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# Hypercyclic tuples of the adjoint of the weighted composition operators

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**Abstract**

An $n$-tuple of commuting operators, $(T_1, T_2, ..., T_n)$ on a Hilbert space $\mathcal{H}$ is said to be hypercyclic, if there exists a vector $x \in \mathcal{H}$ such that the set $\{{T_1}^{k_1}{T_2}^{k_2}...{T_n}^{k_n}x : k_i \geq 0, i = 1, 2, ...n\}$ is dense in $\mathcal{H}$. In this paper, we give sufficient conditions under which the adjoint of an $n$-tuple of a weighted composition operator on a Hilbert space of analytic functions is hypercyclic.

**Key Words:** Hypercyclicity, tuples, weighted composition operators

## 1. Introduction

An $n$-tuple of operators is a finite sequence of length $n$ of commuting continuous linear operators $T_1, T_2, ..., T_n$ acting on a locally convex topological vector space $X$. Hypercyclic tuples of operators were introduced in [5, 7] and [12]. A tuple $(T_1, T_2, ..., T_n)$ is said to be hypercyclic, if there exists a vector $x \in X$ such that the set $\{{T_1}^{k_1}{T_2}^{k_2}...{T_n}^{k_n}x : k_i \geq 0, i = 1, 2, ..., n\}$ is dense in $X$. This definition generalizes the hypercyclicity of a single operator to a tuple of operators. Like Feldman in [7], we denote the semigroup generated by a tuple $T = (T_1, ..., T_n)$ by $\mathcal{F}_T = \{{T_1}^{k_1}{T_2}^{k_2}...{T_n}^{k_n} : k_i \geq 0, i = 1, 2, ..., n\}$ and the orbit of $x$ under the tuple $T$ by $orb(T, x) = \{Sx : S \in \mathcal{F}_T\}$.

Consider a Hilbert space $\mathcal{H}$ of functions analytic on the open unit disc $\mathbb{D}$ such that for each $\lambda \in \mathbb{D}$ the linear functional $e_\lambda$ of evaluation at $\lambda$ is bounded on $\mathcal{H}$. Moreover, the constant function 1 and the identity function $f(z) = z$ are in $\mathcal{H}$. The weighted Hardy space is the well-known example of such $\mathcal{H}$. Let $(\beta(n))_n$ be a sequence of positive numbers with $\beta(0) = 1$. The weighted Hardy space $H^2(\beta)$ is defined as the space of analytic functions $f = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ on $\mathbb{D}$ satisfying

$$\|f\|_\beta^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2|\beta(n)|^2 < \infty.$$

The classical Hardy space, the Bergman space and the Dirichlet space are weighted Hardy spaces with $\beta(n) = 1$, $\beta(n) = (n+1)^{-\frac{1}{2}}$ and $\beta(n) = (n+1)^{\frac{1}{2}}$, respectively. Reference [4] is a good source on properties of

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weighted Hardy spaces. The continuity of point evaluations along with the Riesz representation theorem imply that for each $\lambda \in \mathbb{D}$ there is a unique function $K_\lambda \in \mathcal{H}$ such that $f(\lambda) = \langle f, K_\lambda \rangle$, $f \in \mathcal{H}$. The function $K_\lambda$ is the reproducing kernel for the point $\lambda$.

A complex-valued function $\omega$ on $\mathbb{D}$ for which $\omega f \in \mathcal{H}$ for every $f \in \mathcal{H}$ is called a multiplier of $\mathcal{H}$ and the collection of all multipliers is denoted by $M(\mathcal{H})$. Each multiplier $\omega$ of $\mathcal{H}$ determines a multiplication operator $M_\omega$ on $\mathcal{H}$ by $M_\omega f = \omega f$, $f \in \mathcal{H}$. Each multiplier is a bounded analytic function on $\mathbb{D}$. In fact, since the constant functions are in $\mathcal{H}$, every function in $M(\mathcal{H})$ is analytic on $\mathbb{D}$. Moreover, if $\lambda \in \mathbb{D}$ then

$$|\omega(\lambda)K_\lambda(\lambda)| = |\langle M_\omega K_\lambda, K_\lambda \rangle| \leq \|M_\omega\| \|K_\lambda\|^2.$$

