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An oscillation theorem for second-order nonlinear differential equations of Euler type

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Abstract

We consider the nonlinear equation $t^2x'' + g(x) = 0$, where $g(x)$ satisfies $xg(x) > 0$ for $x \neq 0$, but is not assumed to be sublinear or superlinear. We study the problem whether all nontrivial solutions of the equation are oscillatory in some critical cases.

Key Words: Oscillation, nonlinear differential equations, Liénard system

1. Introduction

The existence of oscillatory and periodic solutions plays a key role in characterizing the behavior of differential equations. The dynamic behaviors of second order differential equation have been widely investigated due to their application in many fields such as physics, mechanics and engineering technique fields. In such applications, it is important to know the existence of oscillatory and periodic solutions of equations. The oscillation problem for second order nonlinear differential equations has been studied in many papers (for example, see [1-16] and the references cited therein). In this paper we consider the second order nonlinear differential equation of Euler type

$$t^2x'' + g(x) = 0, \quad t > 0, \tag{1.1}$$

and give sufficient conditions for all nontrivial solutions of this system to be oscillatory. Here, $g(x)$ is locally Lipschitz, continuous on $\mathbb{R}$, and

$$xg(x) > 0 \quad if \quad x \neq 0. \tag{1.2}$$

A nontrivial solution of (1.1) is said to be *oscillatory* if it has arbitrarily large zeros. Otherwise, the solution is said to be *nonoscillatory.* Because of Sturm's separation theorem, the solutions of second order linear differential equations are either all oscillatory or all nonoscillatory, but cannot be both. Thus, we can classify second order linear differential equations into two types. However, the oscillation problem for (1.1) is not so easy, because $g(x)$ is nonlinear.

Euler differential equation is a special case of (1.1). In fact if we let $g(x) = \lambda x$, then (1.1) is called Euler differential equation. In this case, the number $1/4$ is called the oscillation constant and it is well known that if $\lambda > 1/4$, then all nontrivial solutions of (1.1) are oscillatory and otherwise they are nonoscillatory. In other words, $1/4$ is the lower bound for all nontrivial solutions of (1.1) to be oscillatory. Other results on the oscillation constant for linear differential equations can be found in [9–13] and the references cited therein.

Wong [16] studied the equation

$$x'' + a(t)g(x) = 0, \quad t > 0, \tag{1.3}$$

which includes the Emden Fowler differential equation. Using Sturm's comparison theorem, he proved Theorems A and B as follows

Theorem A Assume that $a(t)$ is continuously differentiable and satisfies

$$t^2 a(t) \geq 1 \tag{1.4}$$

for t sufficiently large, and that there exists a λ with $\lambda > 1/4$ such that

$$\frac{g(x)}{x} \geq \frac{1}{4} + \frac{\lambda}{(\log|x|)^2}, \tag{1.5}$$

for $|x|$ sufficiently large. Then all nontrivial solutions of (1.5) are oscillatory.

Theorem B Assume that $a(t)$ is continuously differentiable and satisfies

$$0 \leq t^2 a(t) \leq 1 \tag{1.6}$$

for t sufficiently large and

$$A(t) := \frac{a'(t)}{2a^{\frac{3}{2}}(t)} + 1 = o(t) \quad \text{as} \quad t \to \infty. \tag{1.7}$$

If, in addition, $A(t) \leq 0$ and there exists a λ with $0 < \lambda \leq 1/16$ such that

$$\frac{g(x)}{x} \leq \frac{1}{4} + \frac{\lambda}{(\log|x|)^2} \tag{1.8}$$

for $x > 0$ or $x < 0$, $|x|$ sufficiently large, then all nontrivial solutions of (1.5) are nonoscillatory.

Theorems A and B are complete extensions of the result for the linear case and can be applied to sublinear and superlinear cases. Since (1.1) is nonlinear, we cannot use Sturm's separation theorem. So, oscillatory solutions and nonoscillatory solutions maybe exist together in (1.1). But, Theorems A and B show that it is impossible.

