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On normality of meromorphic functions with multiple zeros and sharing values

You-Ming Wang

Abstract

In this paper we study the problem of normal families of meromorphic functions concerning shared values. Let F be a family of meromorphic functions in the plane domain $D \subseteq \mathbb{C}$ and n be a positive integer. Let a, b be two finite complex constants such that $a \neq 0$. If $n \geq 3$ and $f + a(f')^n$ and $g + a(g')^n$ share b in D for every pair of functions $f, g \in F$, then F is normal in D. And some examples are provided to show the result is sharp.

Key words and phrases: Meromorphic functions, shared value, normal family.

1. Introduction and main results

In this paper, we denote by $\mathbb{C}$ the whole complex plane. Let f be a meromorphic function in a domain $D \subset \mathbb{C}$. For $a \in \mathbb{C}$, set $\overline{E}_f(a) = \{z \in D : f(z) = a\}$. We say that two meromorphic functions f and g share the value a provided that $\overline{E}_f(a) = \overline{E}_g(a)$ in D. When $a = \infty$ the zeros of $f - a$ mean the poles of f (see [3]).

Let h be a meromorphic function in a domain $D \subset \mathbb{C}$. We say h is a normal function if there exists a positive number M such that $h^\sharp(z) \leq M$ for all $z \in D$, where $h^\sharp(z) = \frac{|h'(z)|}{1+|h(z)|^2}$ denotes the spherical derivative of h.

Let F be a family of meromorphic functions in a domain $D \subseteq \mathbb{C}$. We say that F is normal in D if every sequence $\{f_n\} \subseteq F$ contains a subsequence which converges spherically uniformly on the compact subsets of D (see [10]).

According to Bloch′s principle, every condition which reduces a meromorphic function in the plane $\mathbb{C}$ to a constant, makes a family of meromorphic functions in a domain D normal. Although the principle is false in general (see [9]), many authors proved normality criteria for families of meromorphic functions by starting from Picard type theorems (see [6, 13, 14]). It is also more interesting to find normality criteria from the point of view of shared values. In this area, Schwick [11] first proved an interesting result that a family of meromorphic functions, in a domain in which every function shares three distinct finite complex numbers with its first derivative, is normal. And later, Sun [12] proved that a family of meromorphic functions, in a domain in

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which each pair of functions share three fixed distinct values, is normal. This is an improvement of the famous Montel′s Normal Criterion [7] by the idea of shared values. More results about normality criteria concerning shared values can be found, for instance, in [4, 8, 16] and so on.

In [15], Ye proved the following.

Theorem A *Let f be a transcendental meromorphic function and a be a nonzero finite complex number. Then $f+a(f')^n$ assumes every finite complex value infinitely often for each positive integer $n \geq 3$.*

Ye asked whether Theorem A remains valid for $n=2$. Fang and Zalcman, in [5], gave an affirmative answer to this question and obtained the following.

Theorem B *Let f be a transcendental meromorphic function and a be a nonzero finite complex number. Then $f+a(f')^n$ assumes every finite complex value infinitely often for each positive integer $n \geq 2$.*

Corresponding to Theorem B there are the following theorems about normal families in [5].

Theorem C *Let F be a family of meromorphic functions on the plane domain D, let $n \geq 2$ be a positive integer, and let $a \neq 0, b$ be two complex numbers. If, for each $f \in F$, all zeros of f are multiple and $f+a(f')^n \neq b$ on D, then F is normal in D.*

It is natural to ask whether Theorem C can be improved by the idea of shared values. In this paper, we study the problem and obtain the following theorem.

Theorem 1 *Let F be a family of meromorphic functions in the plane domain D and n be a positive integer. Let a, b be two finite complex constants such that $a \neq 0$. If $n \geq 3$ and $f+a(f')^n$ and $g+a(g')^n$ share b in D for every pair of functions $f, g \in F$, then F is normal in D.*

Example 1 Let $D=\{z:|z|<1\}$ and $F=\{f_n(z)\}$, where

$$f_n(z)=nz^2, \quad z \in D, \quad n=1,2,\ldots.$$

Clearly, $f+(f')^2=(n+4n^2)z^2$. So for each pair m,n, $f_n+(f_n')^2$ and $f_m+(f_m')^2$ share the value 0 in D, however, F fails to be normal in D since $f_n^\sharp(\frac{1}{\sqrt{n}})=\sqrt{n} \to \infty$ as $n \to \infty$.

