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Invariants of symmetric algebras associated to graphs

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Abstract

In this work we deal with the symmetric algebra of monomial ideals that arise from graphs, the edge ideals. The notion of s-sequence is explored for such ideals in order to compute standard algebraic invariants of their symmetric algebra in terms of the corresponding invariants of special quotients of the polynomial ring related to the graphs.

Key Words: Edge ideals, symmetric algebra, s-sequence

1. Introduction

In this article we study the symmetric algebra of monomial ideals ([1], [4]), in particular of some ideals arising from graphs. In order to compute standard invariants of such symmetric algebra, we investigate some cases for which the monomial ideals are generated by s-sequences. In [2] the notion of s-sequence is employed to compute the invariants of the symmetric algebra of finitely generated modules. Our proposal is to compute standard invariants of the symmetric algebra in terms of the corresponding invariants of special quotients of the polynomial ring related to the graph. This computation can be obtained for finitely generated modules generated by an s-sequence.

Let G be a graph with no cycles. An algebraic object attached to G is the edge ideal $I(G)$ that is a monomial ideal of $R = K[X_1, \dots, X_n]$, K a field, n the number of vertices of G. $I(G)$ is generated by square-free monomials of degree two in the polynomial ring R, $I(G) = \big(\{X_iX_j \mid \{v_i, v_j\} \text{ is an edge of } G\}\big)$. In [6] there are some results about monomial ideals of R that can arise from the edges of a simple graph.

The aim of this paper is to investigate classes of simple graphs and to prove that the notion of s-sequence can be explored in this family of monomial ideals in order to compute algebraic invariants of their symmetric algebra.

The work is organized as follows. In section 2 some preliminary notions about the theory of s-sequences are given. In sections 3 and 4 the notion of s-sequence is investigated for edge ideals associated to trees and forests. In section 5 we give the structure of the annihilator ideals of these edge ideals generated by an s-sequence and we compute the invariants: (a) the dimension, $\dim_R(Sym_R(I(G)))$; (b) the multiplicity, $\mathrm{e}(Sym_R(I(G)))$; and (c) the Castelnuovo-Mumford regularity, $\mathrm{reg}_R(Sym_R(I(G)))$. More precisely, we achieve formulas for (a), (b) and when $I(G)$ is generated by a strong s-sequence we give bounds for (c) in terms of the annihilator ideals.

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2. Preliminaries and notations

Let's recall the theory of s-sequences in order to apply it to our classes of monomial ideals.

Let M be a finitely generated module on a Noetherian ring R, and $f_1, \dots, f_t$ be the generators of M. Let (a_{ij}), for $i = 1, \dots, t,\ j = 1, \dots, p$, be the relation matrix of M. Let $Sym_R(M)$ be the symmetric algebra of M, then $Sym_R(M) = R[T_1, \dots, T_t]/J$, where $R[T_1, \dots, T_t]$ is a polynomial ring in the variables $T_1, \dots, T_t$ and J is its relation ideal, generated by $g_j = \sum_{i,j} a_{ij} T_i$, for $i = 1, \dots, t,\ j = 1, \dots, p$.

If we assign degree 1 to each variable T_i and degree 0 to the elements of R, then J is a graded ideal and $Sym_R(M)$ is a graded algebra on R.

Set $S = R[T_1, \dots, T_t]$ and let $\prec$ be a monomial order on the monomials of S in the variables T_i. With respect to this term order, if $f = \sum a_\alpha \underline{T}^\alpha$, where $\underline{T}^\alpha = T_1^{\alpha_1} \cdots T_t^{\alpha_t}$ and $\alpha = (\alpha_1, \dots, \alpha_t) \in \mathbb{N}^t$, we put $\mathrm{in}_\prec(f) = a_\alpha \underline{T}^\alpha$, where $\underline{T}^\alpha$ is the largest monomial in f such that $a_\alpha \neq 0$.
So we can define the monomial ideal $\mathrm{in}_\prec(J) = (\{\mathrm{in}_\prec(f) \mid f \in J\})$.

For every $i = 1, \dots, t$, we set $M_{i-1} = Rf_1 + \cdots + Rf_{i-1}$ and let $\mathcal{I}_i = M_{i-1} :_R f_i$ be the colon ideal. Since $M_i/M_{i-1} \simeq R/\mathcal{I}_i$, $\mathcal{I}_i$ is the annihilator of the cyclic module $R/\mathcal{I}_i$. $\mathcal{I}_i$ is called an *annihilator* ideal of the sequence $f_1, \dots, f_t$.

It is $(\mathcal{I}_1 T_1, \mathcal{I}_2 T_2, \dots, \mathcal{I}_t T_t) \subseteq \mathrm{in}_\prec(J)$, and the two ideals coincide in degree 1.

Definition 2.1 The sequence $f_1, \dots, f_t$ is said to be an *s-sequence* for M if

$$(\mathcal{I}_1 T_1, \mathcal{I}_2 T_2, \dots, \mathcal{I}_t T_t) = \mathrm{in}_\prec(J).$$

When $\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \cdots \subseteq \mathcal{I}_t$, $f_1, \dots, f_t$ is said to be a *strong* s-sequence.

If $R = K[X_1, \dots, X_n]$ is the polynomial ring over a field K, we can use the Gröbner bases theory to compute $\mathrm{in}_\prec(J)$. Let $\prec$ be any term order on $K[X_1, \dots, X_n; T_1, \dots, T_t]$ with $X_i \prec T_j$ for all i, j. Then for any Gröbner basis B for $J \subset K[X_1, \dots, X_n, T_1, \dots, T_t]$ with respect to $\prec$, we have $\mathrm{in}_\prec(J) = (\{\mathrm{in}_\prec(f) \mid f \in B\})$. If the elements of B are linear in the T_i, it follows that $f_1, \dots, f_t$ is an s-sequence for M.

Let $M = I = (f_1, \dots, f_t)$ be a monomial ideal of $R = K[X_1, \dots, X_n]$. Set $f_{ij} = \frac{f_i}{[f_i, f_j]}$ for $i \neq j$, where $[f_i, f_j]$ is the greatest common divisor of the monomials f_i and f_j. J is generated by $g_{ij} = f_{ij} T_j - f_{ji} T_i$ for $1 \leqslant i < j \leqslant t$. The monomial sequence $f_1, \dots, f_t$ is an s-sequence if and only if g_{ij} for $1 \leqslant i < j \leqslant t$ is a Gröbner basis for J for any term order in $K[X_1, \dots, X_n; T_1, \dots, T_t]$ with $X_i \prec T_j$ for all i, j.

Notice that the annihilator ideals of the monomial sequence $f_1, \dots, f_t$ are the ideals $I_i = (f_{1i}, f_{2i}, \dots, f_{i-1,i})$, for $i = 1, \dots, t$ ([2]).

Remark 2.1 ([2], Lemma 1.4)
From the theory of Gröbner bases, if $f_1, \dots, f_t$ is a monomial s-sequence with respect to some admissible term order $\prec$, then $f_1, \dots, f_t$ is an s-sequence for any other admissible term order.

We now study the symmetric algebra of a class of monomial modules over the polynomial ring $R = K[X_1, \dots, X_n]$ that are monomial ideals arising from graphs.

Let G be a graph, $V(G)$ and $E(G)$ be the sets of its vertices and edges respectively. G is said to be *simple* if, for all $\{v_i, v_j\} \in E(G)$, it is $v_i \neq v_j$. G is *connected* if it has no isolated subgraphs.

A *forest* is an acyclic graph. A *tree* is a connected acyclic graph.

If $V(G) = \{v_1, \dots, v_n\}$ and $R = K[X_1, \dots, X_n]$ is the polynomial ring over a field K such that each variable X_i corresponds to the vertex v_i, the *edge ideal* $I(G)$ associated to G is the ideal $\big(\{X_iX_j \mid \{v_i, v_j\} \in E(G)\}\big) \subset R$.

3. Trees and s-sequences

In this section we give a study of edge ideals associated to connected acyclic graphs. The results show that the generators of the edge ideal of a tree form an s-sequence.

