

# Generalization of some properties of Banach algebras to fundamental locally multiplicative topological algebras

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## Abstract

In this article we generalized some properties of Banach algebras, to a new class of topological algebras namely fundamental and fundamental locally multiplicative topological algebras (abbreviated by *FLM*). Also the new notion of sub-multiplicatively metrizable topological algebra is given and some well known spectral properties of Banach algebras are generalized to such kind of algebras.

**Key Words:** *FLM* algebras, fundamental topological algebras, holomorphic function, multiplicative linear functionals, semi-simple algebras, spectral radius

## 1. Introduction

The notion of fundamental topological spaces (also algebras) has been introduced in [1] in 1990 extending the meaning of both local convexity and local boundedness.

A topological linear space  $\mathcal{A}$  is said to be fundamental one if there exists  $b > 1$  such that for every sequence  $(x_n)$  of  $\mathcal{A}$ , the convergence of  $b^n(x_n - x_{n-1})$  to zero in  $\mathcal{A}$  implies that  $(x_n)$  is Cauchy.

A fundamental topological algebra is an algebra whose underlying topological linear space is fundamental. The famous Cohen factorization theorem for complete metrizable fundamental topological algebras is proved in [1] and the  $n^{\text{th}}$  roots and quasi square roots in fundamental topological algebras are studied in [4].

The fundamental locally multiplicative topological algebras (abbreviated by *FLM*) with a property very similar to the normed algebras is also introduced in [2]. A fundamental topological algebra is called locally multiplicative if there exists a neighborhood  $U_0$  of zero such that, for every neighborhood  $V$  of zero, the sufficiently large powers of  $U_0$  lie in  $V$ .

Also in [2] a topological structure is defined on the algebraic dual space of an *FLM* algebra to make it a normed space, and some of the famous theorems of Banach algebras are extended for complete metrizable *FLM* algebras. In this paper we have studied the linear multiplicative functionals on *FLM* algebras and proved some results on them in section 2. In section 3, we introduced the new notion of sub-multiplicative metrizable topological algebras and by using it we generalized some properties of Banach algebras to *FLM* algebras.

**2. Multiplicative functional on *FLM* algebras**

A version of the *Gleason, Kahane-Zelazko theorem* is proved for *FLM* algebras in [3]. In theorem 5.5 of [3],  $T : \mathcal{A} \rightarrow \mathbb{C}$  is assumed a non-zero linear functional on  $\mathcal{A}$  and proved that  $T$  is multiplicative if and only if  $T(a) \in Sp(a)$  for all  $a \in \mathcal{A}$ . Now by replacing  $\mathbb{C}$  by a semi-simple complete metrizable *FLM* algebra  $\mathcal{B}$  we generalized it as follow.

**Theorem 2.1** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be two commutative complete metrizable *FLM* algebras, with unit elements, and let  $\mathcal{B}$  be semi-simple. If  $T : \mathcal{A} \rightarrow \mathcal{B}$  is a linear mapping, such that  $Sp(Tx) \subset Sp(x)$ , for any  $x \in \mathcal{A}$ , then  $T$  is a multiplicative mapping.*

**Proof.** Let  $f$  be a multiplicative and linear functional on  $\mathcal{B}$  and put  $F(x) = f(Tx)$  for any  $x \in \mathcal{A}$ . So  $F$  is a linear functional on  $\mathcal{A}$ , and also by theorem 5.5 [3],

$$F(x) = f(Tx) \in Sp(Tx) \subset Sp(x),$$

and so by using again theorem 5.5 [3]  $F$  is multiplicative and linear functional on  $\mathcal{A}$ . It follows that

$$F(xy) = F(x)F(y),$$

or

$$f(Txy) = f(Tx)f(Ty) = f(TxTy).$$

Since  $f$  is arbitrary multiplicative linear functional on  $\mathcal{B}$  and  $\mathcal{B}$  is semi-simple, thus  $T$  is multiplicative. □