Example 2 Let $D=\{z:|z|<1\}$ and $F=\{f_n(z)\}$, where

$$f_n(z)=nz, \quad z \in D, \quad n=1,2,\ldots.$$

Clearly, $f-(f')^3=n(z-n^2)$. So for each pair m,n, $f_n+(f_n')^3$ and $f_m+(f_m')^3$ share the value 0 in D, but, F fails to be normal in D since $f_n^\sharp(\frac{1}{n})=\frac{n}{2} \to \infty$ as $n \to \infty$.

Example 1 shows that Theorem 1 is not valid when $n=2$, so the condition $n=3$ is best possible for Theorem 1. And Example 2 shows that Theorem 1 is not valid when f has no multiple zeros, so the condition that f has only multiple zeros is best possible for Theorem 1.

2. Some Lemmas

Lemma 1 [1] *Let F be a family of meromorphic functions in the unit disk $\triangle \subseteq \mathbb{C}$ and let k be a positive integer. Suppose that all zeros of f have multiplicity at least k for every $f \in F$, and suppose that there exists a number $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. If F is not normal at $z_0 \in \triangle$, then for any $0 \leq \alpha \leq k$, there exist*
(1) a number $r \in (0,1)$;
(2) a sequence of complex numbers $z_n \to z_0, |z_n| \leq r$;
(3) a sequence of functions $f_n \in F$;
(4) a sequence of positive numbers $\rho_n \to 0$
such that $g_n(\xi) = \rho_n^{-\alpha} f_n(z_n + \rho_n \xi)$ converges locally uniformly (with respect to spherical metric) to a non-constant meromorphic function $g(\xi)$ on $\mathbb{C}$, and moreover, the zeros of $g(\xi)$ are of multiplicity at least k, $g^\sharp(\xi) \leq g^\sharp(0) = kA + 1$.

Remark 1 In Lemma 1, if $0 \leq \alpha < k$, then the hypothesis of $f^{(k)}$ can be dropped, and $kA+1$ can be replaced by an arbitrary positive number (see [1]).

Lemma [2] *A normal function has order at most two. A normal entire function is of exponential type, and thus has order at most one.*

Lemma 3 *Let $n \geq 3$ be a positive integer and f be a non-constant rational meromorphic function with multiple zeros, then $f + a(f')^n$ has at least two distinct zeros.*

Proof. **Case 1.** If $f + a(f')^n$ has no zeros.

Case 1.1. Since $n \geq 3$ and f is a non-constant function, it is easily obtained that f is not a polynomial.

Case 1.2. If f is rational but not a polynomial. Set $f(z) = \frac{p(z)}{q(z)}$ and use $\deg(g)$ to denote the degree of a polynomial g, where $p(z), q(z)$ are polynomials. Put $\deg(p(z)) = p \geq 2$ and $\deg(q(z)) = q$. Then

$$f + a(f')^n = \frac{p(z)q^{2n-1}(z) + a[p'(z)q(z) - p(z)q'(z)]^n}{q^{2n}(z)} \tag{2.1}$$

has no zeros. Recall that $\deg[p(z)q^{2n-1}(z)] = 2nq + p - q$ and $\deg[(p'(z)q(z) - p(z)q'(z))^n] \leq n(p + q - 1)$.

Case 1.2.1. If $q \geq p-1$, so $2nq+p-q > n(p+q-1)$ and then $\deg[p(z)q^{2n-1}(z)] > \deg[(p'(z)q(z)-p(z)q'(z))^n]$. Hence (2.1) means that $f + a(f')^n$ has zeros, which contradicts that $f + a(f')^n$ has no zeros.