Let G be a connected acyclic graph with n vertices and define the edge ideal in $R = K[X_1, \dots, X_n]$,

$$I(G) = (X_1X_r, X_2X_r, \dots, X_{r-1}X_r, X_1X_{r+1}, X_{r+1}X_{r+2}, \dots, X_{r+s_1-1}X_{r+s_1}, \\ X_2X_{r+s_1+1}, \dots, X_{r-1}X_{r+s_1+\dots+s_{r-2}+1}, \dots, X_{n-1}X_n).$$

Proposition 3.1 *Let* $I(G)=(X_1X_r, X_2X_r, \dots, X_{r-1}X_r, X_1X_{r+1}, X_{r+1}X_{r+2}, \dots, X_{r+s_1-1}X_{r+s_1}, X_2X_{r+s_1+1}, \dots, X_{r-1}X_{r+s_1+\dots+s_{r-2}+1}, \dots, X_{n-1}X_n) \subset R$ *be the edge ideal of a graph* G *with* $n = r + \dots + s_{r-1}$ *vertices and* $n-1$ *edges. If* $Sym_R(I(G)) = R[T_1, \dots, T_{n-1}]/J$, *then* $J = (\{g_{ij}, 1 \leqslant i < j \leqslant n-1\})$, *where*

$$g_{ij} = \begin{cases} X_iT_j - X_jT_i & \text{if } 1 \leqslant i < j \leqslant r-1 \\ X_rT_j - X_{j+1}T_i & \text{if } i=1; j=r \text{ or } i=2,\dots,r-1; j=r+s_1+\dots+s_{i-1} \\ X_iT_{j+1} - X_{j+2}T_j & \text{if } i=1; j=r \text{ or } i=2,\dots,r-1; j=r+s_1+\dots+s_{i-1} \\ X_iT_j - X_{j+1}T_i & \text{if } j=i+1; i=r+1,\dots,r+s_1-2 \text{ or } j=i+1; i=r+ \\ & +s_1+\dots+s_{h-1}+k, h=2,\dots,r-1, k=1,\dots,s_h-2 \\ f_iT_j - f_jT_i & \text{otherwise, with } f_i \text{ the generators of } I(G). \end{cases}$$

Proof. Observe that G is a graph having only the vertex corresponding to the variable X_r of degree > 2. The generators of $I(G)$ are the following:

$f_1 = X_1X_r, f_2 = X_2X_r, \dots, f_{r-1} = X_{r-1}X_r,$
$f_r = X_1X_{r+1}, f_{r+1} = X_{r+1}X_{r+2}, \dots, f_{r+s_1-1} = X_{r+s_1-1}X_{r+s_1},$
$f_{r+s_1} = X_2X_{r+s_1+1}, \dots, f_{r+s_1+s_2-1} = X_{r+s_1+s_2-1}X_{r+s_1+s_2}, \dots\dots,$
$f_{r+s_1+\dots+s_{r-2}} = X_{r-1}X_{r+\dots+s_{r-2}+1}, \dots, f_{n-1} = f_{r+\dots+s_{r-1}-1} = X_{n-1}X_n.$

Put $r = t_1,\ r+s_1 = t_2,\ r+s_1+s_2 = t_3, \dots, n = r+s_1+\dots+s_{r-1} = t_r$.

Set $f_{ij} = \frac{f_i}{[f_i, f_j]}$ for $i < j = 1, \dots, t_r - 1$.

We compute:

$f_{12} = f_{13} = \dots = f_{1,t_1-1} = X_1,\ f_{1,t_1} = X_{t_1},\ f_{1,t_1+1} = \dots = f_{1,t_r-1} = X_1X_{t_1} = f_1,$

$f_{23} = f_{24} = \dots = f_{2,t_1-1} = X_2,\ f_{2,t_1} = \dots = f_{2,t_2-1} = X_2X_{t_1} = f_2,\ \ f_{2,t_2} = X_{t_1},$

$f_{2,t_2+1} = \dots = f_{2,t_r-1} = X_2X_{t_1} = f_2, \dots\dots,$

$f_{t_1-2,t_1-1} = X_{t_1-2}, f_{t_1-2,t_1} = \dots = f_{t_1-2,t_{r-2}-1} = X_{t_1-2}X_{t_1} = f_{t_1-2},\ f_{t_1-2,t_{r-2}} = X_{t_1},$

$f_{t_1-2,t_{r-2}+1} = \dots = f_{t_1-2,t_r-1} = X_{t_1-2}X_{t_1} = f_{t_1-2},$

$f_{t_1-1,t_1} = \dots = f_{t_1-1,t_{r-1}-1} = X_{t_1-1}X_{t_1} = f_{t_1-1},\ \ f_{t_1-1,t_{r-1}} = X_{t_1},$

$f_{t_1-1,t_{r-1}+1} = \dots = f_{t_1-1,t_r-1} = X_{t_1-1}X_{t_1} = f_{t_1-1},$

$f_{t_1,t_1+1} = X_1,\ f_{t_1,t_1+2} = \dots = f_{t_1,t_r-1} = X_1X_{t_1+1} = f_{t_1},$

$f_{t_1+1,t_1+2} = X_{t_1+1},\ f_{t_1+1,t_1+3} = \ldots = f_{t_1+1,t_r-1} = X_{t_1+1}X_{t_1+2} = f_{t_1+1}, \ldots\ldots,$
$f_{t_2-2,t_2-1} = X_{t_2-2},\ f_{t_2-2,t_2} = \ldots = f_{t_2-2,t_r-1} = X_{t_2-2}X_{t_2-1} = f_{t_2-2},$
$f_{t_2-1,t_2} = \ldots = f_{t_2-1,t_r-1} = X_{t_2-1}X_{t_2} = f_{t_2-1}, \ldots\ldots,$
$f_{t_{r-1},t_{r-1}+1} = X_{t_1-1},\ f_{t_{r-1},t_{r-1}+2} = \ldots = f_{t_{r-1},t_r-1} = X_{t_1-1}X_{t_{r-1}+1} = f_{t_{r-1}}, \ldots,$
$f_{t_r-2,t_r-1} = X_{t_r-2}.$

In a general form we write:

$f_{ij} = f_{t_i,t_{i+1}} = X_i$, for $1 \leqslant i < j \leqslant r-1$; $f_{t_{r-1},t_{r-1}+1} = X_{t_1-1}$;
$f_{i,t_i} = X_{t_1}$; $f_{t_i+k,t_i+k+1} = X_{t_i+k}$, for $i = 1, \ldots, r-1$, $k = 1, \ldots, s_i-2$;
$f_{ij} = f_i$ otherwise, $i < j$.

In a similar way we can obtain:

$f_{j\,i} = X_j$, for $1 \leqslant i < j \leqslant r-1$;
$f_{t_i,i} = X_{t_i+1}$; $f_{t_i+k,t_i+k-1} = X_{t_i+k+1}$, for $i = 1, \ldots, r-1$, $k = 1, \ldots, s_i-1$;
$f_{j\,i} = f_j$ otherwise, $i < j$.

Then the generators of J are the linear forms:

$g_{ij} = X_iT_j - X_jT_i$, for $1 \leqslant i < j \leqslant r-1$;
$g_{i,t_i} = X_{t_1}T_{t_i} - X_{t_i+1}T_i$; $g_{t_i,t_i+1} = X_iT_{t_i+1} - X_{t_i+2}T_{t_i}$, for $i = 1, \ldots, r-1$;
$g_{t_i+k,t_i+k+1} = X_{t_i+k}T_{t_i+k+1} - X_{t_i+k+2}T_{t_i+k}$, for $i = 1, \ldots, r-1$, $k = 1, \ldots, s_i-2$;
$g_{ij} = f_iT_j - f_jT_i$ otherwise, $i < j$. □

Theorem 3.1 *Let G be a connected acyclic graph with n vertices. Let $R = K[X_1, \ldots, X_n]$. The edge ideal*

$$I(G) = (X_1X_r, X_2X_r, \ldots, X_{r-1}X_r, X_1X_{r+1}, X_{r+1}X_{r+2}, \ldots, X_{r+s_1-1}X_{r+s_1}, X_2X_{r+s_1+1}, \ldots, X_{r-1}X_{r+s_1+\ldots+s_{r-2}+1}, \ldots, X_{n-1}X_n)$$

is generated by an s-sequence.