This implies that $|\omega(\lambda)| \leq \|M_\omega\|$ for every $\lambda \in \mathbb{D}$ and so $\omega \in H^\infty$. If $\omega \in M(\mathcal{H})$ and $\varphi$ is a mapping from $\mathbb{D}$ into $\mathbb{D}$ such that $f \circ \varphi$ is in $\mathcal{H}$ for every $f \in \mathcal{H}$, then an application of the closed graph theorem shows that the weighted composition operator $C_{\omega,\varphi}$ defined by $C_{\omega,\varphi}(f)(z) = M_\omega C_\varphi(f)(z) = \omega(z)f(\varphi(z))$ is bounded. From now on, we assume that $\omega$ and $\varphi$ satisfy these properties. For a positive integer $n$, the $n$th iterate of $\varphi$, denoted by $\varphi_n$, is the function obtained by composing $\varphi$ with itself $n$ times; also, $\varphi_0$ is defined to be the identity function. Moreover, when $\varphi$ is invertible, we define the iterates $\varphi_{-n} = \varphi^{-1} \circ \varphi^{-1} \circ ... \circ \varphi^{-1}$ ($n$ times). Also, $C^*_{w,\,\varphi} K_\lambda = \overline{w(\lambda)} K_{\varphi(\lambda)}$ for every $\lambda$ in $\mathbb{D}$ which implies that $C^{*n}_{w,\,\varphi} K_\lambda = \prod_{j=0}^{n-1} \overline{w(\varphi_j(\lambda))} K_{\varphi_n(\lambda)}$. Moreover, $C^n_{w,\,\varphi}(f) = (\prod_{k=0}^{n-1} w \circ \varphi_k) f \circ \varphi_n$ for every $f \in \mathcal{H}$. The properties of composition and weighted composition operators on various spaces of analytic functions have been investigated by many authors; see monographs [4, 15] and, for example, the following recent papers [9, 10, 11] and references therein.

In this paper, we give sufficient conditions for the $n$-tuple of the adjoint of a weighted composition operator to be hypercyclic. Hypercyclicity of operators have been widely studied. It was shown by Rolewicz [13] that twice the backward shift on the space $\ell^2(\mathbb{N})$ is hypercyclic. Many natural operators are hypercyclic. For example, certain operators in the classes of weighted shifts [14], composition operators [2], and the adjoint of subnormal, hyponormal and multiplication operators [6, 3], and the weighted composition operators and their adjoint operators [16, 17, 11] are hypercyclic. A good source on this topic is [1].

**Proposition 1** *([7], Proposition 2.4) Suppose that $T = (T_1, ..., T_n)$ is a hypercyclic tuple on a separable Banach space $X$. Then every non-zero orbit of $T^* = ({T_1}^*, ..., {T_n}^*)$ is unbounded.*

**Proposition 2** *If $\varphi_1$ and $\varphi_2$ are analytic maps of the disc into itself then $(C^*_{\varphi_1}, C^*_{\varphi_2})$ is not hypercyclic on $\mathcal{H}$.*

**Proof.** Since ${C_{\varphi_1}}^{k_1} {C_{\varphi_2}}^{k_2} 1 = 1$, then the orbit of 1 under $(C_{\varphi_1}, C_{\varphi_2})$ is bounded. Thus, using Proposition 1, the result follows. □

## 2. Tuples of weighted composition operators

We begin this section with a lemma that gives a necessary and sufficient condition for two weighted composition operators to commute.

**Lemma 1** *If $\omega_1(z)$ and $\omega_2(z)$ are nonzero for all $z \in \mathbb{D}$, then $C_{\omega_1,\varphi_1}$ and $C_{\omega_2,\varphi_2}$ commute if and only if $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$ and $\omega_1 \cdot (\omega_2 \circ \varphi_1) = \omega_2 \cdot (\omega_1 \circ \varphi_2)$.*

**Proof.** Suppose that $C_{\omega_1,\varphi_1}$ and $C_{\omega_2,\varphi_2}$ commute. Then

$$\omega_1 \cdot (\omega_2 \circ \varphi_1) = C_{\omega_1,\varphi_1} C_{\omega_2,\varphi_2} 1 = C_{\omega_2,\varphi_2} C_{\omega_1,\varphi_1} 1 = \omega_2 \cdot (\omega_1 \circ \varphi_2).$$

Moreover, since

$$\begin{aligned} (\omega_1 \cdot (\omega_2 \circ \varphi_1) \cdot (\varphi_2 \circ \varphi_1))(z) &= (C_{\omega_1,\varphi_1} C_{\omega_2,\varphi_2} g)(z) \\ &= (C_{\omega_2,\varphi_2} C_{\omega_1,\varphi_1} g)(z) = (\omega_2 \cdot (\omega_1 \circ \varphi_2) \cdot (\varphi_1 \circ \varphi_2))(z), \end{aligned}$$

where $g(z) = z$ we have $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$. For the converse, take $f \in \mathcal{H}$. Then

$$\begin{aligned} C_{\omega_1,\varphi_1} C_{\omega_2,\varphi_2} f &= \omega_1 \cdot (\omega_2 \cdot (f \circ \varphi_2)) \circ \varphi_1 \\ &= \omega_1 \cdot (\omega_2 \circ \varphi_1) \cdot f \circ \varphi_2 \circ \varphi_1 \\ &= \omega_2 \cdot (\omega_1 \circ \varphi_2) \cdot f \circ \varphi_1 \circ \varphi_2 \\ &= \omega_2 \cdot (\omega_1 \cdot (f \circ \varphi_1)) \circ \varphi_2 \\ &= C_{\omega_2,\varphi_2} C_{\omega_1,\varphi_1} f. \end{aligned}$$

