gamma\left(\frac{\beta_i + 1}{\lambda}\right)\right) \Gamma\left(s - \frac{1}{\lambda}\sum_{i=1}^{n-1}(\beta_i + 1)\right).$$

Proof. Let J denotes the left-hand side of the identity (3.2). By using the substitution $u_i = t_i^{\lambda}$, $i = 1, \dots, n-1$, we find that

$$J = \frac{1}{\lambda^{n-1}} \int_{(0,\infty)^{n-1}} \frac{\prod_{i=1}^{n-1} u_i^{\frac{\beta_i+1}{\lambda}-1}}{\left(1 + \sum_{i=1}^{n-1} u_i\right)^s} du_1 \dots du_{n-1}.$$

Applying Lemma 1 we get

$$J = \frac{1}{\Gamma(s)\lambda^{n-1}} \prod_{i=1}^{n} \Gamma(r_i),$$

where $r_i = \frac{\beta_i+1}{\lambda}$, $i = 1, \dots, n-1$, and $r_n = s - \frac{1}{\lambda}\sum_{i=1}^{n-1}(\beta_i + 1)$. In this way we prove (3.2). □

Remark 1 *It is easy to see that Theorem 2 is the generalization of Theorem B. Namely, let us define $\widetilde{A}_i = \lambda_i - 1$, $i = 1, \dots, n$, and $K(x_1, \dots, x_n) = (x_1 + \dots + x_n)^{-s}$. By using Lemma 1 we have $L = k(\lambda_1, \dots, \lambda_n) = \frac{1}{\Gamma(s)} \prod_{i=1}^{n} \Gamma(\lambda_i)$.*

We proceed with some special homogeneous function. Since the function $K(x_1, \dots, x_n) = \left(\sum_{i=1}^{n} x_i^{s(\lambda-1)}\right) / \left(\sum_{i=1}^{n} x_i^{\lambda}\right)^s$, $\lambda > 1$, is homogeneous of degree $-s$, by using Theorem 2 we obtain:

Corollary 1 *Let $n \geq 2$ be an integer and let $p_1, \dots, p_n$ be conjugate parameters such that $p_i > 1$, $i = 1, \dots, n$ and let $\frac{1}{q} = \sum_{i=1}^{n-1} \frac{1}{p_i}$. If $a > 1$, $f_i > 0$, $i = 1, \dots, n$, measurable functions, then the following inequalities hold and are equivalent:*

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\sum_{i=1}^{n} a^{s(\lambda-1)t_i}}{\left(\sum_{i=1}^{n} a^{\lambda t_i}\right)^s} \prod_{i=1}^{n} f_i(t_i) dt_1 \dots dt_n$$
$$< L_1 \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} a^{-st_i} f_i^{p_i}(t_i) dt_i \right)^{\frac{1}{p_i}} \tag{3.3}$$

and

$$\left[\int_{-\infty}^{\infty} a^{\frac{st_n}{p_n - 1} - t_n} \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\sum_{i=1}^{n} a^{s(\lambda-1)t_i}}{\left(\sum_{i=1}^{n} a^{\lambda t_i}\right)^s} \prod_{i=1}^{n-1} f_i(t_i) dt_1 \cdots dt_{n-1} \right)^q dt_n \right]^{\frac{1}{q}}$$
$$< L_1 \prod_{i=1}^{n-1} \left(\int_{-\infty}^{\infty} a^{-st_i} f_i^{p_i}(t_i) dt_i \right)^{\frac{1}{p_i}}, \tag{3.4}$$

where the constant

$$L_1 = \frac{(\lambda \ln a)^{1-n}}{\Gamma(s)} \sum_{j=1}^{n} \left[\left(\prod_{i=1, i\neq j}^{n} \Gamma\left(\frac{s}{p_i\lambda}\right) \right) \cdot \Gamma\left(\frac{sp_j(\lambda-1)+s}{p_j\lambda}\right) \right] \tag{3.5}$$

is the best possible in the inequalities (3.3) and (3.4).

Proof. Set $K(x_1, \ldots, x_n) = \left(\sum_{i=1}^n x_i^{s(\lambda-1)}\right) / \left(\sum_{i=1}^n x_i^{\lambda}\right)^s$, $\lambda > 0$, and $x_i = u_i(t_i) := a^{t_i}$, $i = 1, \ldots, n$, in Theorem 2. Then, using the notation of Theorem 1 we have $a_i = -\infty$ and $b_i = \infty$.
It is easy to see that the parameters A_{ij}, $i, j = 1, \ldots, n$, defined by

$$A_{ij} = \frac{s - p_j}{p_i p_j}$$

satisfy the condition (2.6). Therefrom, from the statement of Theorem 2 follows $\widetilde{A}_i = \frac{s-p_i}{p_i}$, $i = 1, \ldots, n$. Now, by using the definition of $u_i(t_i)$ and the parameters $\widetilde{A}_i$ we get

$$(u_i(t_i))^{-1-p_i\widetilde{A}_i}(u_i'(t_i))^{1-p_i} = (\ln a)^{1-p_i} a^{-st_i}$$

and

$$(u_n(t_n))^{(1-q)(-1-p_n\widetilde{A}_n)} = (a^{t_n})^{\frac{s}{p_n-1}-1}.$$

Further, it is enough to calculate the constant $L_1 = (\ln a)^{1-n} \cdot L$, where $L = k\left(\frac{s-p_2}{p_2}, \ldots, \frac{s-p_n}{p_n}\right)$. Using the definition of the function $k(\alpha_1, \ldots, \alpha_{n-1})$ given by (1.2) we have

$$L = \int_{(0,\infty)^{n-1}} \frac{1 + t_1^{s(\lambda-1)} + \ldots + t_{n-1}^{s(\lambda-1)}}{\left(1 + \sum_{i=1}^n t_i^{\lambda}\right)^s} t_1^{\frac{s}{p_2}-1} \ldots t_{n-1}^{\frac{s}{p_n}-1} dt_1 \ldots dt_{n-1} = \sum_{k=0}^{n-1} I_k, \tag{3.6}$$

where

$$I_0 = \int_{(0,\infty)^{n-1}} \frac{t_1^{\frac{s}{p_2}-1} \ldots t_{n-1}^{\frac{s}{p_n}-1}}{\left(1 + \sum_{i=1}^n t_i^{\lambda}\right)^s} dt_1 \ldots dt_{n-1}$$

and

$$I_k = \int_{(0,\infty)^{n-1}} \frac{t_1^{\frac{s}{p_2}-1} \ldots t_k^{s(\lambda-1)+\frac{s}{p_{k+1}}-1} \ldots t_{n-1}^{\frac{s}{p_n}-1}}{\left(1 + \sum_{i=1}^n t_i^{\lambda}\right)^s} dt_1 \ldots dt_{n-1}, \quad \text{for } k = 1, \ldots, n-1.$$

By using Lemma 2 we get

$$I_0 = \frac{1}{\Gamma(s)\lambda^{n-1}} \left(\prod_{i=2}^{n} \Gamma\left(\frac{s}{p_i\lambda}\right) \right) \cdot \Gamma\left(\frac{sp_1(\lambda-1)+s}{p_1\lambda}\right),$$

and similarly

$$I_k = \frac{1}{\Gamma(s)\lambda^{n-1}} \left(\prod_{i=1, i\neq k+1}^{n} \Gamma\left(\frac{s}{p_i\lambda}\right) \right) \cdot \Gamma\left(\frac{sp_{k+1}(\lambda-1)+s}{p_{k+1}\lambda}\right),$$

for $k = 1, \ldots, n-1$. Now, from (3.6) we get (3.5). □

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