Proof. Following the steps of the proof of Proposition 3.1, let

$f_1 = X_1X_{t_1}, f_2 = X_2X_{t_2}, \ldots, f_{t_1-1} = X_{t_1-1}X_{t_1}$,
$f_{t_1} = X_1X_{t_1+1}, f_{t_1+1} = X_{t_1+1}X_{t_1+2}, \ldots, f_{t_2-1} = X_{t_2-1}X_{t_2}, \ldots\ldots,$
$f_{t_{r-1}} = X_{t_1-1}X_{t_{r-1}+1}, \ldots, f_{t_r-1} = X_{t_r-1}X_{t_r}$

be the generators of $I(G)$. They form an s-sequence if $B = \{g_{ij} = f_{ij}T_j - f_{j\,i}T_i \mid 1 \leqslant i < j \leqslant t_r - 1\}$ is a Gröbner basis for J. For a suitable term order $\prec$, we want to prove that the S-pairs $S(g_{ij}, g_{hl})$, with $i, j, h, l \in \{1, \ldots, t_r - 1\}, i < j,\ i < h < l$, have a standard expression with respect to B with remainder 0. Note that, to get a standard expression of $S(g_{ij}, g_{hl})$ is equivalent to find some $g_{st} \in B$ whose initial term divides the initial term of $S(g_{ij}, g_{hl})$ and substitute a multiple of g_{st} such that the remaindered polynomial has a smaller initial term and so on up to the remainder is 0. We have

$$S(g_{ij}, g_{hl}) = \frac{f_{ij}f_{lh}}{[f_{ij}, f_{hl}]}T_jT_h - \frac{f_{hl}f_{j\,i}}{[f_{ij}, f_{hl}]}T_iT_l.$$

Let $V(G) = \{v_1, \ldots, v_n\}$ be the vertex set of G with $\deg(v_r) > 2$, $v_r \in V(G)$. If $[\mathrm{in}_\prec(g_{ij}), \mathrm{in}_\prec(g_{hl})] = 1$, then $S(g_{ij}, g_{hl}) = f_{lh}g_{ij}T_h - f_{j\,i}g_{hl}T_i$. If $[\mathrm{in}_\prec(g_{ij}), \mathrm{in}_\prec(g_{hl})] \neq 1$, the following standard expressions occur:

- When the path from v_i to v_j and the one from v_h to v_l do not contain v_r,

$$S(g_{ij}, g_{hl}) = [f_{ji}, f_{lh}]\left(\frac{f_{jl}}{[f_{ih}, f_{jl}]} g_{ih} T_l - \frac{f_{ih}}{[f_{ih}, f_{jl}]} g_{jl} T_h\right),$$

with $g_{j\,l} = -g_{lj}$ if $l < j$ and $g_{ih} = g_{jl} = 0, f_{ih} = f_{jl} = 1$ if $i = h, j = l$;

- When the path from v_h to v_l contains v_r,

$$S(g_{ij}, g_{hl}) = [f_{ji}, f_{lh}]\left(\frac{f_{lj}}{[f_{hi}, f_{lj}]} g_{ih} T_j - \frac{f_{hi}}{[f_{hi}, f_{lj}]} g_{jl} T_i\right);$$

- When the path from v_i to v_j and the one from v_h to v_l contain v_r,

$$S(g_{ij}, g_{hl}) = g_{ij} T_l - g_{hl} T_j,$$

such that $\mathrm{in}_\prec(g_{ij}T_l)$ and $\mathrm{in}_\prec(g_{hl}T_j)$ are smaller than $\mathrm{in}_\prec(S(g_{ij}, g_{hl}))$, for some monomial order $\prec$ and for an ordering fixed on the variables.

Hence all the S-pairs $S(g_{ij}, g_{hl})$ reduce to 0 with respect to B. □

Theorem 3.2 *Let G be a tree with n vertices. The edge ideal of it $I(G) \subset R = K[X_1, \ldots, X_n]$ is generated by an s-sequence.*

Proof. A tree G can be intended as an extension of the connected acyclic graph examined in the present section in which there are further vertices of degree >2. Namely, G may have vertices v_p, $p \neq r$, where three or more edges begin or end. In this way, for each of these vertices in G, we can take in account the previous considerations.

Let $f_1, \ldots, f_{n-1}$ denote the generators of the edge ideal $I(G)$. Following a procedure as in Proposition 3.1 , we are able to obtain the generators $g_{ij} = f_{ij}T_j - f_{j\,i}T_i$, $1 \leqslant i < j \leqslant n-1$ of the relation ideal J of the symmetric algebra of $I(G)$.

To show that $f_1, \ldots, f_{n-1}$ is an s-sequence, it is enough to see that the set of g_{ij} is a Gröbner basis for J, i.e. the S-pairs $S(g_{ij}, g_{hl})$ such that $i, j, h, l \in \{1, \ldots, n-1\}, i<j, i<h<l$, have a standard expression with respect to $\{g_{ij}\}$ with remainder 0.

Through a generalization of the reasoning of Theorem 3.1, similar formulas hold for the $S(g_{ij}, g_{hl})$ by iterating the computation for getting standard expressions of the S-pairs in every vertex of degree >2 .

In conclusion, all the S-pairs reduce to 0 with respect to $\{g_{ij}\}$. □

Remark 3.1 In the following we will examine interesting classes of connected acyclic graphs with n vertices that are certain trees, so their edge ideals in $R = K[X_1, \ldots, X_n]$ are generated by s-sequences. In particular:

$I(G) = (X_1X_n, X_2X_n, \ldots, X_{n-1}X_n)$, the star with $n-1$ edges,

$I(G) = (X_1X_2, X_2X_3, \ldots, X_{n-1}X_n)$, the line with n points,

$I(G) = (X_1X_{n-1}, X_2X_{n-1}, \ldots, X_{n-2}X_{n-1}, X_\ell X_n)$, $\ell = 1, \ldots, n-2$.

The first two cases are considered in [3].

4. Forests and s-sequences

In this section we consider the following edge ideals of $R = K[X_1, \ldots, X_n]$ associated to forests:

a) $I(G) = (X_1X_2, X_3X_4, \ldots, X_{m-1}X_m, X_{m+1}X_n, \ldots, X_{n-1}X_n)$,

b) $I(G) = (X_1X_m, X_2X_m, \ldots, X_{m-1}X_m, X_{m+1}X_n, X_{m+2}X_n, \ldots, X_{n-1}X_n)$.

Proposition 4.1 *Let $I(G) = (X_1X_2, X_3X_4, \ldots, X_{m-1}X_m, X_{m+1}X_n, \ldots, X_{n-1}X_n)$ be the edge ideal of a graph G with n vertices and $t = n - \frac{m}{2} - 1$ edges. If $Sym_R(I(G)) = R[T_1, \ldots, T_t]/J$, then $J = (\{g_{ij}, 1 \leqslant i < j \leqslant t\})$, where*

$$g_{ij} = \begin{cases} X_{2i-1}X_{2i}T_j - X_{2j-1}X_{2j}T_i & \text{if } 1 \leqslant i < j \leqslant \frac{m}{2} \\ X_{i+\frac{m}{2}}T_j - X_{j+\frac{m}{2}}T_i & \text{if } \frac{m}{2}+1 \leqslant i \leqslant n-\frac{m}{2}-2, \\ & \frac{m}{2}+2 \leqslant j \leqslant n-\frac{m}{2}-1 \\ X_{2i-1}X_{2i}T_j - X_{j+\frac{m}{2}}X_nT_i & \text{if } 1 \leqslant i \leqslant \frac{m}{2},\ \frac{m}{2}+1 \leqslant j \leqslant n-\frac{m}{2}-1. \end{cases}$$

Proof. $I(G)$ is generated by $t = n - \frac{m}{2} - 1$ monomials as follows: $f_1 = X_1X_2$, $f_2 = X_3X_4$, $\ldots$, $f_{\frac{m}{2}} = X_{m-1}X_m$, $f_{\frac{m}{2}+1} = X_{m+1}X_n$, $\ldots$, $f_t = X_{n-1}X_n$. Set $f_{ij} = \frac{f_i}{[f_i,f_j]}$ for $i<j$, $i,j = 1,\ldots,t$.
For $i < j$, we compute $f_{ij} = X_{2i-1}X_{2i}$ for $1 \leqslant i \leqslant \frac{m}{2}$ and $2 \leqslant j \leqslant n - \frac{m}{2} - 1$ and $f_{ij} = X_{i+\frac{m}{2}}$ for $\frac{m}{2}+1 \leqslant i < j \leqslant n-\frac{m}{2}-1$.
Similarly, we have $f_{ji} = X_{2j-1}X_{2j}$ for $1 \leqslant i < j \leqslant \frac{m}{2}$, $f_{ji} = X_{j+\frac{m}{2}}$ for $\frac{m}{2}+1 \leqslant i < j \leqslant n-\frac{m}{2}-1$ and $f_{ji} = X_{j+\frac{m}{2}}X_n$ for $1 \leqslant i \leqslant \frac{m}{2}$, $\frac{m}{2}+1 \leqslant j \leqslant n-\frac{m}{2}-1$.
Being J generated by the linear forms $g_{ij} = f_{ij}T_j - f_{ji}T_i$ for $1 \leqslant i < j \leqslant t$, the thesis follows. □