Let  $\mathcal{A}$  and  $\mathcal{B}$  be two commutative complete metrizable *FLM* algebras, with unit elements  $e_{\mathcal{A}}$  and  $e_{\mathcal{B}}$ , respectively. If  $T$  is a multiplicative linear mapping from  $\mathcal{A}$  into  $\mathcal{B}$ , such that  $Te_{\mathcal{A}} \neq e_{\mathcal{B}}$ , then it may be  $Sp(Tx)$  is not a subset of  $Sp(x)$ . Furthermore  $Sp(x) \subset Sp(Tx)$ . For example, let  $A_1$  and  $A_2$  be commutative complete metrizable *FLM* algebras,  $B = A_1 \oplus A_2$  and  $T : A_1 \rightarrow B$ . Then we have the following theorem.

**Theorem 2.2** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be two commutative complete metrizable *FLM* algebras, with unit elements  $e_{\mathcal{A}}$  and  $e_{\mathcal{B}}$ , respectively. Let  $T$  be a linear multiplicative mapping from  $\mathcal{A}$  to  $\mathcal{B}$ , such that  $Te_{\mathcal{A}} = e_{\mathcal{B}}$ . Then for any  $x \in \mathcal{A}$ ,  $Sp(Tx) \subset Sp(x)$ .*

**Proof.** By assumption we have

$$e_{\mathcal{B}} = Te_{\mathcal{A}} = Txx^{-1} = TxTx^{-1},$$

for any invertible element  $x \in \mathcal{A}$ . This shows that, for any such  $x$ ,  $Tx$  is invertible in  $\mathcal{B}$  and

$$T(x^{-1}) = (Tx)^{-1}.$$

Now if  $\lambda \notin Sp(x)$ , then  $x - \lambda e_{\mathcal{A}}$  is invertible in  $\mathcal{A}$  and so  $T(x - \lambda e_{\mathcal{A}}) = Tx - \lambda e_{\mathcal{B}}$  is invertible in  $\mathcal{B}$ . Therefore  $\lambda \notin Sp(Tx)$ . □

**3. New results on *FLM* algebras**

In this section, by introducing the new notion of sub-multiplicative metrizable topological algebra, we generalize some well known spectral properties of Banach algebras to complete metrizable *FLM* algebras.

By  $\Omega_{\mathcal{A}}$  we mean the set of all elements  $a \in \mathcal{A}$  such that  $\rho(a) < 1$ , where  $\rho(a)$  is the spectral radius of  $a \in \mathcal{A}$ . We denote the center of topological algebra  $\mathcal{A}$ , by  $Z(\mathcal{A})$ , such that

$$Z(\mathcal{A}) = \{a \in \mathcal{A} : ax = xa, \quad \text{for all } x \in \mathcal{A}\}.$$

**Definition 3.1** *Let  $(\mathcal{A}, d)$  be a metrizable topological algebra. We say  $\mathcal{A}$  is a sub-multiplicative metrizable topological algebra if*

$$d(0, xy) \leq d(0, x)d(0, y)$$

for each  $x, y \in \mathcal{A}$ .

It is clear that, when  $\mathcal{A}$  is a sub-multiplicatively metrizable topological algebra, the meter  $d_{\mathcal{A}}$  is not a discrete meter; for example, if  $d_{\mathcal{A}}$  is a Dirac meter on some ideal  $E$  of  $\mathcal{A}$ , the sub-multiplicatively of the meter fails. For abbreviation we denote  $d_{\mathcal{A}}(0, x)$  by  $D_{\mathcal{A}}(x)$  for any  $x \in \mathcal{A}$ .

The following lemma is proved for Banach algebras and has a similar proof for *FLM* algebras (see theorem 3.2.6, [5]). Therefore, we remove its proof, because it is well known and clear.