□

**Proposition 3** *If $T = (C_{\omega_1,\varphi_1}, C_{\omega_2,\varphi_2})$ is a hypercyclic tuple then*

*(1) $\omega_1(z)$ and $\omega_2(z)$ are both nonzero for every $z \in \mathbb{D}$.*

*(2) $(\varphi_1, \varphi_2)$ is univalent.*

**Proof.** (1) If $\omega_1(z) = 0$ for some $z$, then $C^*_{\omega_1,\varphi_1} K_z = \overline{\omega_1(z)} K_{\varphi_1(z)} = 0$. Thus,

$$C^{*n_j}_{\omega_2,\varphi_2} C^{*m_j}_{\omega_1,\varphi_1} K_z = 0$$

for every $m_j \geq 0$ and $n_j \geq 0$ which implies that an orbit of $T^* = (C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ is bounded. Therefore, by Proposition 1, $T = (C_{\omega_1,\varphi_1}, C_{\omega_2,\varphi_2})$ is not hypercyclic. Hence, $\omega_1(z) \neq 0$ for every $z \in \mathbb{D}$. Similarly, $\omega_2(z) \neq 0$ for every $z \in \mathbb{D}$.

(2) Let $f$ be a hypercyclic vector for $T$. Suppose that $(\varphi_1(z), \varphi_2(z)) = (\varphi_1(\lambda), \varphi_2(\lambda))$. Then

$$
\begin{aligned}
& \langle C^m_{\omega_2,\varphi_2} C^n_{\omega_1,\varphi_1} f, K_z\rangle = \langle C^n_{\omega_1,\varphi_1} f, C^{*m}_{\omega_2,\varphi_2} K_z\rangle \\
= \;& \langle \prod_{i=0}^{n-1} \omega_1 \circ (\varphi_1)_i \cdot f \circ (\varphi_1)_n, [\prod_{i=0}^{m-1} \overline{\omega_2 \circ (\varphi_2)_i(z)}] K_{(\varphi_2)_m(z)}\rangle \\
= \;& \prod_{i=0}^{m-1} \omega_2((\varphi_2)_i(z)) \prod_{i=0}^{n-1} (\omega_1 \circ (\varphi_1)_i)((\varphi_2)_m(z)) \cdot (f \circ (\varphi_1)_n)((\varphi_2)_m(z)) \\
= \;& \omega_1((\varphi_2)_m(z))\omega_2(z) \prod_{i=1}^{m-1} \omega_2((\varphi_2)_i(\lambda)) \prod_{i=1}^{n-1} (\omega_1 \circ (\varphi_1)_i)((\varphi_2)_m(\lambda)).(f \circ (\varphi_1)_n)((\varphi_2)_m(\lambda)) \\
= \;& \frac{\omega_1((\varphi_2)_m(z))\omega_2(z)}{\omega_1((\varphi_2)_m(\lambda))\omega_2(\lambda)} \prod_{i=0}^{m-1} \omega_2((\varphi_2)_i(\lambda)) \prod_{i=0}^{n-1} (\omega_1 \circ (\varphi_1)_i)((\varphi_2)_m(\lambda)) \cdot (f \circ (\varphi_1)_n)((\varphi_2)_m(\lambda)) \\
= \;& \frac{\omega_1((\varphi_2)_m(z))\omega_2(z)}{\omega_1((\varphi_2)_m(\lambda))\omega_2(\lambda)} \langle C^m_{\omega_2,\varphi_2} C^n_{\omega_1,\varphi_1} f, K_\lambda\rangle,
\end{aligned}
$$

where $m$ and $n$ are non-negative integers so that $m^2 + n^2 \neq 0$. Thus,

$$\langle g, K_z\rangle = \frac{\omega_1((\varphi_2)_m(z))\omega_2(z)}{\omega_1((\varphi_2)_m(\lambda))\omega_2(\lambda)} \langle g, K_\lambda\rangle$$

for every $g \in \mathcal{H}$ . Set $g \equiv 1$. Therefore,

$$\langle h, K_z\rangle = \langle h, K_\lambda\rangle$$

for every $h \in \mathcal{H}$. Now, taking $h(s) = s$, we get $z = \lambda$. □

We remark that it follows from the Denjoy-Wolff theorem [4] that if $\varphi$ is a self map of $\mathbb{D}$ and has a fixed point in $\mathbb{D}$ then it is unique.