Proposition 4.2 *Let $I(G) = (X_1X_m, \ldots, X_{m-1}X_m, X_{m+1}X_n, X_{m+2}X_n, \ldots, X_{n-1}X_n)$ be the edge ideal of a graph G with n vertices and $n-2$ edges. If $Sym_R(I(G)) = R[T_1, \ldots, T_{n-2}]/J$, then $J = (\{g_{ij}, 1 \leqslant i < j \leqslant n-2\})$, where*

$$g_{ij} = \begin{cases} X_iT_j - X_jT_i & \text{if } 1 \leqslant i < j \leqslant m-1 \\ X_iX_mT_j - X_{j+1}X_nT_i & \text{if } 1 \leqslant i \leqslant m-1,\ m \leqslant j \leqslant n-2 \\ X_{i+1}T_j - X_{j+1}T_i & \text{if } m \leqslant i < j \leqslant n-2. \end{cases}$$

Proof. $I(G)$ is generated by $n-2$ elements as follows: $f_1 = X_1X_m$, $f_2 = X_2X_m$, $\ldots$, $f_{m-1} = X_{m-1}X_m$, $f_m = X_{m+1}X_n$, $\ldots$, $f_{n-2} = X_{n-1}X_n$. Set $f_{ij} = \frac{f_i}{[f_i,f_j]}$ for $i<j$, $i,j = 1,\ldots,n-2$.
For $i<j$, we compute $f_{ij} = X_i$ for $1 \leqslant i < j \leqslant m-1$, $f_{ij} = X_iX_m$ for $1 \leqslant i \leqslant m-1$, $m \leqslant j \leqslant n-2$, and $f_{ij} = X_{i+1}$ for $m \leqslant i < j \leqslant n-2$.
Similarly, we have $f_{ji} = X_j$ for $1 \leqslant i < j \leqslant m-1$, $f_{ji} = X_{j+1}X_n$ for $1 \leqslant i \leqslant m-1$, $m \leqslant j \leqslant n-2$ and $f_{ji} = X_{j+1}$ for $m \leqslant i < j \leqslant n-2$.
Being J generated by the linear forms $g_{ij} = f_{ij}T_j - f_{ji}T_i$ for $1 \leqslant i < j \leqslant n-2$, then the assertion follows. □

Next result states that the above ideals are generated by an s-sequence.

Theorem 4.1 *Let $R = K[X_1, \ldots, X_n]$. The edge ideals*

a) $I(G) = (X_1X_2, X_3X_4, \ldots, X_{m-1}X_m, X_{m+1}X_n, \ldots, X_{n-1}X_n)$

b) $I(G)=(X_1X_m, X_2X_m,\dots,X_{m-1}X_m, X_{m+1}X_n, X_{m+2}X_n,\dots,X_{n-1}X_n)$

are generated by an s*-sequence.*

Proof. a) Let $f_1 = X_1X_2$, $f_2 = X_3X_4$, $\dots$, $f_{\frac{m}{2}} = X_{m-1}X_m$, $f_{\frac{m}{2}+1} = X_{m+1}X_n$, $\dots$, $f_t = X_{n-1}X_n$, $t = n-\frac{m}{2}-1$, be the generators of $I(G)$. One has: if $f_{ij} = X_{2i-1}X_{2i}$ for $1 \leqslant i < j \leqslant \frac{m}{2}$, then $[f_{ij}, f_{hl}] = 1$ for $i<j$, $h<l$, $i\neq h$, $j\neq l$ with $i,j,h,l \in \{1,\dots,\frac{m}{2}\}$; if $f_{ij} = X_{i+\frac{m}{2}}$ for $\frac{m}{2}+1 \leqslant i < j \leqslant t$, then $f_{ij}\neq f_{hl}$ if $i\neq h$ and $j\neq l$, hence $[f_{ij}, f_{hl}] = 1$ for $i<j$, $h<l$, $i\neq h$, $j\neq l$ with $i,j,h,l\in\{\frac{m}{2}+1,\dots,t\}$; if $f_{ij} = X_{2i-1}X_{2i}$ for $1\leqslant i<j\leqslant \frac{m}{2}$ and $f_{hl} = X_{h+\frac{m}{2}}$ for $\frac{m}{2}+1\leqslant h<l\leqslant t$, then $[f_{ij}, f_{hl}] = 1$ for all $i<j$, $h<l$, $i\neq h$, $j\neq l$. Hence by [2] (Prop. 1.7) it follows that $f_1,\dots,f_t$ is an s-sequence.

b) Let $f_1 = X_1X_m$, $f_2 = X_2X_m$, $\dots$, $f_{m-1} = X_{m-1}X_m$, $f_m = X_{m+1}X_n$, $\dots$, $f_{n-2} = X_{n-1}X_n$ be the generators of $I(G)$.

We observe that if $B = \{g_{ij} = f_{ij}T_j - f_{ji}T_i \mid 1\leqslant i<j\leqslant n-2\}$ is a Gröbner basis for J then $f_1,\dots,f_{n-2}$ is an s-sequence. Hence we prove that $S(g_{ij}, g_{hl})$, with $i,j,h,l\in\{1,\dots,n-2\}$, has a standard expression with respect B with remainder 0. We have:

$$S(g_{ij}, g_{hl}) = \frac{f_{ij}f_{lh}}{[f_{ij}, f_{hl}]}T_jT_h - \frac{f_{hl}f_{ji}}{[f_{ij}, f_{hl}]}T_iT_l. \quad (*)$$

Then we compute a standard expression of $S(g_{ij}, g_{hl})$ with respect to B with remainder 0. If $[\mathrm{in}_\prec(g_{ij}), \mathrm{in}_\prec(g_{hl})] = 1$, then $S(g_{ij}, g_{hl}) = f_{lh}T_hg_{ij} - f_{ji}T_ig_{hl}$ for all $i,j,h,l\in\{1,\dots,n-2\}$. If $[\mathrm{in}_\prec(g_{ij}), \mathrm{in}_\prec(g_{hl})]\neq 1$, then we compute a standard expression for all S-polynomials $S(g_{ij}, g_{hl})$ using $(*)$:

- $S(g_{ij}, g_{il}) = -[f_{ji}, f_{li}]g_{jl}T_i$
- $S(g_{ij}, g_{hj}) = [f_{ji}, f_{jh}]g_{ih}T_j$
- $S(g_{ij}, g_{hl}) = [f_{ji}, f_{lh}](f_{lj}g_{ih}T_j + f_{hi}g_{lj}T_i)$ if $j>l$.
- $S(g_{ij}, g_{hl}) = [f_{ji}, f_{lh}](f_{lj}g_{ih}T_j - f_{hi}g_{jl}T_i)$ if $j<l$.

Hence all S-polynomials $S(g_{ij}, g_{hl})$ reduce to 0 with respect to B. □

Remark 4.1 A subclass of the case a) of this section is considered in [3], precisely the forest with edge ideal $I(G) = (X_1X_2, X_3X_4,\dots,X_{n-1}X_n)$.

5. Invariants of the symmetric algebra

In this section we use the theory of s-sequences in order to compute standard algebraic invariants of the symmetric algebra of the examined edge ideals in terms of their annihilator ideals.

We analyze the following classes of edge ideals associated to a connected graph G (see Remark 3.1):

1) $I(G) = (X_1X_n, X_2X_n,\dots,X_{n-1}X_n)$,

2) $I(G) = (X_1X_2, X_2X_3,\dots,X_{n-1}X_n)$,

3) $I(G) = (X_1X_{n-1}, X_2X_{n-1},\dots,X_{n-2}X_{n-1}, X_\ell X_n)$, $\ell=1,\dots,n-2$.

Proposition 5.1 *Let G be the graph with n vertices having edge ideal $I(G)=(X_1X_n, X_2X_n, \dots, X_{n-1}X_n) \subset R=K[X_1,\dots,X_n]$. The annihilator ideals of the generators of $I(G)$ are*

$$\mathcal{I}_1=(0), \quad \mathcal{I}_i=(X_1,\dots,X_{i-1}), \quad \textit{for} \;\; i=2,\dots,n-1.$$

Proof. Let $I(G)=(f_1,\dots,f_{n-1})$, where $f_1=X_1X_n$, $f_2=X_2X_n$, $\dots$, $f_{n-1}=X_{n-1}X_n$. Set $f_{hk}=\frac{f_h}{[f_h,f_k]}$ for $h<k$, $h,k=1,\dots,n-1$. Then the annihilator ideals of the monomial sequence $f_1,\dots,f_{n-1}$ are $\mathcal{I}_i=(f_{1i},f_{2i},\dots,f_{i-1,i})$ for $i=1,\dots,n-1$. For $i=1$ we have $\mathcal{I}_1=(0)$ and by the structure of these monomials it follows $\mathcal{I}_2=(f_{12})=(X_1)$, $\mathcal{I}_3=(f_{13},f_{23})=(X_1,X_2)$, $\dots,\mathcal{I}_{n-1}=(f_{1,n-1},f_{2,n-1},\dots,f_{n-2,n-1})=(X_1,X_2,\dots,X_{n-2})$.