Since (1.3) coincides with (1.1) when $a(t) = 1/t^2$, it seems reasonable to assume (1.4) and (1.6) in Theorems A and B, respectively. But condition (1.7) on $A(t)$ is considerably strict. Although it is known that all nontrivial solutions of (1.3) are nonoscillatory if $a(t) = 1/t^3$ and $g(x)$ is linear or sublinear, condition (1.7) is not satisfied. Thus, the oscillation problem for (1.1) has been solved completely when

$$\limsup_{|x|\to\infty} \frac{g(x)}{x} < \frac{1}{4} \quad \textit{or} \quad \liminf_{|x|\to\infty} \frac{g(x)}{x} > \frac{1}{4}.$$

The purpose of this paper is to give sufficient conditions for all nontrivial solutions of (1.1) to be oscillatory which can be applied when in the following case:

$$\liminf_{|x|\to\infty} \frac{g(x)}{x} \le \frac{1}{4} \le \limsup_{|x|\ \to\infty} \frac{g(x)}{x}. \tag{1.9}$$

In the next section, we will introduce a Liénard system which is equivalent to (1.1). To study the oscillation problem for (1.1) the significant point is to find conditions for deciding whether all orbits intersect the vertical isocline $y = -x$ in the equivalent Liénard system.

2. The equivalent Liénard system

The change of variable $t = e^s$ reduces (1.1) to the equation

$$\ddot{x} - \dot{x} + g(x) = 0, \quad s \in \mathbf{R},$$

where $\dot{} = \frac{d}{ds}$. This equation is equivalent to the system

$$\begin{aligned} \dot{x} &= y + x \\ \dot{y} &= -g(x), \end{aligned} \tag{2.10}$$

which is of Liénard type. Hereafter we denote s by t again. Sugie and Hara in [12] showed that each solution of (1.1) exists in the future, thus every solution of (2.1) exists in the future.

We say that system (2.1) has property (X^+) in the right half plane (resp. in the left half plane), if for every point (x_0, y_0) with $y_0 > x_0$ and $x_0 \ge 0$ (resp. $y_0 < x_0$ and $x_0 \le 0$), the positive semitrajectory of (2.1) passing through (x_0, y_0) crosses the vertical isocline $y = -x$.

Several interesting sufficient conditions for property (X^+) have been presented by M. Gyllenberg, P. Yan and J. Jiang [6], M. Gyllenberg, P. Yan [7], Hara and Yoneyama [8], Villary and Zanolin [14]. The following theorems can be applied when none of their results are applicable.

Consider the Liénard system

$$\dot{x} = y - F(x), \quad \dot{y} = -g(x), \tag{2.11}$$

where $F(x)$ and $g(x)$ are continuous on $\mathbb{R}$ with $F(0) = 0$ and $g(x)$ satisfies (1.2). Let

$$G(x) = \int_0^x g(\xi)d\xi. \tag{2.12}$$

The following two theorems proved in [2] about property (X^+) in the right and left half-plane.

Theorem E ([3, Theorem 2.3]) Assume $G(+\infty) = +\infty$. Then, system (2.2) has property (X^+) in the right half-plane if

$$\limsup_{x\to+\infty} \left(\int_b^x \left(\frac{F(\xi)g(\xi)}{(2G(\xi))^{\frac{3}{2}}} + \frac{g(\xi)}{G(\xi)} \right) d\xi + \frac{F(x)}{\sqrt{2G(x)}} \right) = +\infty, \quad \text{for some } b > 0. \tag{2.13}$$

Theorem F ([3, Theorem 2.4]) Assume $G(-\infty) = +\infty$. Then, system (2.2) has property (X^+) in the left half-plane if

$$\liminf_{x\to-\infty}\left(\int_x^b \left(-\frac{F(\xi)g(\xi)}{(2G(\xi))^{3/2}} + \frac{g(\xi)}{G(\xi)}\right)d\xi + \frac{F(x)}{\sqrt{2G(x)}}\right) = -\infty \quad \text{for some } b < 0. \tag{2.14}$$

We have the following results which are special cases of two theorems above, by letting $F(x) = -x$ in (2.4) and (2.5).

Corollary 2.1 *Assume $G(+\infty) = +\infty$. Then, system (2.1) has property (X^+) in the right half plane if*

$$\limsup_{x\to+\infty}\left(\int_b^x \left(\frac{-\xi g(\xi)}{(2G(\xi))^{\frac{3}{2}}} + \frac{g(\xi)}{G(\xi)}\right)d\xi - \frac{x}{\sqrt{2G(x)}}\right) = +\infty \quad \textit{for some } b > 0. \tag{2.15}$$

Corollary 2.2 *Assume $G(-\infty) = +\infty$. Then system (2.1) has property (X^+) in the left half plane if*

$$\liminf_{x\to-\infty}\left(\int_x^b \left(\frac{\xi g(\xi)}{(2G(\xi))^{\frac{3}{2}}} + \frac{g(\xi)}{G(\xi)}\right)d\xi - \frac{x}{\sqrt{2G(x)}}\right) = -\infty \quad \textit{for some } b < 0. \tag{2.16}$$