**Lemma 3.2** *Let  $\mathcal{A}$  be a complete metrizable *FLM* algebra and  $x \in \mathcal{A}$ . Then for every nonconstant polynomial  $P$  with complex coefficients we have*

$$Sp(P(x)) = P(Sp(x)).$$

Let  $\mathcal{A}$  be a complete metrizable fundamental topological algebra with unit  $e$  and  $x \in \mathcal{A}$ . If for some  $b > 1$ ,  $b^n x^n \rightarrow 0$  in  $\mathcal{A}$ , then  $e - x$  is invertible and

$$(e - x)^{-1} = \sum_{n=0}^{\infty} x^n$$

and also if for some  $b > 1$ ,  $b^n(e - x)^n \rightarrow 0$ , then  $x$  is invertible (theorem 4.1, [2]). If  $\mathcal{A}$  is a complete metrizable *FLM* algebra with meter  $d_{\mathcal{A}}$ , then  $(e - x)$  is invertible for  $d_{\mathcal{A}}(0, x) = D_{\mathcal{A}}(x) < 1$ . Now if  $\mathcal{A}$  is a complete metrizable *FLM* algebra with sub-multiplicatively meter  $d_{\mathcal{A}}$  and  $\lambda \neq 0$ , then  $(e - \lambda x)$  is invertible for  $d_{\mathcal{A}}(0, x) = D_{\mathcal{A}}(x) < |\lambda|$ .

**Theorem 3.3** *Let  $\mathcal{A}$  be a complete metrizable *FLM* algebra with sub-multiplicative meter  $d_{\mathcal{A}}$ . Then  $\rho(x) = \lim_{n \rightarrow \infty} D_{\mathcal{A}}(x^n)^{1/n}$ .*

**Proof.** From above discussion, we have  $\rho(x) \leq D_{\mathcal{A}}(x)$ , for any  $x \in \mathcal{A}$ . Now if applied the previous lemma to  $x^n$ , we have  $\rho(x)^n \leq D_{\mathcal{A}}(x^n)$ . Let  $f$  be a linear functional on  $\mathcal{A}$ , then the map from  $\mathbb{C} \setminus Sp(x)$  to  $\mathbb{C}$ , which  $\lambda \mapsto f((\lambda e - x)^{-1})$  is holomorphic. By theorem 4.1 of [2], we have

$$f((\lambda e - x)^{-1}) = \frac{1}{\lambda}(f(e) + \frac{f(x)}{\lambda} + \dots + \frac{f(x^n)}{\lambda^n} + \dots).$$

Fix  $\lambda$ , such that  $|\lambda| > \rho(x)$ . Then for every linear functional  $f$  on  $\mathcal{A}$ , we have  $\sup_n | \frac{f(x^n)}{\lambda^n} | < \infty$ . By applying the Banach-Steinhaus theorem (theorem 2.8, [7]) to the space of all continuous linear functional on  $\mathcal{A}$  and to the sequence of  $T_n$  from that to  $\mathbb{C}$ , defined by  $T_n(f) = \frac{f(x^n)}{\lambda^n}$ , we conclude that there exists a constant  $C$ , depending to  $\lambda$ , such that  $D_{\mathcal{A}}(x^n) \leq C|\lambda|^n$  for all  $n \geq 1$ . Then

$$\lim_{n \rightarrow \infty} \sup D_{\mathcal{A}}(x^n)^{1/n} \leq |\lambda|,$$

for all  $|\lambda| \geq \rho(x)$ . Hence we conclude

$$\rho(x) \leq \lim_{n \rightarrow \infty} \inf D_{\mathcal{A}}(x^n)^{1/n} \leq \lim_{n \rightarrow \infty} \sup D_{\mathcal{A}}(x^n)^{1/n} \leq \rho(x),$$

therefore  $\rho(x) = \lim_{n \rightarrow \infty} D_{\mathcal{A}}(x^n)^{1/n}$ . □

Let  $E(\mathcal{A})$  be the set of all elements  $x \in \mathcal{A}$  for which  $E(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!}$ , can be defined. If  $\mathcal{A}$  be a complete metrizable *FLM* algebra, then  $E(\mathcal{A}) = \mathcal{A}$  (theorem 5.4, [3]).