Hence $\mathcal{I}_i=(X_1,\dots,X_{i-1})$, for $i=2,\dots,n-1$. □

Remark 5.1 By Proposition 5.1 it follows $\mathrm{in}_{\prec}(J)=((X_1)T_2,(X_1,X_2)T_3, \dots,(X_1,X_2,\dots,X_{n-2})T_{n-1})$.

Theorem 5.1 *Let $G, I(G)$ be as in Proposition 5.1. For the symmetric algebra of $I(G)\subset R$ it holds:*

a) $\dim(Sym_R(I(G)))=n+1$,

b) $e(Sym_R(I(G))=n-1$,

c) $reg(Sym_R(I(G))=1$.

Proof. By Proposition 5.1 the s-sequence that generates $I(G)$ is strong.

a) By [6](Thm. 8.2.8), $\dim(Sym_R(I(G)))=\sup\{n+1,n-1\}=n+1$, where $n-1$ is the number of the edges of G.

b) By [2](Prop. 2.4), it follows that $\mathrm{e}(Sym_R(I(G)))=\sum_{i=1}^{n-1}\mathrm{e}(R/\mathcal{I}_i)$. By Proposition 5.1 the annihilator ideals I_i are generated by a regular sequence, then by [5](Thm. 4.8), $\mathrm{e}(R/\mathcal{I}_i)=1$, for $i=2,\dots,n-1$ and $\mathrm{e}(R/(0))=1$. Hence $\mathrm{e}(Sym_R(I(G)))=\sum_{i=1}^{n-1}e(R/\mathcal{I}_i)=n-1$.

c) $\mathrm{reg}(Sym_R(I(G)))=\mathrm{reg}(R[T_1,\dots,T_{n-1}]/J)\leqslant\mathrm{reg}(R[T_1,\dots,T_{n-1}]/\mathrm{in}_{\prec}(J))\leqslant\max_{2\leqslant j\leqslant n-1}\{\sum_{i=1}^{j-1}\deg(f_{ij})-(j-2)\}$, by [5](Thm. 4.8). Then one has $\mathrm{reg}(Sym_R(I(G)))\leqslant\max_{2\leqslant j\leqslant n-1}\{\sum_{i=1}^{j-1}\deg(X_i)-(j-2)\}=(j-1)-(j-2)=1$.

Moreover J is generated by the linear forms of degree two $X_iT_j-X_jT_i$, for $i,j=1,\dots,n-1$. Then $\mathrm{reg}(Sym_R(I(G)))=\mathrm{reg}(R[T_1,\dots,T_{n-1}]/J)\geq 1$. It follows that $\mathrm{reg}(Sym_R(I(G)))=1$. □

Proposition 5.2 *Let G be the graph with n vertices having edge ideal $I(G)=(X_1X_2,X_2X_3,\dots,X_{n-1}X_n)\subset R=K[X_1,\dots,X_n]$. The annihilator ideals of the generators of $I(G)$ are*

$$\mathcal{I}_1=(0), \mathcal{I}_2=(X_1), \mathcal{I}_3=(X_2), \mathcal{I}_i=(X_1X_2,X_2X_3,\dots,X_{i-3}X_{i-2},X_{i-1}),$$

for $i=4,\dots,n-1$.

Proof. Let $I(G)=(f_1,\dots,f_{n-1})$ where $f_1=X_1X_2$, $f_2=X_2X_3$, $\dots$, $f_{n-1}=X_{n-1}X_n$. Set $f_{hk}=\frac{f_h}{[f_h,f_k]}$ for $h<k$, $h,k=1,\dots,n-1$. The annihilator ideals of the monomial sequence $f_1,\dots,f_{n-1}$ are $\mathcal{I}_i=(f_{1i},f_{2i},\dots,f_{i-1,i})$ for $i=1,\dots,n-1$. We have $\mathcal{I}_1=(0)$, $\mathcal{I}_2=(f_{12})=(X_1)$, $\mathcal{I}_3=(f_{13},f_{23})=(X_1X_2,X_2)=$

(X_2), $\mathcal{I}_4 = (f_{14}, f_{24}, f_{34}) = (X_1X_2, X_3), \ldots, \mathcal{I}_{n-1} = (f_{1,n-1}, \ f_{2,n-1}, \ldots, f_{n-2,n-1}) = (X_1X_2, X_2X_3, \ldots, X_{n-4}X_{n-3}, X_{n-2})$.

Hence $\mathcal{I}_i = (X_1X_2, X_2X_3, \ldots, X_{i-3}X_{i-2}, X_{i-1})$, for $i = 4, \ldots, n-1$. □

Theorem 5.2 *Let $G, I(G)$ be as in Proposition 5.2. For the symmetric algebra of $I(G) \subset R$ it holds:*

a) $dim(Sym_R(I(G))) = n+1$

b) $e(Sym_R(I(G)) = \binom{n-1}{1} + \binom{n-2}{2} + \binom{n-3}{3} + \ldots$

Proof. a) By [6](Thm. 8.2.8), $\dim(Sym_R(I(G))) = \sup\{n+1, n-1\} = n+1$, where $n-1$ is the number of the edges of G.

b) By [2] (Prop. 2.4), $\mathrm{e}(Sym_R(I(G))) = \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant n-1} \mathrm{e}(R/(\mathcal{I}_{i_1}, \ldots, \mathcal{I}_{i_r}))$ with $\dim(R/(\mathcal{I}_{i_1}, \ldots, \mathcal{I}_{i_r})) = d - r$, where $d = \dim(Sym_R(I(G))) = n+1$ and $1 \leqslant r \leqslant n-1$. Set $d' = \dim(R/(\mathcal{I}_{i_1}, \ldots, \mathcal{I}_{i_r})) = n+1-r$.

The multiplicity $\mathrm{e}(Sym_R(I(G)))$ is given by the sum of the following terms:

$r = 1, \quad \mathrm{e}(R/\mathcal{I}_1) = 1,$

$r = 2, \quad \mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_2)) = \mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_3)) = 1,$

$r = 3, \quad \mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3)) = \mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_4)) = 1$

$\mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_3 + \mathcal{I}_4)) = \mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_3 + \mathcal{I}_5)) = 1,$

$r = 4, \quad \mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4)) = \mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_5)) = 1$

$\mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_4 + \mathcal{I}_5)) = \mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_4 + \mathcal{I}_6)) = 1$

$\mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5)) = \mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_6)) = 1$

$\mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_3 + \mathcal{I}_5 + \mathcal{I}_6)) = \mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_3 + \mathcal{I}_5 + \mathcal{I}_7)) = 1,$

and so on, where, for $r \leqslant n-1$, the number of $\mathrm{e}(R/(\mathcal{I}_{i_1}, \ldots, \mathcal{I}_{i_r}))$ such that $\dim(R/(\mathcal{I}_{i_1}, \ldots, \mathcal{I}_{i_r})) = n+1-r$ is in general double with respect to that of the preceding case $r-1$, namely from each $\mathrm{e}(R/(\mathcal{I}_{i_1}, \ldots, \mathcal{I}_{i_{r-1}})) = 1$ it comes $\mathrm{e}(R/(\mathcal{I}_{i_1}, \ldots, \mathcal{I}_{i_{r-1}}, \mathcal{I}_{i_{r-1}+1})) = 1$ and $\mathrm{e}(R/(\mathcal{I}_{i_1}, \ldots, \mathcal{I}_{i_{r-1}}, \mathcal{I}_{i_{r-1}+2})) = 1$.

But if the index i_{r-1} is equal to $n-2$, it derives only $\mathrm{e}(R/(\mathcal{I}_{i_1}, \ldots, \mathcal{I}_{i_{r-1}+1})) = 1$, nothing if $i_{r-1} = n-1$.