**Proposition 4** *If $T = (C_{\omega_1,\varphi_1}, C_{\omega_2,\varphi_2})$ is a hypercyclic tuple and $a$ is an interior fixed point of $\varphi_1$ or $\varphi_2$, then $|\omega_1(a)| > 1$ or $|\omega_2(a)| > 1$.*

**Proof.** Suppose that $\varphi_1(a) = a$. Then $\varphi_1(\varphi_2(a)) = \varphi_2(\varphi_1(a)) = \varphi_2(a)$, which implies that $\varphi_2(a) = a$. So

$$C^{*m}_{\omega_2,\varphi_2} C^{*n}_{\omega_1,\varphi_1} K_a = (\overline{\omega_1(a)})^n C^{*m}_{\omega_2,\varphi_2} K_a = (\overline{\omega_1(a)})^n (\overline{\omega_2(a)})^m K_a$$

Now, if $|\omega_1(a)| \leq 1$ and $|\omega_2(a)| \leq 1$, then $orb(T^*, K_a)$ is bounded. Thus, by Proposition 1, $T$ is not hypercyclic, which is a contradiction. □

**Corollary 1** *If $\varphi_1$ or $\varphi_2$ has an interior fixed point then $(C_{\varphi_1}, C_{\varphi_2})$ is not hypercyclic.*

**Proof.** Put $\omega_1(z) \equiv 1$ and $\omega_2(z) \equiv 1$ in Proposition 4. □

An argument similar to the proof of Proposition 2.5 of [7] shows the next proposition.

**Proposition 5** *(Hypercyclicity Criterion) Suppose that* $(T_1, T_2, ..., T_n)$ *is an* $n$*-tuple of operators on a separable Banach space* $Z$*. Suppose further that there exist* $n$ *strictly increasing sequences of positive integers* $\{k_{1j}\}_j$*,* $\{k_{2j}\}_j$ *,...., and* $\{k_{nj}\}_j$*, dense sets* $X$ *and* $Y$ *in* $Z$ *and functions* $S_j : Y \longrightarrow Z$ *such that*

*(1) For each* $x \in X$*,* $T_1^{k_{1j}} T_2^{k_{2j}} ... T_n^{k_{nj}} x \longrightarrow 0$ *as* $j \longrightarrow \infty$*;*

*(2) for each* $y \in Y$*,* $S_j y \longrightarrow 0$ *as* $j \longrightarrow \infty$*;*

*(3) for each* $y \in Y$*,* $T_1^{k_{1j}} T_2^{k_{2j}} ... T_n^{k_{nj}} S_j y \longrightarrow y$ *as* $j \longrightarrow \infty$*.*

*Then* $(T_1, T_2, ..., T_n)$ *is hypercyclic.*

It follows from Lemma 1 that if

$$\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1, \;\; \omega_1 = \omega_1 \circ \varphi_2 \;\; and \;\; \omega_2 = \omega_2 \circ \varphi_1, \qquad (*)$$

then $C_{\omega_1, \varphi_1}$ and $C_{\omega_2, \varphi_2}$ commute. We give some examples of such functions. Suppose that $\varphi_r(z) = e^{ir\pi} z$ where $r = \frac{p}{q}$, $p$ and $q$ are integers so that $(p, q) = 1$. Define $\omega_r(z) = \sum_{n=0}^{\infty} a_n z^n$, where

$$a_n = \begin{cases} \frac{1}{2^n} & (n = \frac{2kq}{p} \; for \; some \; k \in \mathbb{Z}), \\ 0 & \text{otherwise}; \end{cases}$$

then $\omega_r \in H^\infty$. Moreover, $\omega_r \circ \varphi_r(z) = \omega_r(z)$ for all $z \in \mathbb{D}$ and $\varphi_r \circ \varphi_s = \varphi_s \circ \varphi_r$.

**Theorem 1** *Let* $\varphi_1$ *and* $\varphi_2$ *be two disc automorphism such that* $(*)$ *holds and*

$$\sup_{n \in \mathbb{Z}} \|K_{(\varphi_2)_n \circ (\varphi_1)_n (z)}\| < \infty$$

*for every* $z \in \mathbb{D}$*. If the sets*

$$A = \{z \in \mathbb{D} : \lim_{n \longrightarrow +\infty} \prod_{j=0}^{n-1} (\omega_1 \circ (\varphi_1)_j)(z) \cdot (\omega_2 \circ (\varphi_2)_j)(z) = 0\}$$

*and*

$$B = \{z \in \mathbb{D} : \lim_{n \longrightarrow +\infty} \prod_{j=0}^{n} [(\omega_1 \circ (\varphi_1)_{-j})(z) \cdot (\omega_2 \circ (\varphi_2)_{-j})(z)]^{-1} = 0\}$$

*have limit points in* $\mathbb{D}$*, then* $(C^*_{\omega_1, \varphi_1}, C^*_{\omega_2, \varphi_2})$ *is hypercyclic.*

**Proof.** We will show that the hypercyclicity criterion holds. To see this take $T_i = C^*_{\omega_i, \varphi_i}$ for $i = 1, 2$. Since

$$T_i^n K_z = [\prod_{j=0}^{n-1} \overline{(\omega_i \circ (\varphi_i)_j)(z)}] K_{(\varphi_i)_n (z)}$$

for $i = 1, 2$ and $n \geq 1$, we have

$$T_2^n T_1^n K_z = [\prod_{j=0}^{n-1} \overline{\omega_1 \circ (\varphi_1)_j (z)}][\prod_{j=0}^{n-1} \overline{\omega_2 \circ (\varphi_2)_j \circ (\varphi_1)_n (z)}] K_{(\varphi_2)_n \circ (\varphi_1)_n (z)}$$

for every $n \geq 1$.