The following theorem is a version of Zemánek theorem (theorem 5.3.1, [5]) for *FLM* algebras:

**Theorem 3.4** *Let  $\mathcal{A}$  be a complete metrizable *FLM* algebra with sub-multiplicatively meter  $d_{\mathcal{A}}$ . Then the following statements are equivalent:*

- (i)  $a$  is in the Jacobson radical of  $\mathcal{A}$ ;
- (ii)  $Sp(a + x) = Sp(x)$ , for all  $x \in \mathcal{A}$ ;
- (iii)  $\rho(a + x) = 0$ , for all quasi-nilpotent elements  $x$  in  $\mathcal{A}$ ;
- (iv)  $\rho(a + x) = 0$ , for all quasi-nilpotent elements  $x$  in a neighborhood 0 in  $\mathcal{A}$ ;
- (v) there exists  $C > 0$  such that  $\rho(x) \leq CD_{\mathcal{A}}(x - a)$ , for all  $x \in \mathcal{A}$  in a neighborhood of  $a$  in  $\mathcal{A}$ .

**Proof.** Straightforward. □

**Theorem 3.5** *Let  $\mathcal{A}$  be a complete semi-simple metrizable *FLM* algebra with sub-multiplicative meter. If  $g : \Omega_{\mathcal{A}} \rightarrow \Omega_{\mathcal{A}}$  be a holomorphic map satisfying  $g(0) = 0$  and  $g'(0) = I$ , then  $g(c) = c$  for all  $c \in \Omega_{\mathcal{A}} \cap Z(\mathcal{A})$ .*

To prove this theorem we need to the following lemma.

**Lemma 3.6** *Let  $\mathcal{A}$  be a complete metrizable *FLM* algebra with sub-multiplicative meter  $d_{\mathcal{A}}$ . Suppose that  $x, y \in \mathcal{A}$  satisfy  $xy = yx$ . Then  $\rho(x + y) \leq \rho(x) + \rho(y)$  and  $\rho(xy) \leq \rho(x)\rho(y)$ .*

**Proof.** Since  $xy = yx$ , then  $(xy)^n = x^n y^n$  for each integer  $n \geq 1$ . By theorem 3.3, we have

$$\begin{aligned} \rho(xy) &= \lim_{n \rightarrow \infty} D_{\mathcal{A}}((xy)^n)^{1/n} = \lim_{n \rightarrow \infty} D_{\mathcal{A}}(x^n y^n)^{1/n} \\ &\leq \lim_{n \rightarrow \infty} D_{\mathcal{A}}(x^n)^{1/n} \lim_{n \rightarrow \infty} D_{\mathcal{A}}(y^n)^{1/n} \\ &= \rho(x)\rho(y). \end{aligned}$$

Let  $\rho(x) < \alpha$ ,  $\rho(y) < \beta$  and  $a = x/\alpha$ ,  $b = y/\beta$ . Then  $\rho(a) < 1$  and  $\rho(b) < 1$ . Therefore there exists some integer  $N$  such that for  $n \geq N$ , we have  $\max(D_{\mathcal{A}}(a^{2^n}), D_{\mathcal{A}}(b^{2^n})) < 1$ . Now let  $\gamma_n = \max_{0 \leq k \leq 2^n} D_{\mathcal{A}}(a^k)D_{\mathcal{A}}(b^{2^n-k})$ , then we have

$$\begin{aligned} D_{\mathcal{A}}((a+b)^{2^n})^{1/2^n} &= D_{\mathcal{A}}\left(\sum_{k=0}^{2^n} \binom{2^n}{k} x^k y^{2^n-k}\right)^{1/2^n} \\ &\leq \left(\sum_{k=0}^{2^n} \binom{2^n}{k} \alpha^k \beta^{2^n-k} D_{\mathcal{A}}(a^k)D_{\mathcal{A}}(b^{2^n-k})\right)^{1/2^n} \\ &\leq (\alpha + \beta)\gamma_n^{1/2^n}. \end{aligned}$$