Case 1.2.2. If $q < p - 1$, a simple calculation implies that $\deg[p'(z)q(z) - p(z)q'(z)]^n = n(p + q - 1)$ and $2nq + p - q < n(p + q - 1)$, therefore, $\deg[p(z)q^{2n-1}(z)] < \deg[(p'(z)q(z) - p(z)q'(z))^n]$. It follows from (2.1) that $f + a(f')^n$ has zeros. This is a contradiction, again.

Case 2. Suppose that $f + a(f')^n$ has exactly one zero z_0.

Case 2.1. If f is a non-constant polynomial.

Set $f+a(f')^n = A(z-z_0)^l$, where A is a non-zero constant, l is a positive integer and $l \geq 2$. Then $f'[1+an(f')^{n-2}f''] = Al(z-z_0)^{l-1}$. Recall that f has only multiple zeros. But f' has exactly the same zero z_0, so f has the same zero z_0 and z_0 is the unique zero of f. Thus $f(z) = A_0(z-z_0)^k$, where A_0 is non-zero constant, k is a positive integer and $k \geq 2$. Thus $f+a(f')^n = A_0(z-z_0)^k[1+aA_0^{n-1}(z-z_0)^{nk-n-k}]$ has at least two distinct zeros since $nk-n-k \geq 1$. This contradicts that our assumptions.

Case 2.2. If f is rational but not a polynomial. Suppose that $f+a(f')^n$ has exactly one zero z_0 with multiplicity l. So we deduce that f has has exactly one zero z_0 and then z_0 is the unique zero of f. Otherwise $f+a(f')^n$ has at least two distinct zeros, which contradicts that our assumptions.

We set

$$f(z) = \frac{A(z-z_0)^k}{(z-z_1)^{l_1}(z-z_2)^{l_2}\cdots(z-z_m)^{l_m}}, \tag{2.2}$$

where A is a non-zero constant and $l_i \geq 1 (i=1,2,\ldots,m), k \geq 2$.

For simplicity, we denote

$$l_1 + l_2 + \cdots + l_m = q.$$

From (2.2), it follows that

$$f'(z) = \frac{A(z-z_0)^{k-1}g(z)}{(z-z_1)^{l_1+1}(z-z_2)^{l_2+1}\cdots(z-z_m)^{l_m+1}}, \tag{2.3}$$

where $g(z) = k(z-z_1)(z-z_2)\cdots(z-z_m)+(z-z_0)[l_1(z-z_2)(z-z_3)\cdots(z-z_m)+\cdots+l_m(z-z_1)(z-z_2)\cdots(z-z_{m-1})]$.

From (2.2) and (2.3), then

$$\begin{aligned} f+a(f')^n &= \frac{A(z-z_0)^k(z-z_1)^{(n-1)l_1+n}(z-z_2)^{(n-1)l_2+n}\cdots(z-z_m)^{(n-1)l_m+n}}{(z-z_1)^{n(l_1+1)}(z-z_2)^{n(l_2+1)}\cdots(z-z_m)^{n(l_m+1)}} \\ &\quad + \frac{aA^n(z-z_0)^{n(k-1)}g^n(z)}{(z-z_1)^{n(l_1+1)}(z-z_2)^{n(l_2+1)}\cdots(z-z_m)^{n(l_m+1)}}. \end{aligned} \tag{2.4}$$

Since $n(k-1) > k$ for $n \geq 3$ and $k \geq 2$, then (2.4) implies that

$$f+a(f')^n = \frac{A(z-z_0)^k g_1(z)}{(z-z_1)^{n(l_1+1)}(z-z_2)^{n(l_2+1)}\cdots(z-z_m)^{n(l_m+1)}}, \tag{2.5}$$

here $g_1(z) = [(z-z_1)^{(n-1)l_1+n}(z-z_2)^{(n-1)l_2+n}\cdots(z-z_m)^{(n-1)l_m+n} + aA^{n-1}(z-z_0)^{n(k-1)-k}g^n(z)]$. By the assumption that $f+a(f')^n$ has exactly one zero z_0 with multiply l, we deduce from (2.5) that

$$f+a(f')^n = \frac{C(z-z_0)^l}{(z-z_1)^{n(l_1+1)}(z-z_2)^{n(l_2+1)}\cdots(z-z_m)^{n(l_m+1)}}. \tag{2.6}$$

Then (2.5) and (2.6) mean that

$$C(z-z_0)^l \equiv A(z-c_0)^k g_1(z), \tag{2.7}$$

where C is a non-zero constant.