Put $S_A = \text{span}\{K_z : z \in A\}$ and $S_B = \text{span}\{K_z : z \in B\}$. Therefore, $\overline{S_A} = \overline{S_B} = \mathcal{H}$ thanks to $(S_A)^{\perp} = (S_B)^{\perp} = (0)$.

Since $\sup_{n\in\mathbb{Z}} \|K_{(\varphi_2)_n \circ (\varphi_1)_n(z)}\| < \infty$, $\omega_2 \circ \varphi_1 = \omega_2$ and $\varphi_1 \circ \varphi_2 = \varphi_2 \circ \varphi_1$, we conclude that for every $f \in S_A$

$$T_2^n T_1^n f \longrightarrow 0$$

as $n \longrightarrow \infty$.

On the other hand, $\omega_2 \circ \varphi_1^{-1} = \omega_2$, $\omega_1 \circ \varphi_2^{-1} = \omega_1$ and $\varphi_1^{-1} \circ \varphi_2^{-1} = \varphi_2^{-1} \circ \varphi_1^{-1}$; therefore, if $z \in B$ then $\varphi_1^{-1} \circ \varphi_2^{-1}(z) \in B$. So we can define

$$S : \{K_z : z \in B\} \longrightarrow S_B$$

by

$$SK_z = \overline{(\omega_1((\varphi_1)^{-1}(z)) \cdot \omega_2((\varphi_2)^{-1}(z)))}^{-1} K_{\varphi_1^{-1} \circ \varphi_2^{-1}(z)}$$

and extend it linearly to $S_B$. Now, $T_2 T_1 S K_z = K_z$, and so $T_2^n T_1^n S^n$ is the identity on $S_B$ for every $n \geq 0$. Moreover, it is easily seen that

$$S^n K_z = \prod_{j=1}^{n} \overline{[(\omega_1 \circ (\varphi_1)_{-j})(z) \cdot (\omega_2 \circ (\varphi_2)_{-j})(z)]}^{-1} K_{((\varphi_1)_{-n} \circ (\varphi_2)_{-n})(z)}$$

for every $n \geq 1$; thus, $S^n$ converges pointwise to zero on the dense subset $S_B$. Hence, hypercyclicity criterion implies that $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ is hypercyclic. □

**Corollary 2** *If the sets*

$$\{z \in \mathbb{D} : \lim_{n\longrightarrow+\infty} (\omega_1(z)\omega_2(z))^n = 0\}$$

*and*

$$\{z \in \mathbb{D} : \lim_{n\longrightarrow+\infty} \frac{1}{(\omega_1(z)\omega_2(z))^n} = 0\}$$

*have limit points in* $\mathbb{D}$ *then* $(M^*_{\omega_1}, M^*_{\omega_2})$ *is hypercyclic on* $\mathcal{H}$.

**Proof.** Put $\varphi_1(z) = \varphi_2(z) = z$ and apply the preceding theorem. □

**Example 1** *Let* $\omega_1(z) = z$ *and* $\omega_2(z) = z + 5$. *It is easily seen that*

$$\{x : 0 \leq x \leq \frac{-5 + \sqrt{29}}{2}\} \subseteq \{z \in \mathbb{D} : \lim_{n\longrightarrow+\infty} (z(z+5))^n = 0\}$$

*and*

$$\{x : -1 < x < \frac{-5 + \sqrt{21}}{2}\} \subseteq \{z \in \mathbb{D} : \lim_{n\longrightarrow+\infty} \frac{1}{(z(z+5))^n} = 0\}$$

*hence* $(M^*_{\omega_1}, M^*_{\omega_2})$ *is hypercyclic on* $\mathcal{H}$. *Also, since* $ran\omega_i \cap \partial\mathbb{D} = \emptyset$ *for* $i = 1, 2$, *the operators* $M^*_{\omega_i}$, $i = 1, 2$ *are not hypercyclic on* $\mathcal{H}$ *(see [[8], Theorem 4.9]).*

Recall that if $\varphi$ is a hyperbolic automorphism then by the Denjoy-Wolff Theorem, one of its fixed point is the Denjoy-Wolff point of $\varphi$ and the other is repulsive; i.e., it is the Denjoy-Wolff point of $\varphi^{-1}$. Furthermore, the angular derivative of $\varphi$ at the Denjoy-Wolff point $a$, $\varphi'(a)$ is less than 1 (see [[2], Page 24]).