The sequence  $(\gamma_n)$  is decreasing and therefore

$$\begin{aligned} \rho(x+y) &= \lim_{n \rightarrow \infty} (D_{\mathcal{A}}(x+y)^{2^n})^{1/2^n} \\ &\leq (\alpha + \beta) \lim_{n \rightarrow \infty} \sup \gamma_n^{1/2^n} \\ &\leq (\alpha + \beta) \lim_{n \rightarrow \infty} \sup \gamma_N^{1/2^n} \\ &= \alpha + \beta, \end{aligned}$$

for arbitrary  $\rho(x) < \alpha$ ,  $\rho(y) < \beta$ . The proof is complete. □

**Proof.** [Proof of theorem 3.5]

Fix  $c \in \Omega_A \cap Z(A)$ . Define  $f : \mathbb{C} \rightarrow \Omega_A$  with  $f(\lambda) = g(\lambda c)$ .  $f$  is holomorphic on

$$\{\lambda \in \mathbb{C} : |\lambda| < \frac{1}{\rho(c)}\}.$$

Then  $g$  has Taylor expansion about 0 and we have

$$g(\lambda c) = \lambda c + \sum_{j=2}^{\infty} \lambda^j a_j \quad (|\lambda| < \frac{1}{\rho(c)}).$$

Now we have to prove that  $a_j = 0$ , for all  $j$ ; if not the case, suppose for contradiction, that there is some  $j$  with  $a_j \neq 0$  and let  $k$  be the smallest integer such that  $a_k \neq 0$ . Take  $q \in A$  with  $\rho(q) = 0$  and let  $n \geq 1$ . Then, writing  $g^n$  for the  $n$ -fold composition  $g \circ \dots \circ g$ , we have

$$g^n(\lambda c + \lambda^k nq) = \lambda c + \lambda^k n(ak + q) + O(\lambda^{k+1}) \quad (\lambda \rightarrow 0).$$

Now as  $c$  and  $q$  commute, it follows that  $\rho(\lambda c + \lambda^k nq) \leq \rho(\lambda c) + \rho(\lambda^k nq) = |\lambda|\rho(c)$  (lemma 3.4), and so we can define a holomorphic function  $h : \{0 < |\lambda| < 1/\rho(c)\} \rightarrow \mathcal{A}$  by

$$h(\lambda) = \frac{g^n(\lambda c + \lambda^k nq) - \lambda c}{n\lambda^k} \quad (0 < |\lambda| < 1/\rho(c)),$$

isolated singularity at  $\lambda = 0$  can be removed by setting  $h(0) = a_k + q$ . By Vesentini's theorem (theorem 3.4.7, [5]), the composition  $\rho \circ h$  is a subharmonic function on  $\{0 < |\lambda| < 1/\rho(c)\}$ , and so by the maximum principle

$$\rho(h(0)) \leq \max_{|\lambda|=1} \rho(h(\lambda)).$$

Making use of lemma 3.4 again to estimate the right-hand side, it follows that

$$\rho(a_k + q) \leq 2/n.$$

As this is true for each  $n$ , we can let  $n \rightarrow \infty$  deduce that  $\rho(a_k + q) = 0$ . And as this holds for each  $q \in \mathcal{A}$  with  $\rho(q) = 0$ , Zemánek's characterization of the radical (theorem 3.5), implies that  $a_k$  belongs to the radical of  $\mathcal{A}$ , which is zero since  $\mathcal{A}$  is semi-simple. Thus  $a_k = 0$ , and we have arrived at a contradiction. We conclude that indeed  $a_j = 0$  for all  $j \geq 2$ , and hence from (1) that  $g(c) = c$ .  $\square$

In theorem 3.5 the property of sub-multiplicativity of *FLM* algebras is essential but in the next theorem we do not need it.