We will need the following lemmas (refer to [12, Lemma 3.1 and 3.2] for the proof) in the next section

Lemma 2.1 *For each point $C = (c, -c)$ with $c > 0$, the positive semitrajectory of (2.1) passing through C crosses the negative y-axis.*

Lemma 2.2 *For each point $C = (-c, c)$ with $c > 0$, the positive semitrajectory of (2.1) passing through C crosses the positive y-axis.*

3. Main results

In this section we will present our main results and present some examples to illustrate our results. The main theorem is as follows.

Theorem 3.1 *Suppose that (2.6) and (2.7) hold. Then, all nontrivial solutions of (1.1) are oscillatory.*

Proof. Each solution of (1.1) exists in the future [12, Proposition 2.1]. Since (2.6) and (2.7) hold, system (2.1) which is equivalent to (1.1) has property (X^+) in the right and left half plane. Thus, it follows from Lemmas 2.1 and 2.2 that every solution of (2.1) keeps on rotating around the origin except the zero solution. Hence, all nontrivial solutions of (1.1) are oscillatory.

Theorem 3.2 *Let $\lambda > 0$. Then all nontrivial solutions of (1.1) are oscillatory if*

$$G(x) \geq \frac{1}{8}\left(\frac{x\ln(|x|)}{\ln(|x|) - \lambda}\right)^2 \tag{3.17}$$

for $|x| > R$ with a sufficiently large $R > 0$.

Proof. Suppose that (3.1) holds, then for $|x| > R$

$$1 - \frac{|x|}{2\sqrt{2G(x)}} \geq \frac{\lambda}{\ln(|x|)},$$

hence, for $x > R$ we have

$$\int_R^x \left(\frac{-\xi g(\xi)}{(2G(\xi))^{\frac{3}{2}}} + \frac{g(\xi)}{G(\xi)} \right) d\xi = \int_R^x \frac{g(\xi)}{G(\xi)} \left(1 - \frac{\xi}{2\sqrt{2G(\xi)}} \right) d\xi$$
$$\geq \lambda \int_R^x \frac{g(\xi)}{G(\xi)\ln(\xi)} d\xi = \lambda \left(\frac{\ln(G(x))}{\ln(x)} - \frac{\ln(G(R))}{\ln(R)} + \int_R^x \frac{\ln(G(\xi))}{\xi \ln^2(\xi)} d\xi \right)$$
$$\geq \lambda \left(\frac{\ln(G(x))}{\ln(x)} - \frac{\ln(G(R))}{\ln(R)} + \int_b^x \frac{1}{\xi \ln(\xi)} d\xi \right) \quad \text{for some } \; b > R.$$

Therefore,

$$\lim_{x \to +\infty} \int_R^x \left(\frac{-\xi g(\xi)}{(2G(\xi))^{\frac{3}{2}}} + \frac{g(\xi)}{G(\xi)} \right) d\xi = +\infty.$$

Notice that $\frac{x}{\sqrt{2G(x)}}$ is bounded, thus (2.6) holds. Similarly, we can conclude that (2.7) holds. Therefore, Theorem 3.1 implies that all nontrivial solutions of (1.1) are oscillatory.

Corollary 3.1 *Suppose that*

$$\liminf_{|x| \to \infty} \frac{G(x)}{x^2} > \frac{1}{8}. \tag{3.18}$$

Where, $G(x)$ is defined by (2.3)*. Then all nontrivial solutions of* (1.1) *are oscillatory.*

Proof. Suppose that (3.2) holds, then,

$$\alpha = \liminf_{|x| \to \infty} \left(1 - \frac{x}{2\sqrt{2G(x)}} \right) > 0.$$

Hence, for $x > R$ with sufficiently large $R > 0$ we have

$$\int_R^x \left(\frac{-\xi g(\xi)}{(2G(\xi))^{\frac{3}{2}}} + \frac{g(\xi)}{G(\xi)} \right) d\xi = \int_R^x \frac{g(\xi)}{G(\xi)} \left(1 - \frac{\xi}{2\sqrt{2G(\xi)}} \right) d\xi$$
$$\geq \frac{\alpha}{2} \int_R^x \frac{g(\xi)}{G(\xi)} d\xi = \frac{\alpha}{2} \left(\ln G(x) - \ln G(R) \right).$$