**Corollary 3** *Suppose that $\{K_\lambda : \lambda \in \mathbb{D}\}$ is bounded in which $K_\lambda$ is the reproducing kernel at $\lambda$, and $\varphi_1$ and $\varphi_2$ are two hyperbolic automorphisms with the Denjoy-Wolff points $a_1$ and $a_2$ and repulsive fixed points $b_1$ and $b_2$, respectively. Moreover, suppose that $(*)$ holds, $\omega_1$ and $\omega_2$ have non-tangential limits $\omega_1(a_1)$ at $a_1$, $\omega_2(a_2)$ at $a_2$, $\omega_1(b_1)$ at $b_1$ and $\omega_2(b_2)$ at $b_2$. If $|\omega_1(a_1)\omega_2(a_2)| < 1 < |\omega_1(b_1)\omega_2(b_2)|$ then $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ is hypercyclic.*

**Proof.** Since $\varphi_1'(a_1) < 1$ for every $z \in \mathbb{D}$, there is a non-tangential approach region containing all iterates $(\varphi_1)_n(z)$ (see [[4], Lemma 2.66]), so $\lim_{n\to\infty} \omega_1((\varphi_1)_n(z)) = \omega_1(a_1)$. Similarly, $\lim_{n\to\infty} \omega_2((\varphi_2)_n(z)) = \omega_2(a_2)$. Thus,

$$\lim_{n\to\infty} (\omega_1 \circ (\varphi_1)_n(z)) \cdot (\omega_2 \circ (\varphi_2)_n(z)) = \omega_1(a_1)\omega_2(a_2),$$

which implies that

$$\sum_{j=0}^{\infty} (1 - |(\omega_1 \circ (\varphi_1)_j(z)) \cdot (\omega_2 \circ (\varphi_2)_j(z))|) = \infty.$$

Therefore,

$$\lim_{n\to\infty} \prod_{j=0}^{n-1} |(\omega_1 \circ (\varphi_1)_j(z)) \cdot (\omega_2 \circ (\varphi_2)_j(z))| = 0.$$

Thus, the set $A$ in Theorem 1, has a limit point in $\mathbb{D}$. Similarly, since $((\varphi_1)_{-1})'(b_1) < 1$ and $((\varphi_2)_{-1})'(b_2) < 1$,

$$\sum_{j=1}^{\infty} (1 - |(\omega_1 \circ (\varphi_1)_{-j}(z)) \cdot (\omega_2 \circ (\varphi_2)_{-j}(z))|^{-1}) = \infty,$$

and so the set $B$ in Theorem 1, has a limit point in $\mathbb{D}$. Hence, the proof is completed by applying Theorem 1. □

Note that if $\varphi_1$ and $\varphi_2$ are two elliptic automorphisms so that $C_{\varphi_1}$ and $C_{\varphi_2}$ commute then they have the same interior fixed points. For $\alpha \in \mathbb{D}$, consider an automorphism of $\mathbb{D}$ defined by $\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}$, $(z \in \mathbb{D})$. Some spaces such as the Hardy, Bergman and Dirichlet spaces contain $\psi_\alpha$ for every $\alpha \in \mathbb{D}$.

**Theorem 2** *Suppose that $\mathcal{H}$ contains $\psi_\alpha$ for every $\alpha \in \mathbb{D}$. Let $\varphi_1$ and $\varphi_2$ be two elliptic automorphisms with an interior fixed point $a$ such that $(*)$ holds. If the sets $A$ and $B$ in Theorem 1 have limits in $\mathbb{D}$, then $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ is hypercyclic.*

**Proof.** First, assume that $a = 0$. Then $\varphi_1(z) = e^{i\theta_1} z$ and $\varphi_2(z) = e^{i\theta_2} z$ for some $\theta_1, \theta_2 \in [0, 2\pi]$. Thus, for $z \in \mathbb{D}$, $\{(\varphi_2)_n \circ (\varphi_1)_n(z) : n \in \mathbb{Z}\} \subseteq z\partial\mathbb{D}$. But $z\partial\mathbb{D}$ is a compact subset of $\mathbb{D}$ and so for $f \in \mathcal{H}$ the continuity of $f$ implies that

$$(f((\varphi_2)_n \circ (\varphi_1)_n(z)))_{n\in\mathbb{Z}}$$

is a bounded sequence; this along with the principle of uniform boundedness shows that for every $z \in \mathbb{D}$

$$(K_{(\varphi_2)_n \circ (\varphi_1)_n(z)})_{n\in\mathbb{Z}}$$

is also bounded. Hence, by applying Theorem 1, $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ is hypercyclic.