Case 2.2.1. If $l > k$, By (2.7) we conclude that g_1 has a zero z_0 and then $(z-z_1)^{(n-1)l_1+n}(z-z_2)^{(n-1)l_2+n}\cdots(z-z_m)^{(n-1)l_m+n}$ has a zero z_0, which is impossible.

Case 2.2.2. If $l = k$, (2.7) implies that

$$h_1(z) + h_2(z) \equiv \frac{C}{A}, \tag{2.8}$$

where $h_1(z) = (z-z_1)^{(n-1)l_1+n}(z-z_2)^{(n-1)l_2+n}\cdots(z-z_m)^{(n-1)l_m+n}$ and $h_2(z) = aA^{n-1}(z-z_0)^{n(k-1)-k}g^n(z)$.

We easily obtain from (2.8) that $\deg(h_1) = \deg(h_2)$. On the other hand, we deduce from Case 1.2 that

$$\deg[p(z)q^{2n-1}(z)] \neq \deg[(p'(z)q(z) - p(z)q'(z))^n].$$

Then (2.5) and the definitions of g_1, h_1 and h_2 yield that $\deg(h_1) \neq \deg(h_2)$. We thus have a contradiction again. □

The proof is complete.

3. Proof of Theorem

Proof of Theorem 1. Suppose that F is not normal in D. Then there exists at least one point z_0 such that F is not normal at the point z_0. Without loss of generality we assume that $z_0 = 0$ and $D = \triangle$. We consider two cases.

Case 1. $b = 0$. Since the zeros of f have multiplicity at least 2, then we may apply Lemma 1 with any positive value of α. Take $\alpha = \frac{n}{n-1}$, there exist:

(1) a real number r, $r < 1$;

(2) points $z_k \to 0$, $|z_k| < r$;

(3) positive numbers ρ_k, $\rho_k \to 0$; and

(4) functions f_k, $f_k \in F$ such that

$$g_k(\xi) = \rho_k^{-\frac{n}{n-1}} f_k(z_k + \rho_k\xi) \to g(\xi) \tag{3.1}$$

locally uniformly with respect to spherical metric on $\mathbb{C}$, where $g(\xi)$ is a non-constant meromorphic function and all of whose zeros are multiple.

From (3.1) we obtain

$$g_k' = \rho_k^{-\frac{1}{n-1}} f_k' \to g',$$

and

$$\rho_k^{-\frac{n}{n-1}}[f_k + a(f_k')^n] = g_k + a(g_k')^n \to g + a(g')^n, \tag{3.2}$$

also locally uniformly with respect to the spherical metric.

If $g+a(g')^n\equiv 0$, then g clearly has no poles and is not any polynomial with order at least 2, so g is a transcendental entire function. By Lemma 2, g is of exponential type. Since $a(g')^{n-1}\equiv -\frac{g}{g'}$, by Nevanlinna′s First Fundamental Theorem, it means that

$$\begin{aligned}(n-1)m(r,g) &\le (n-1)m\left(r,g'\right)+(n-1)m\left(r,\frac{g}{g'}\right)+O(1)\\ &= m\left(r,(g')^{n-1}\right)+(n-1)m\left(r,\frac{g}{g'}\right)+O(1)=nm\left(r,\frac{g}{g'}\right)+O(1)\\ &= n\left[m\left(r,\frac{g'}{g}\right)+N\left(r,\frac{g'}{g}\right)-N\left(r,\frac{g}{g'}\right)\right]+O(1)\\ &\le n\overline{N}\left(r,\frac{1}{g}\right)+S(r,g)\le \frac{n}{2}N\left(r,\frac{1}{g}\right)+S(r,g)\\ &\le \frac{n}{2}T\left(r,\frac{1}{g}\right)+S(r,g)=\frac{n}{2}T(r,g)+S(r,g).\end{aligned}$$

Then $\frac{n-2}{2}T(r,g)\le S(r,g)$ and thus $T(r,g)=S(r,g)$ since $n\ge 3$. This is a contradiction.