Consequently, some of the $\mathrm{e}(R/(\mathcal{I}_{i_1}, \ldots, \mathcal{I}_{i_r}))$ cannot be considered, those having maximum index greater or equal than n. In particular,

$r = n-2$,

$\mathrm{e}(R/(\mathcal{I}_1 + \ldots + \mathcal{I}_{n-3} + \mathcal{I}_{n-2})) = \mathrm{e}(R/(\mathcal{I}_1 + \ldots + \mathcal{I}_{n-3} + \mathcal{I}_{n-1})) =$
$\mathrm{e}(R/(\mathcal{I}_1 + \ldots + \mathcal{I}_{n-4} + \mathcal{I}_{n-2} + \mathcal{I}_{n-1})) = \ldots = \mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_3 + \ldots \mathcal{I}_{n-1})) = 1$

$r = n-1, \quad \mathrm{e}(R/\mathcal{I}_1 + \ldots + \mathcal{I}_{n-1})) = 1.$

Let $F_0 = 0$, $F_1 = 1$, …, $F_i = F_{i-2} + F_{i-1}$, $i \geqslant 2$, be the Fibonacci sequence. It results:

- if $n = 2 \quad \mathrm{e}(Sym_R(I(G))) = \mathrm{e}(R/(\mathcal{I}_1)) = 1,$

- if $n = 3 \quad \mathrm{e}(Sym_R(I(G))) = \mathrm{e}(R/(\mathcal{I}_1)) + \mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_2)) = 2 = 1 + F_2,$

- if $n = 4$ $\quad \mathrm{e}(Sym_R(I(G))) = \mathrm{e}(R/(\mathcal{I}_1)) + \mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_2)) +$
$+ \mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_3)) + \mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3)) = 4 = 2 + F_3,$

- if $n = 5$ $\quad \mathrm{e}(Sym_R(I(G))) = \mathrm{e}(R/(\mathcal{I}_1)) + \mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_2)) + \mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_3)) +$
$+ \mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3)) + \mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_4)) + \mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_3 + \mathcal{I}_4)) +$
$+ \mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4)) = 7 = 4 + F_4,$

and so on.

Hence $\mathrm{e}(Sym_R(I(G)))$ is the sum of the first $n-1$ terms of the Fibonacci sequence, that is $F_{n+1} - 1$, so the assertion follows taking in consideration the Lucas' formula. □

Proposition 5.3 *Let G be the graph with n vertices having edge ideal $I(G) = (X_1X_{n-1}, X_2X_{n-1}, \ldots, X_{n-2}X_{n-1}, X_\ell X_n)$, $\ell = 1, \ldots, n-2$, $I(G) \subset R = K[X_1, \ldots, X_n]$. The annihilator ideals of the generators of $I(G)$ are*

$$\mathcal{I}_1 = (0), \quad \mathcal{I}_i = (X_1, \ldots, X_{i-1}), \ \textit{for } i = 2, \ldots, n-2, \quad \mathcal{I}_{n-1} = (X_{n-1}).$$

Proof. Let $I(G) = (f_1, f_2, \ldots, f_{n-2}, f_{n-1})$, where $f_1 = X_1X_n$, $f_2 = X_2X_n, \ldots,$ $f_{n-2} = X_{n-2}X_{n-1}$, $f_{n-1} = X_\ell X_n$, $\ell = 1, \ldots, n-2$. Set $f_{hk} = \frac{f_h}{[f_h, f_k]}$ for $h < k$, $h, k = 1, \ldots, n-1$. The annihilator ideals of the monomial sequence $f_1, \ldots, f_{n-1}$ are $\mathcal{I}_i = (f_{1i}, f_{2i}, \ldots, f_{i-1,i})$, for $i = 1, \ldots, n-1$. Hence we have $\mathcal{I}_1 = (0)$, $\mathcal{I}_2 = (f_{12}) = (X_1)$, $\mathcal{I}_3 = (f_{13}, f_{23}) = (X_1, X_2), \ldots,$ $\mathcal{I}_{n-2} = (f_{1,n-2}, \ldots, f_{n-3,n-2}) = (X_1, X_2, \ldots, X_{n-3})$, $\mathcal{I}_{n-1} = (f_{1,n-1}, \ldots, f_{n-2,n-1}) = (X_1X_{n-1}, \ldots, X_{\ell-1}X_{n-1}, X_{n-1}, X_{\ell+1}X_{n-1}, \ldots, X_{n-2}X_{n-1}) = (X_{n-1})$. □

Theorem 5.3 *Let $G, I(G)$ be as in Proposition 5.3. For the symmetric algebra of $I(G) \subset R$ it holds:*

a) $\dim(Sym_R(I(G))) = n + 1$,

b) $e(Sym_R(I(G)) = 2(n-2)$.

Proof. a) By [6](Thm. 8.2.8), $\dim(Sym_R(I(G))) = \sup\{n+1, n-1\} = n+1$, where $n-1$ is the number of the edges of G.

b) By [2] (Prop. 2.4), $\mathrm{e}(Sym_R(I(G))) = \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant n-1} \mathrm{e}(R/(\mathcal{I}_{i_1}, \ldots, \mathcal{I}_{i_r}))$ with $\dim(R/(\mathcal{I}_{i_1}, \ldots, \mathcal{I}_{i_r})) = d - r$, where $d = \dim(Sym_R(I(G))) = n+1$ and $1 \leqslant r \leqslant n-1$. Set $d' = \dim(R/(\mathcal{I}_{i_1}, \ldots, \mathcal{I}_{i_r})) = n + 1 - r$.

The multiplicity $\mathrm{e}(Sym_R(I(G)))$ is given by the sum of the following terms:

$r = 1, \quad \mathrm{e}(R/\mathcal{I}_1) = 1,$

$r = 2, \quad \mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_2)) = \mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_{n-1})) = 1,$

$r = 3, \quad \mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3)) = \mathrm{e}(R/(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_{n-1})) = 1,$

....

$r = n-2, \quad \mathrm{e}(R/(\mathcal{I}_1 + \ldots + \mathcal{I}_{n-2})) = \mathrm{e}(R/(\mathcal{I}_1 + \ldots + \mathcal{I}_{n-3} + \mathcal{I}_{n-1})) = 1,$

$r = n-1, \quad \mathrm{e}(R/(\mathcal{I}_1 + \ldots + \mathcal{I}_{n-1})) = 1,$

and the assertion easily follows. □

Now we analyze the following classes of edge ideals:

1) $I(G) = (X_1X_2, X_3X_4, \dots, X_{n-1}X_n)$,

2) $I(G) = (X_1X_2, X_3X_4, \dots, X_{m-1}X_m, X_{m+1}X_n, \dots, X_{n-1}X_n)$,

3) $I(G) = (X_1X_m, X_2X_m, \dots, X_{m-1}X_m, X_{m+1}X_n, X_{m+2}X_n, \dots, X_{n-1}X_n)$,

where G is a non connected graph (see section 3).

Proposition 5.4 *Let G be a graph with n vertices and edge ideal $I(G) = (X_1X_2, X_3X_4, \dots, X_{n-1}X_n) \subset R = K[X_1, \dots, X_n]$. The annihilator ideals of the generators of $I(G)$ are*

$$\mathcal{I}_1 = (0), \mathcal{I}_i = (X_1X_2, X_3X_4, \dots, X_{2i-3}X_{2i-2}), \quad \textit{for} \;\; i = 2, \dots, \frac{n}{2}.$$

Proof. Let $I(G) = (f_1, \dots, f_{\frac{n}{2}})$ where $f_1 = X_1X_2$, $f_2 = X_3X_4$, $\dots$, $f_{\frac{n}{2}} = X_{n-1}X_n$. Then the annihilator ideals of the monomial sequence $f_1, \dots, f_{\frac{n}{2}}$ are $\mathcal{I}_i = (f_{1i}, f_{2i}, \dots, f_{i-1,i})$ for $i = 1, \dots, \frac{n}{2}$. For $i = 1$ we have $\mathcal{I}_1 = (0)$. Moreover, $\mathcal{I}_2 = (f_{12}) = (X_1X_2)$, $\mathcal{I}_3 = (f_{13}, f_{23}) = (X_1X_2, X_3X_4)$, $\dots$, $\mathcal{I}_{\frac{n}{2}} = (f_{1,\frac{n}{2}}, f_{2,\frac{n}{2}}, \dots, f_{\frac{n}{2}-1,\frac{n}{2}}) = (X_1X_2, X_3X_4, \dots, X_{n-3}X_{n-2})$.
Hence $\mathcal{I}_i = (X_1X_2, X_3X_4, \dots, X_{2i-3}X_{2i-2})$, for $i = 2, \dots, \frac{n}{2}$. □

Remark 5.2 By Proposition 5.4 it follows that
$\mathrm{in}_{\prec}(J) = ((X_1X_2)\,T_2, (X_1X_2, X_3X_4)\,T_3, \dots, (X_1X_2, \dots, X_{n-3}X_{n-2})\,T_{\frac{n}{2}})$.