Now, for the general case put $\mathcal{K} = \{f \circ \psi_a^{-1} : f \in \mathcal{H}\}$ endowed with inner product

$$\langle f, g\rangle_{\mathcal{K}} = \langle f \circ \psi_a, g \circ \psi_a\rangle_{\mathcal{H}}.$$

Then $\mathcal{K}$ is a Hilbert space of analytic functions on $\mathbb{D}$ and $C_{\psi_a} : \mathcal{K} \longrightarrow \mathcal{H}$ defined by $C_{\psi_a} f = f \circ \psi_a$ is a linear isometric isomorphism. Furthermore, $\tilde{\varphi_1} = \psi_a \circ \varphi_1 \circ \psi_a^{-1}$ and $\tilde{\varphi_2} = \psi_a \circ \varphi_2 \circ \psi_a^{-1}$ are automorphisms with the interior fixed point zero, and $\tilde{\omega_1} = \omega_1 \circ \psi_a^{-1}$ and $\tilde{\omega_2} = \omega_2 \circ \psi_a^{-1}$ are in $M(\mathcal{K})$. Finally, since by the first step $(C^*_{\tilde{\omega_1},\tilde{\varphi_1}}, C^*_{\tilde{\omega_2},\tilde{\varphi_2}})$ is hypercyclic and $C_{\tilde{\omega_i},\tilde{\varphi_i}} = C^{-1}_{\psi_a} \circ C_{\omega_i,\varphi_i} \circ C_{\psi_a}$ for $i = 1, 2$, one can see that $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ is also hypercyclic. □

**Example 2** *Consider* $\varphi_1(z) = iz$, $\varphi_2(z) = -iz$, $\omega_1(z) = z^4$ *and* $\omega_2(z) = z^4 + 3$. *Then the sets* $A$ *and* $B$ *mentioned in Theorem 1 are*

$$A = \{z \in \mathbb{D} : \lim_{n \longrightarrow +\infty} z^{4n}(z^4 + 3)^n = 0\}$$

*and*

$$B = \{z \in \mathbb{D} : \lim_{n \longrightarrow +\infty} \frac{1}{z^{4n}(z^4 + 3)^n} = 0\}.$$

*It is easily seen that* $[0, \frac{1}{2}) \subseteq A$ *and* $(\frac{1}{\sqrt[4]{2}}, 1) \subseteq B$. *Hence,* $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ *is hypercyclic.*

**Proposition 6** *Let* $\varphi_1$ *and* $\varphi_2$ *be two elliptic automorphisms with an interior fixed point* $a$ *and* $\omega_1, \omega_2 : \mathbb{D} \longrightarrow \mathbb{C}$ *satisfy the inequality*

$$|\omega_1(a)\omega_2(a)| < 1 < \liminf_{|z| \longrightarrow 1^-} |\omega_1(z)\omega_2(z)|.$$

*If* $(*)$ *holds then* $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ *is a hypercyclic pair.*

**Proof.** As it is seen in the preceding theorem, we can assume that $a = 0$ and $\varphi_1(z) = e^{i\theta_1} z$ and $\varphi_2(z) = e^{i\theta_2} z$ for some $\theta_1, \theta_2 \in [0, 2\pi]$. Therefore, $(\varphi_2)_n \circ (\varphi_1)_n(z) = e^{in\theta_2} e^{in\theta_1} z$ for every $n \in \mathbb{Z}$. This along with the principle of uniform boundedness implies that $\sup_{n\in\mathbb{Z}} \|K_{(\varphi_2)_n \circ (\varphi_1)_n(z)}\| < \infty$ for every $z \in \mathbb{D}$. On the other hand, since $|\omega_1(0)\omega_2(0)| < 1$ there exist a constant $\lambda_1$ and a positive number $\delta_1$ such that $|\omega_1(z)\omega_2(z)| < \lambda_1 < 1$ whenever $|z| < \delta_1$. This, in turn, implies that if $|z| < \delta_1$ then

$$|\prod_{j=0}^{n-1} \omega_1((\varphi_1)_j(z))\omega_2((\varphi_2)_j(z))| < \lambda_1^n \longrightarrow 0$$

as $n \longrightarrow +\infty$. Consequently, $\{z : |z| < \delta_1\}$ is a subset of the set $A$ in Theorem 1. Moreover, since $1 < \liminf_{|z| \longrightarrow 1^-} |\omega_1(z)\omega_2(z)|$ there exist a constant $\lambda_2$ and a positive number $\delta_2 < 1$ such that $|\omega_1(z)\omega_2(z)| > \lambda_2 > 1$ when $|z| > 1 - \delta_2$. Therefore, if $|z| > 1 - \delta_2$ then

$$\prod_{j=1}^{n} |(\omega_1 \circ (\varphi_1)_{-j})(z)(\omega_2 \circ (\varphi_2)_{-j})(z)|^{-1} < \frac{1}{\lambda_2^n} \longrightarrow 0$$

as $n \longrightarrow +\infty$. Thus, $\{z : |z| > 1-\delta_2\}$ is a subset of the set $B$ in Theorem 1. Hence $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ is hypercyclic. □