**Theorem 3.7** *Let  $\mathcal{A}$  be a semi-simple complete metrizable FLM algebra. Given  $a \in \Omega_{\mathcal{A}} \setminus Z_{\mathcal{A}}$ , then there exists a holomorphic map  $g : \Omega_{\mathcal{A}} \rightarrow \Omega_{\mathcal{A}}$  satisfying  $g(0) = 0$  and  $g'(0) = I$  such that  $g(a) \neq a$ .*

**Proof.** Let  $a \in \Omega_{\mathcal{A}} \setminus Z_{\mathcal{A}}$ . Then there exists  $u \in \mathcal{A}$ , such that  $au \neq ua$ . Suppose that  $d_{\mathcal{A}}(0, u) < 1$ , where  $d_{\mathcal{A}}$  is a meter on  $\mathcal{A}$ . Then  $v := \log(e - u)$  satisfies  $e^{-v}ae^v \neq a$ . Define  $g : \Omega_{\mathcal{A}} \rightarrow \Omega_{\mathcal{A}}$  by

$$g(x) = e^{-\frac{xv}{a}} x e^{\frac{xv}{a}} \quad (x \in \Omega_{\mathcal{A}}).$$

Then  $g$  is a holomorphic function,  $g(0) = 0$  and  $g'(0) = I$ , but  $g(a) = e^{-v}ae^v \neq a$ .  $\square$

By combination of theorems 3.5 and 3.7, we have the following theorem.

**Theorem 3.8** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be semi-simple complete metrizable FLM algebras with sub-multiplicatively meter. If  $f : \Omega_{\mathcal{A}} \rightarrow \Omega_{\mathcal{B}}$  is a biholomorphic map, then  $f(\Omega_{\mathcal{A}} \cap Z_{\mathcal{A}}) = \Omega_{\mathcal{B}} \cap Z_{\mathcal{B}}$ .*

**Proof.** Let  $c \in \Omega_{\mathcal{A}} \cap Z_{\mathcal{A}}$  be an arbitrary. Without loss of generality suppose that  $c \neq 0$ . Then from assumption about  $f$ ,  $f(c) \neq f(0)$ . Take  $b \in \mathcal{B}$ , and define  $h : \Omega_{\mathcal{B}} \rightarrow \Omega_{\mathcal{B}}$  by

$$h(y) = e^{-\left(\frac{y-f(0)}{f(c)-f(0)}\right)^2 b} y e^{\left(\frac{y-f(0)}{f(c)-f(0)}\right)^2 b} \quad (y \in \Omega_{\mathcal{B}}).$$

By above definition  $h$  is a holomorphic function,  $h(f(0)) = f(0)$  and  $h'(f(0)) = I$ . Now set  $g = f^{-1} \circ h \circ f$ , then  $g$  is a holomorphic function from  $\Omega_{\mathcal{A}}$  into  $\Omega_{\mathcal{A}}$ , such that  $g(0) = 0$  and  $g'(0) = I$ . Therefore  $g(c) = c$  (theorem 2.1), and from definition of  $g$ , we have  $h(f(c)) = f(c)$ . By proof of theorem 2.2,  $f(c) \in Z_{\mathcal{B}}$ . In case  $c = 0$ , by continuity of  $f$ , the proof remains true. Hence  $f(\Omega_{\mathcal{A}} \cap Z_{\mathcal{A}}) \subset \Omega_{\mathcal{B}} \cap Z_{\mathcal{B}}$ .

Let  $c \in \Omega_{\mathcal{B}} \cap Z_{\mathcal{B}}$ , with applying the same argument to  $f^{-1}$ , we have  $c \in f(\Omega_{\mathcal{A}} \cap Z_{\mathcal{A}})$ . Therefore  $\Omega_{\mathcal{B}} \cap Z_{\mathcal{B}} \subset f(\Omega_{\mathcal{A}} \cap Z_{\mathcal{A}})$ .  $\square$

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