Since g is a non-constant meromorphic function, by Theorem B and Lemma 3, we deduce that $g+a(g')^n$ has at least two distinct zeros.

We conclude that $g+a(g')^n$ has just a unique zero.

Suppose that there exist two distinct zeros ξ_0 and ξ_0^* and choose $\delta(\delta>0)$ small enough such that $D(\xi_0,\delta)\bigcap D(\xi_0^*,\delta)=\emptyset$, where $D(\xi_0,\delta)=\{\xi:|\xi-\xi_0|<\delta\}$ and $D(\xi_0^*,\delta)=\{\xi:|\xi-\xi_0^*|<\delta\}$.

From (3.1) and (3.2), by Hurwitz′s theorem, there exist points $\xi_k\in D(\xi_0,\delta)$, $\xi_k^*\in D(\xi_0^*,\delta)$ such that for sufficiently large k

$$f_k(z_k+\rho_k\xi_k)+a[f_k'(z_k+\rho_k\xi_k)]^n=0,$$
$$f_k(z_k+\rho_k\xi_k^*)+a[f_k'(z_k+\rho_k\xi_k^*)]^n=0.$$

By the hypothesis that for each pair of functions f and g in F, $f+a(f')^n$ and $g+a(g')^n$ share 0, we know that for any positive integer m

$$f_m(z_k+\rho_k\xi_k)+a[f_m'(z_k+\rho_k\xi_k)]^n=0,$$
$$f_m(z_k+\rho_k\xi_k^*)+a[f_m'(z_k+\rho_k\xi_k^*)]^n=0.$$

Fix m, take $k\to\infty$, and note $z_k+\rho_k\xi_k\to 0, z_k+\rho_k\xi_k^*\to 0$, then

$$f_m(0)+a(f_m')^n(0)=0.$$

Since the zeros of $f_m+a(f_m')^n$ has no accumulation point, so

$$z_k+\rho_k\xi_k=0,\quad z_k+\rho_k\xi_k^*=0.$$

Hence

$$\xi_k=-\frac{z_k}{\rho_k},\quad \xi_k^*=-\frac{z_k}{\rho_k}.$$

This contradicts the fact that $\xi_k \in D(\xi_0, \delta)$, $\xi_k^* \in D(\xi_0^*, \delta)$ and $D(\xi_0, \delta) \bigcap D(\xi_0^*, \delta) = \emptyset$. So $g + a(g')^n$ has just a unique zero. This contradicts the fact that $g + a(g')^n$ has at least two distinct zeros.

Case 2. $b \neq 0$. By Lemma 1 again, there exist:
(1) a real number r, $r < 1$;
(2) points $z_k \to 0, |z_k| < r$;
(3) positive numbers $\rho_k, \rho_k \to 0$; and
(4) functions $f_k, f_k \in F$ such that

$$g_k(\xi) = \rho_k^{-1} f_k(z_k + \rho_k \xi) \to g(\xi) \tag{3.3}$$

locally uniformly with respect to spherical metric on $\mathbb{C}$, where $g(\xi)$ is a non-constant meromorphic function, all of whose zeros are multiple.

From (3.3) we obtain

$$g_k' = f_k' \to g',$$

and

$$f_k + a(f_k')^n - b = \rho_k g_k + a(g_k')^n - b \to a(g')^n - b \tag{3.4}$$

also locally uniformly with respect to the spherical metric.

If $g + a(g')^n \equiv b$. The argument in this case is completely analogous to the proof of $g + a(g')^n \equiv 0$ and then we have a contradiction. So we omit its proof.

We conclude that $a(g')^n - b$ has at most one zero.