Theorem 5.4 *Let $R = K[X_1, \dots, X_n]$, $I(G) = (X_1X_2, X_3X_4, \dots, X_{n-1}X_n)$. Then:*

a) $dim(Sym_R(I(G))) = n+1$,

b) $e(Sym_R(I(G)) = \sum_{i=1}^{\frac{n}{2}} 2^{i-1}$,

c) $reg(Sym_R(I(G)) \leqslant \frac{n}{2}$.

Proof. By Proposition 5.4, $I(G)$ is generated by a strong s-sequence.

a) By [5] (Thm. 4.8), $Sym_R(I(G))$ is Cohen-Macaulay having dimension $\dim(R) + 1 = n + 1$.

b) By [2] (Prop. 2.4), it follows that $\mathrm{e}(Sym_R(I(G))) = \sum_{i=1}^{\frac{n}{2}} \mathrm{e}(R/\mathcal{I}_i)$. By Proposition 5.4 we compute $\mathrm{e}(R/\mathcal{I}_1) = 1$, $\mathrm{e}(R/\mathcal{I}_2) = 2$, $\mathrm{e}(R/\mathcal{I}_3) = 4$, $\mathrm{e}(R/\mathcal{I}_4) = 8, \dots, \mathrm{e}(R/\mathcal{I}_i) = 2^{\,i-1}$. Hence $\mathrm{e}(Sym_R(I(G))) = \sum_{i=1}^{\frac{n}{2}} 2^{\,i-1}$.

c) $\mathrm{reg}(Sym_R(I(G))) = \mathrm{reg}(R[T_1, \dots, T_{\frac{n}{2}}]/J) \leqslant \mathrm{reg}(R[T_1, \dots, T_{\frac{n}{2}}]/\,\mathrm{in}_{\prec}(J)) \leqslant \max_{2 \leqslant j \leqslant \frac{n}{2}} \{\sum_{i=1}^{j-1} \deg(f_{ij}) - (j-2)\}$ by [5] (Thm. 4.8). Then one computes: $\mathrm{reg}(Sym_R(I(G))) \leqslant \max_{2 \leqslant j \leqslant \frac{n}{2}} \{\sum_{i=1}^{j-1} \deg(X_{2i-1}X_{2i}) - (j-2)\} = \max_{2 \leqslant j \leqslant \frac{n}{2}} \{2(j-1) - (j-2)\} = \frac{n}{2}$. □

Proposition 5.5 *Let $I(G) = (X_1X_2, X_3X_4, \ \dots, \ X_{m-1}X_m, X_{m+1}X_n, \ \dots, \ X_{n-1}X_n)$ be an ideal of $R = K[X_1, \dots, X_n]$. Then the annihilator ideals of the generators of $I(G)$ are:*

$$\mathcal{I}_1 = (0), \quad \mathcal{I}_i = (X_1X_2, X_3X_4, \dots, X_{2i-3}X_{2i-2}) \quad \textit{for} \;\; i = 2, \dots, \frac{m}{2} + 1,$$

$\mathcal{I}_{\frac{m}{2}+j} = (X_1X_2, \ldots, X_{m-1}X_m, X_{m+1}, \ldots, X_{m+j-1})$ *for* $j = 2, \ldots, n-m-1$.

Proof. Let $f_1 = X_1X_2$, $f_2 = X_3X_4$, ..., $f_{\frac{m}{2}} = X_{m-1}X_m$, $f_{\frac{m}{2}+1} = X_{m+1}X_n$, ..., $f_t = X_{n-1}X_n$, for $t{=}n{-}\frac{m}{2}{-}1$, be the generators of $I(G)$. Then the annihilator ideals of the monomial sequence $f_1, \ldots, f_t$ are the following: $\mathcal{I}_1 = (0)$, and by the structure of the monomials, $\mathcal{I}_2 = (X_1X_2)$, $\mathcal{I}_3 = (X_1X_2, X_3X_4)$, ..., $\mathcal{I}_{\frac{m}{2}} = (X_1X_2, X_3X_4, \ldots, X_{m-3}X_{m-2})$, $\mathcal{I}_{\frac{m}{2}+1} = (X_1X_2,\ X_3X_4, \ldots, X_{m-3}X_{m-2}, X_{m-1}X_m)$, $\mathcal{I}_{\frac{m}{2}+2} = (X_1X_2, X_3X_4, \ldots, X_{m-3}X_{m-2},\ X_{m-1}X_m, X_{m+1})$, ..., $\mathcal{I}_t = (X_1X_2, X_3X_4, \ldots, X_{m-3}X_{m-2}, X_{m-1}X_m, X_{m+1}, \ \ldots, X_{n-2})$. The assertion follows. □

Remark 5.3 By Proposition 5.5 one has
$\mathrm{in}_{\prec}(J) = ((X_1X_2)\,T_2, (X_1X_2, X_3X_4)\,T_3, \ \ldots, \ (X_1X_2, X_3X_4, \ldots, X_{m-3}X_{m-2},\ X_{m-1}X_m, X_{m+1}, \ldots, X_{n-2})\ T_{n-\frac{m}{2}-1})$.

Theorem 5.5 *Let* $R = K[X_1, \ldots, X_n]$, $I(G) = (X_1X_2, X_3X_4, \ldots, X_{m-1}X_m,\ X_{m+1}X_n, \ldots, X_{n-1}X_n)$. *Then:*

a) $\mathit{dim}(\mathit{Sym}_R(I(G))) = n+1$

b) $e(\mathit{Sym}_R(I(G)) = \sum_{i=1}^{\frac{m}{2}} 2^{i-1} + 2^{\frac{m}{2}}(n-m-1)$

c) $\mathit{reg}(\mathit{Sym}_R(I(G)) \leqslant \frac{m}{2}+1$.

Proof. By Proposition 5.5 $I(G)$ is generated by a strong s-sequence.

a) By [5] (Thm. 4.8), $\mathit{Sym}_R(I(G))$ is Cohen-Macaulay having dimension $\dim(R)+1 = n+1$.

b) By [2] (Prop. 2.4), it follows that $\mathrm{e}(\mathit{Sym}_R(I(G))) = \sum_{i=1}^{t} \mathrm{e}(R/I_i)$, $t = n - \frac{m}{2} - 1$. Using Proposition 5.5, we compute $\mathrm{e}(R/\mathcal{I}_1) = 1$, $\mathrm{e}(R/\mathcal{I}_2) = 2$, $\mathrm{e}(R/\mathcal{I}_3) = 4$, $\mathrm{e}(R/\mathcal{I}_4) = 8, \ldots, \mathrm{e}(R/\mathcal{I}_{\frac{m}{2}}) = 2^{\frac{m}{2}-1}$, $\mathrm{e}(R/\mathcal{I}_{\frac{m}{2}+1}) = 2^{\frac{m}{2}}, \ldots,$ $\mathrm{e}(R/\mathcal{I}_t) = 2^{\frac{m}{2}}$. Hence: $\mathrm{e}(R/\mathcal{I}_i) = 2^{i-1}$, for $i = 1, \ldots, \frac{m}{2}$ and $\mathrm{e}(R/\mathcal{I}_j) = 2^{\frac{m}{2}}$, for $j = \frac{m}{2}, \ldots, t$. It follows $\mathrm{e}(\mathit{Sym}_R(I(G))) = \sum_{i=1}^{\frac{n}{2}} 2^{i-1} + (t - (\frac{m}{2}+1)+1)2^{\frac{m}{2}}$. The assertion holds.

c) Let $t = n - \frac{m}{2} - 1$. Then $\mathrm{reg}(\mathit{Sym}_R(I(G))) = \mathrm{reg}(R[T_1, \ldots, T_t]/J) \leqslant \mathrm{reg}(R[T_1, \ldots, T_t]/\,\mathrm{in}_{\prec}(J)) \leqslant \max_{2 \leqslant j \leqslant t}\{\sum_{i=1}^{j-1} \deg(f_{ij}) - (j-2)\}$, by [5] (Thm. 4.8). So it is $\mathrm{reg}(\mathit{Sym}_R(I(G))) \leqslant \max_{2 \leqslant j \leqslant n-\frac{m}{2}-1}\{\sum_{k=1}^{j-1} \deg(X_{2k-1}X_{2k}) - (j-2)\}$. Set $d = \sum_{k=1}^{j-1} \deg(X_{2k-1}X_{2k}) - (j-2)$. One computes: $d = k$, for $k = 1, \ldots, \frac{m}{2}$ and $d = \frac{m}{2}+1$, for $k = \frac{m}{2}+1, \ldots, n-\frac{m}{2}-2$. Hence $\mathrm{reg}(\mathit{Sym}_R(I(G))) \leqslant \frac{m}{2}+1$. □