Therefore,

$$\lim_{x \to +\infty} \int_R^x \left(\frac{-\xi g(\xi)}{(2G(\xi))^{\frac{3}{2}}} + \frac{g(\xi)}{G(\xi)} \right) d\xi = +\infty.$$

Since $\frac{x}{\sqrt{2G(x)}}$ is bounded, (2.6) holds. Similarly, we can conclude that (2.7) holds, thus, all nontrivial solutions of (1.1) are oscillatory. □

Corollary 3.2 *Let* $\lambda > 0$. *Then all nontrivial solutions of* (1.1) *are oscillatory if*

$$\frac{g(x)}{x} \geq \frac{1}{4} + \frac{\lambda}{\log|x|} \tag{3.19}$$

for $|x| > R$ *with a sufficiently large* $R > 0$.

Proof. Define continuous functions $k(x), K(x)$ and $L(x)$ on $\mathbf{R}$ by

$$k(x) = \frac{\lambda x}{\log|x|}, \quad K(x) = \int_0^x k(\xi)d\xi \quad \text{and} \quad L(x) = \frac{\lambda x^2}{2\log|x|}$$

for $|x|$ sufficiently large, respectively. Then we have

$$K(x) \geq L(x) - M \qquad \text{for some} \;\; M > 0$$

and by (3.3)

$$G(x) \geq \frac{1}{8}x^2 + K(x) - N \text{ for some } \; N > 0.$$

Hence, for $|x|$ sufficiently large

$$\begin{aligned} G(x) \qquad & \geq \frac{1}{8}x^2 + L(x) - (N+M) \geq \frac{1}{8}x^2 + \frac{\lambda x^2}{4log|x|} \\ & = \frac{x^2}{8}\left(1 + \frac{2\lambda}{log|x|}\right) \geq \frac{1}{8}\left(\frac{xlog(|x|)}{log(|x|) - \frac{\lambda}{4}}\right)^2. \end{aligned}$$

Hence, by Theorem 3.2 all nontrivial solutions of (1.1) are oscillatory. □

Proposition 3.1 *For every* $\alpha \geq 0$ *there exists a function* $g \in C^\infty(\mathbf{R})$ *such that*

$$xg(x) > 0 \quad if \quad x \neq 0, \quad \liminf_{|x|\to\infty} \frac{g(x)}{x} = \alpha,$$

and all nontrivial solutions of (1.1) *are oscillatory.*

Proof. First let $m > \max\{|n|, \frac{1}{4}\}$. Then we prove all nontrivial solutions of (1.1) are oscillatory if

$$\frac{g(x)}{x} \geq m + n\cos(x).$$

for $|x| > R$ with a sufficiently large $R > 0$.

It is clear that (1.2) holds. We have

$$G(x) \geq \frac{m}{2}x^2 + nx\sin(x) + n\cos(x) - n,$$

thus,

$$\liminf_{|x|\to\infty} \frac{G(x)}{x^2} \geq \frac{m}{2} > \frac{1}{8}.$$

Hence, by Corollary 3.1 all nontrivial solutions of (1.1) are oscillatory. □

Second let $\lambda > 0$, $\beta \in R$ and $\alpha > \frac{1}{2}$. Then we prove all nontrivial solutions of (1.1) are oscillatory if

$$\frac{g(x)}{x} \geq \exp(-\lambda x^2) + \alpha \cos^2(\beta x).$$

for $|x| > R$ with a sufficiently large $R > 0$.

It is clear that (1.2) holds. We have

$$G(x) \geq \frac{1}{8}\left(-\frac{4e^{-\lambda x^2}}{\lambda} + 2\alpha x^2 + \frac{\alpha\cos(2\beta x)}{\beta^2} + \frac{2\alpha x\sin(2\beta x)}{\beta}\right) + \frac{1}{2\lambda} - \frac{\alpha}{\beta^2},$$

thus,

$$\liminf_{|x|\to\infty} \frac{G(x)}{x^2} \geq \frac{\alpha}{4} > \frac{1}{8}.$$

Hence, by Corollary 3.1 all nontrivial solutions of (1.1) are oscillatory.

Now let $g(x) = (\frac{1}{4}+\alpha)x + \frac{1}{4}x\cos(x)$ and $g(x) = x\exp(-x^2) + x\cos^2(x)$ for $\alpha > 0$ and $\alpha = 0$, respectively. The proof is complete.

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