Case 2.1. If $a(g')^n - b$ has no zeros. Suppose then that $a(g')^n - b \neq 0$. Let $c_1, c_2, \ldots, c_n$ be the (distinct) solutions of $w^n = b/a$. By Nevanlinna′s Second Fundamental Theorem,

$$\begin{aligned} T(r, g') &\leq \overline{N}(r, g') + \overline{N}\left(r, \frac{1}{g' - c_1}\right) + \cdots + \overline{N}\left(r, \frac{1}{g' - c_n}\right) + S(r, g') \\ &\leq \overline{N}(r, g') + S(r, g') \leq \frac{1}{2} N(r, g') + S(r, g') \\ &\leq \frac{1}{2} T(r, g') + S(r, g'). \end{aligned}$$

It follows that $T(r, g') = S(r, g')$, a contradiction.

Case 2.2. If $a(g')^n - b$ has zeros, we claim that $a(g')^n - b$ has just a unique zero. Suppose that there exist two distinct zeros ξ_0 and ξ_0^* and choose $\delta (\delta > 0)$ small enough such that $D(\xi_0, \delta) \bigcap D(\xi_0^*, \delta) = \emptyset$, where $D(\xi_0, \delta) = \{\xi : |\xi - \xi_0| < \delta\}$ and $D(\xi_0^*, \delta) = \{\xi : |\xi - \xi_0^*| < \delta\}$.

From (3.3) and (3.4), by Hurwitz′s theorem, there exist points $\xi_k \in D(\xi_0, \delta)$, $\xi_k^* \in D(\xi_0^*, \delta)$ such that for sufficiently large k

$$\begin{aligned} & f_k(z_k + \rho_k \xi_k) + a[f_k'(z_k + \rho_k \xi_k)]^n - b = 0, \\ & f_k(z_k + \rho_k \xi_k^*) + a[f_k'(z_k + \rho_k \xi_k^*)]^n - b = 0. \end{aligned}$$

By the hypothesis that for each pair of functions f and g in F, $f+a(f')^n$ and $g+a(g')^n$ share b, we know that for any positive integer m

$$f_m(z_k+\rho_k\xi_k)+a[f'_m(z_k+\rho_k\xi_k)]^n-b=0,$$
$$f_m(z_k+\rho_k\xi_k^*)+a[f'_m(z_k+\rho_k\xi_k^*)]^n-b=0.$$

Fix m, take $k\to\infty$, and note $z_k+\rho_k\xi_k\to 0, z_k+\rho_k\xi_k^*\to 0$, then

$$f_m(0)+a(f'_m)^n(0)-b=0.$$

Since the zeros of $f_m+a(f'_m)^n-b$ has no accumulation point, so

$$z_k+\rho_k\xi_k=0, \quad z_k+\rho_k\xi_k^*=0.$$

Hence

$$\xi_k=-\frac{z_k}{\rho_k}, \quad \xi_k^*=-\frac{z_k}{\rho_k}.$$

This contradicts the fact that $\xi_k\in D(\xi_0,\delta)$, $\xi_k^*\in D(\xi_0^*,\delta)$ and $D(\xi_0,\delta)\bigcap D(\xi_0^*,\delta)=\emptyset$. So $a(g')^n-b$ has just a unique zero. Let $c_1,c_2,\ldots,c_n$ be the (distinct) solutions of $w^n=b/a$. Hence $g'-c_i$ has at most one zeros and the same zero as $a(g')^n-b$ for the only one of $i\in\{1,2,\ldots,n\}$. By Nevanlinna$'$s Second Fundamental Theorem,

$$\begin{aligned}T(r,g') &\le \overline{N}(r,g')+\overline{N}\left(r,\frac{1}{g'-c_1}\right)+\cdots+\overline{N}\left(r,\frac{1}{g'-c_n}\right)+S(r,g')\\ &\le \overline{N}(r,g')+S(r,g')\le\frac{1}{2}N(r,g')+S(r,g')\\ &\le \frac{1}{2}T(r,g')+S(r,g').\end{aligned}$$

It follows that $T(r,g')=S(r,g')$, which is impossible.

This proves the Theorem 1.

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You-Ming WANG
Department of Applied Mathematics,
College of Science,
Hunan Agricultural University,
ChangSha 410128, P.R. CHINA

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