Proposition 5.6 *Let* $I(G) = (X_1X_m, X_2X_m, \ldots, X_{m-1}X_m, X_{m+1}X_n, \ \ldots, \ X_{n-1}X_n)$ *be an ideal of* $R = K[X_1, \ldots, X_m, \ldots, X_n]$, $m, n > 2$. *Then the annihilator ideals of the generators of* $I(G)$ *are:*

$$\mathcal{I}_1 = (0), \quad \mathcal{I}_i = (X_1, X_2 \ldots, X_{i-1}) \ \ \textit{for} \ \ i = 2, \ldots, m-1,$$

$$\mathcal{I}_m = (X_1X_m, X_2X_m, \ldots, X_{m-1}X_m)$$

$$\mathcal{I}_j = (X_1X_m, X_2X_m, \ldots, X_{m-1}X_m, X_{m+1}, \ldots, X_j) \ \ \textit{for} \ \ j = m+1, \ldots, n-2.$$

Proof. Let $f_1 = X_1X_m$, $f_2 = X_2X_m$, ..., $f_{m-1} = X_{m-1}X_m$, $f_m = X_{m+1}X_n$, ..., $f_{n-2} = X_{n-1}X_n$ be the generators of $I(G)$. Then the annihilator ideals of the monomial sequence $f_1, \ldots, f_{n-2}$ are the following: $\mathcal{I}_1 = (0)$, and by the structure of the monomials, $\mathcal{I}_2 = (X_1)$, $\mathcal{I}_3 = (X_1, X_2)$, $\ldots$, $\mathcal{I}_{m-1} = (X_1, X_2, \ldots, X_{m-2})$, $\mathcal{I}_m = (X_1X_m, X_2X_m, \ldots, X_{m-1}X_m)$, $\mathcal{I}_{m+1} = (X_1X_m, X_2X_m, \ldots, X_{m-1}X_m, X_{m+1})$, $\mathcal{I}_{m+2} = (X_1X_m, X_2X_m, \ldots, X_{m-1}X_m,\ X_{m+1}, X_{m+2})$, $\ldots$, $\mathcal{I}_{n-2} = (X_1X_m, X_2X_m, \ldots, X_{m-1}X_m, X_{m+1}, X_{m+2}, \ldots, X_{n-2})$. The assertion follows. □

Theorem 5.6 *Let* $R = K[X_1, \ldots, X_m, \ldots, X_n]$, $m, n > 2$, *and* $I(G) = (X_1X_m, X_2X_m, \ldots, X_{m-1}X_m, X_{m+1}X_n, X_{m+2}X_n, \ldots, X_{n-1}X_n)$. *Then:*

a) $dim(Sym_R(I(G))) = n + 1$,

b) $e(Sym_R(I(G)) = mn - m^2 - 1$.

Proof. a) By Proposition 5.6, $\mathcal{I}_1 = (0)$, $\mathcal{I}_i = (X_1, X_2 \ldots, X_{i-1})$ if $i = 2, \ldots, m-1$, $\mathcal{I}_m = (X_1X_m, X_2X_m, \ldots, X_{m-1}X_m)$, $\mathcal{I}_i = (X_1X_m, X_2X_m, \ldots,\ X_{m-1}X_m, X_{m+1}, \ldots, X_i)$ if $i = m+1, \ldots, n-2$, and by [2] (Prop. 2.4) we have $\dim(Sym_R(I(G))) = \max_{1 \leqslant r \leqslant n-2}\{\dim(R/(\mathcal{I}_{i_1}, \ldots, \mathcal{I}_{i_r})) + r\}$, for $1 \leqslant i_1 < \cdots < i_r \leqslant n-2$. Hence the maximum dimension is given for $1 \leqslant i_1 < \cdots < i_r \leqslant n-2$: $\dim(R/(\mathcal{I}_{i_1}, \ldots, \mathcal{I}_{i_r})) + r = n - (r-1) + r = n + 1$ and $\dim(Sym_R(I((G))) = n + 1$.

b) By [2](Prop. 2.4), $e(Sym_R(I(G))) = \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant n-1} e(R/(\mathcal{I}_{i_1}, \ldots, \mathcal{I}_{i_r}))$ with $\dim(R/(\mathcal{I}_{i_1}, \ldots, \mathcal{I}_{i_r})) = d - r$, where $d = \dim(Sym_R(I(G))) = n + 1$ and $1 \leqslant r \leqslant n-1$. Set $d' = \dim(R/(\mathcal{I}_{i_1}, \ldots, \mathcal{I}_{i_r})) = n + 1 - r$. The multiplicity $e(Sym_R(I(G)))$ is given by the sum of the following terms:

$r = 1$, $\quad e(R/\mathcal{I}_1) = 1$

$r = 2$, $\quad e(R/(\mathcal{I}_1 + \mathcal{I}_2)) = e(R/(\mathcal{I}_1 + \mathcal{I}_m)) = 1$

$r = 3$, $\quad e(R/(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3)) = e(R/(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_m)) = e(R/(\mathcal{I}_1 + \mathcal{I}_m + \mathcal{I}_{m+1})) = 1$

. . . .

$r = m - 1$, $(m-1$ terms$)$

$e(R/(\mathcal{I}_1 + \ldots + \mathcal{I}_{m-2} + \mathcal{I}_{m-1})) = e(R/(\mathcal{I}_1 + \ldots + \mathcal{I}_{m-2} + \mathcal{I}_m)) =$
$e(R/(\mathcal{I}_1 + \ldots + \mathcal{I}_{m-3} + \mathcal{I}_m + \mathcal{I}_{m+1})) = \ldots = e(R/(\mathcal{I}_1 + \mathcal{I}_m + \ldots + \mathcal{I}_{2m-3})) = 1$

$r = m$, $(m-1$ terms$)$

$e(R/(\mathcal{I}_1 + \ldots + \mathcal{I}_{m-1} + \mathcal{I}_m)) = 2$

$e(R/(\mathcal{I}_1 + \ldots + \mathcal{I}_{m-2} + \mathcal{I}_m + \mathcal{I}_{m+1})) = e(R/(\mathcal{I}_1 + \ldots + \mathcal{I}_{m-3} + \mathcal{I}_m + \mathcal{I}_{m+1} + \mathcal{I}_{m+2})) = \ldots = e(R/(\mathcal{I}_1 + \mathcal{I}_m + \ldots + \mathcal{I}_{2m-2})) = 1$

. . . .

$r = n - m$, $(m-1$ terms$)$

$e(R/(\mathcal{I}_1 + \ldots + \mathcal{I}_m + \ldots + \mathcal{I}_{n-m})) = 2$

$e(R/(\mathcal{I}_1 + \ldots + \mathcal{I}_{m-2} + \mathcal{I}_m + \ldots + \mathcal{I}_{n-m+1})) = e(R/(\mathcal{I}_1 + \ldots + \mathcal{I}_{m-3} + \mathcal{I}_m + \ldots + \mathcal{I}_{n-m+2})) = \ldots = e(R/(\mathcal{I}_1 + \mathcal{I}_m + \ldots + \mathcal{I}_{n-2})) = 1$

$r = n - m + 1$, $(m-2$ terms$)$

$e(R/(\mathcal{I}_1 + \ldots + \mathcal{I}_m + \ldots + \mathcal{I}_{n-m+1})) = 2$

$e(R/(\mathcal{I}_1 + \ldots + \mathcal{I}_{m-2} + \mathcal{I}_m + \ldots + \mathcal{I}_{n-m+2})) = e(R/(\mathcal{I}_1 + \ldots + \mathcal{I}_{m-3} + \mathcal{I}_m + \ldots + \mathcal{I}_{n-m+3})) = \ldots = e(R/(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_m + \ldots + \mathcal{I}_{n-2})) = 1$

. . . .

$r = n - 3$, (2 terms)

$\mathrm{e}(R/(\mathcal{I}_1 + \ldots + \mathcal{I}_m + \ldots + \mathcal{I}_{n-3})) = 2$, $\mathrm{e}(R/(\mathcal{I}_1 + \ldots + \mathcal{I}_{m-2} + \mathcal{I}_m + \ldots + \mathcal{I}_{n-2})) = 1$

$r = n - 2$, $\mathrm{e}(R/(\mathcal{I}_1 + \ldots + \mathcal{I}_m + \ldots + \mathcal{I}_{n-2})) = 2$.

Hence $\mathrm{e}(Sym_R(I(G))) = \dfrac{m(m-1)}{2} + m(n-2m) + \dfrac{m(m+1)}{2} - 3 + 2$, so the assertion follows. □

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