**Example 3** *Consider* $\varphi_1(z) = iz$, $\varphi_2(z) = e^{i\frac{\pi}{4}}z$, $\omega_1(z) = z^8$ *and* $\omega_2(z) = z^4 + c$ *where* $|c| > 2$. *Since*

$$|\omega_1(0)\omega_2(0)| < 1 < \liminf_{|z|\longrightarrow 1^-} |\omega_1(z)\omega_2(z)|$$

$(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$, *is a hypercyclic pair.*

**Proposition 7** *Suppose that there are positive integers* $m_1$ *and* $m_2$ *such that* $(\varphi_1)_{m_1}(z) = z$ *and* $(\varphi_2)_{m_2}(z) = z$ *for all* $z \in \mathbb{D}$. *If*

$$\partial\mathbb{D} \bigcap ran(\prod_{j=0}^{m_1m_2-1} (\omega_1 \circ (\varphi_1)_j \cdot \omega_2 \circ (\varphi_2)_j) \neq \emptyset,$$

*and* $(*)$ *holds then* $(C^*_{\omega_1,\varphi_1}, C^*_{\omega_2,\varphi_2})$ *is hypercyclic.*

**Proof.** We observe that $\{(\varphi_1)_n \circ (\varphi_2)_n(z) : \; n \in \mathbb{Z}\}$ is a subset of $\{(\varphi_1)_j \circ (\varphi_2)_k(z) : j = 0, \ldots, m_1 - 1, \; k = 0, \ldots, m_2 - 1\}$. So the sequence $(K_{(\varphi_1)_n \circ (\varphi_2)_n(z)})_{n\in\mathbb{Z}}$ is bounded for every $z$ in $\mathbb{D}$. Furthermore, since the function

$$(\prod_{j=0}^{m_1m_2-1} \omega_1 \circ (\varphi_1)_j \cdot \omega_2 \circ (\varphi_2)_j)(z)$$

is analytic on $\mathbb{D}$, the open mapping theorem implies that

$$U = \{z \in \mathbb{D} : |(\prod_{j=0}^{m_1m_2-1} \omega_1 \circ (\varphi_1)_j \cdot \omega_2 \circ (\varphi_2)_j)(z)| < 1\}$$

and

$$V = \{z \in \mathbb{D} : |(\prod_{j=0}^{m_1m_2-1} \omega_1 \circ (\varphi_1)_j \cdot \omega_2 \circ (\varphi_2)_j)(z)| > 1\}$$

are non-empty open sets. Fix $z \in U$, and let $P_n = (\prod_{j=0}^{n-1} \omega_1 \circ (\varphi_1)_j \cdot \omega_2 \circ (\varphi_2)_j)(z)$ for $n \in \mathbb{N}$. For $\varepsilon > 0$, since

$$Q_k = (\prod_{j=0}^{m_1m_2-1} \omega_1 \circ (\varphi_1)_j \cdot \omega_2 \circ (\varphi_2)_j(z))^k \longrightarrow 0$$

as $k \longrightarrow \infty$, one can choose $k > 0$ such that $M|Q_k| < \varepsilon$, in which

$$M = \max\{|(\prod_{j=0}^{i} \omega_1 \circ (\varphi_1)_j \cdot \omega_2 \circ (\varphi_2)_j)(z)| : i = 0, 1, ..., m_1m_2 - 1\}.$$

Now, if $n > km_1m_2$ then $|P_n| \leq M|Q_k| < \varepsilon$ which implies that $P_n \longrightarrow 0$ as $n \longrightarrow \infty$. Moreover, since for every $z \in \mathbb{D}$

$$(\prod_{j=0}^{m_1m_2-1} \omega_1 \circ (\varphi_1)_{-j} \cdot \omega_2 \circ (\varphi_2)_{-j})(z) = (\prod_{j=0}^{m_1m_2-1} \omega_1 \circ (\varphi_1)_j \cdot \omega_2 \circ (\varphi_2)_j)(z)$$

by a similar method one can see that

$$\prod_{j=0}^{n-1}[\omega_1 \circ (\varphi_1)_{-j} \cdot \omega_2 \circ (\varphi_2)_{-j})(z)]^{-1} \longrightarrow 0$$

as $n \longrightarrow \infty$ for every $z \in V$. Hence, the result follows using Theorem 1. □

**Corollary 4** *If $ran(\omega_1.\omega_2) \bigcap \partial\mathbb{D} \neq \emptyset$, then $(M^*_{\omega_1}, M^*_{\omega_2})$ is hypercyclic.*

**Proof.** Let $\varphi_1(z) = z$ and $\varphi_2(z) = z$ for all $z \in \mathbb{D}$ in above proposition. □

Taking $\omega_2(z) \equiv 1$ in the above corollary, we get the following result from [8], as a special case.

**Corollary 5 ([8])** *If $ran(\omega_1) \bigcap \partial\mathbb{D} \neq \emptyset$, then $M^*_{\omega_1}$ is hypercyclic.*

**Remark.** By analogous proofs we can show that the results in this paper are also valid for $n$-tuples of the adjoint of the weighted composition operators on $\mathcal{H}$.

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