Turk J Math
36 (2012) , 463 – 474.

doi:10.3906/mat-1003-204

On the biharmonic vector fields

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Abstract

The problem studied in this paper is related to the biharmonicity of a vector field from a Riemannian manifold (M, g) to its tangent bundle TM equipped with the Sasaki metric g^s. We show that a vector field on a compact manifold is biharmonic if and only if is harmonic. We also investigate the biharmonicity of vector field of M, as a map from (M, g) to (TM, g^s).

Key Words: Horizontal lift, vertical lift, harmonic maps, biharmonic maps.

1. Introduction

Biharmonic maps are critical points of bienergy functional defined on the space of smooth maps between Riemannian manifolds, introduced by Eells and Sampson in 1964, which is a generalization of harmonic maps [3].

If $\varphi : (M, g) \to (N, h)$ is a smooth map between Riemannian manifolds, then the tension field of φ is defined as

$$\tau(\varphi) = trace_g \nabla d\varphi.$$

It is said φ is harmonic if the tension field vanishes. The equivalent definition is that φ is a critical point of the energy functional

$$E(\varphi) = \int_M e(\varphi) v_g,$$

where $e(\varphi) = \frac{1}{2} trace_g(\varphi^* h)$ is called energy density of φ.

If M is not compact then the energy $E(\varphi)$ may be defined on its compact subsets.

Definition 1 *A map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds is called biharmonic if it is a critical point of the bienergy functional :*

$$E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 v_g$$

(or over any compact subset $K \subset M$).

2000 *AMS Mathematics Subject Classification:* 53A45, 53C20, 58E20.
The authors would like to thank Professor A.A. Salimov for useful remarks and suggestions.

The Euler-Lagrange equation attached to bienergy is given by the vanishing of the bitension field

$$\tau_2(\varphi) = -J_\varphi(\tau(\varphi)) = -\big(\Delta^\varphi \tau(\varphi) + trace_g R^N(\tau(\varphi), d\varphi)d\varphi\big), \tag{1}$$

where J_φ is the Jacobi operator defined by

$$\begin{aligned} J_\varphi : \Gamma(\varphi^{-1}(TN) &\to \Gamma(\varphi^{-1}(TN) \\ V &\mapsto \Delta^\varphi V + trace_g R^N(V, d\varphi)d\varphi. \end{aligned} \tag{2}$$

(One can refer to [6] for more details.)

2. Some results on horizontal and vertical lifts

Let (M, g) be an n-dimensional Riemannian manifold and (TM, π, M) be its tangent bundle. A local chart $(U, x^i)_{i=1...n}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=1...n}$ on TM. Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.

We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, defined by :

$$\begin{aligned} \mathcal{V}_{(x,u)} &= Ker(d\pi_{(x,u)}) \\ &= \{a^i \frac{\partial}{\partial y^i}|_{(x,u)}; \quad a^i \in \mathbb{R}\} \\ \mathcal{H}_{(x,u)} &= \{a^i \frac{\partial}{\partial x^i}|_{(x,u)} - a^i u^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}|_{(x,u)}; \quad a^i \in \mathbb{R}\}, \end{aligned}$$

where $(x, u) \in TM$, such that $T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$.
Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i \frac{\partial}{\partial y^i} \tag{3}$$

$$X^H = X^i \frac{\delta}{\delta x^i} = X^i \{\frac{\partial}{\partial x^i} - y^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}\}. \tag{4}$$

For consequences, we have $(\frac{\partial}{\partial x^i})^H = \frac{\delta}{\delta x^i}$ and $(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i}$, then $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})_{i=1...n}$ is a local adapted frame in TTM.

Remark 1 *1. If $w = w^i \frac{\partial}{\partial x^i} + \overline{w}^j \frac{\partial}{\partial y^j} \in T_{(x,u)}TM$, then its horizontal and vertical parts are defined by*

$$w^h = w^i \frac{\partial}{\partial x^i} - w^i u^j \Gamma^k_{ij} \frac{\partial}{\partial y^k} \in \mathcal{H}_{(x,u)}$$

$$w^v = \{\overline{w}^k + w^i u^j \Gamma^k_{ij}\} \frac{\partial}{\partial y^k} \in \mathcal{V}_{(x,u)}.$$

2. If $u = u^i \frac{\partial}{\partial x^i} \in T_xM$ then its vertical and horizontal lifts are defined by

$$\begin{aligned} u^V &= u^i \frac{\partial}{\partial y^i} \\ u^H &= u^i \{ \frac{\partial}{\partial x^i} - y^j \Gamma^k_{ij} \frac{\partial}{\partial y^k} \}. \end{aligned}$$

Proposition 1 *(See [10]) Let $F \in \mathfrak{T}^1_p(M)$ be a tensor of type (1,p) (respectively, $G \in \mathfrak{T}^0_p(M)$ a tensor of type (0,p)), then there exist a tensor $\gamma(F) \in \mathfrak{T}^1_{p-1}(TM)$ (respectively, $\gamma(G) \in \mathfrak{T}^0_{p-1}(TM)$), locally defined by*

$$\gamma(F) = F^k_{h_1..h_p} y^{h_1} \frac{\partial}{\partial y^k} \otimes dx^{h_2} \otimes ... \otimes dx^{h_p} \tag{5}$$

$$\gamma(G) = G_{h_1..h_p} y^{h_1} dx^{h_2} \otimes ... \otimes dx^{h_p}, \tag{6}$$

where $F = F^j_{i_1..i_p} \frac{\partial}{\partial x^j} \otimes dx^{i_1} \otimes ... \otimes dx^{i_p}$ and $G = G_{i_1..i_p} dx^{i_1} \otimes ... \otimes dx^{i_p}$.

Proposition 2 *(See [10]). For any $X, Y \in \Gamma(TM)$ and $f \in C^\infty(M)$ we have the following relations*

$(X + Y)^h = (X)^h + (Y)^h$

$(fX)^v = (f)^v X^v$

$(fX)^h = (f)^v X^h$

$X^H f^v = (Xf)^v$

$X^H f^c = (Xf)^c - \gamma(df \circ \nabla X)$

$[X^v, Y^h] = [X, Y]^v - (\nabla_X Y)^v$

$[X^h, Y^h] = [X, Y]^h - \gamma R(X, Y),$

where $f^v = f \circ \pi$, $f^c = \gamma(\nabla df)$ and R is the curvature tensor of ∇.

Definition 2 *The Sasaki metric g^s on the tangent bundle TM of M is given by*

1. $g^s(X^H, Y^H) = g(X, Y) \circ \pi$
2. $g^s(X^H, Y^V) = 0$
3. $g^s(X^V, Y^V) = g(X, Y) \circ \pi,$

for all vector fields $X, Y \in \Gamma(TM)$.

In the more general case, Sasaki metrics and their applications were considered in [2], [9].

Proposition 3 *([10],[4]) Let (M,g) be a Riemannian manifold and $\widehat{\nabla}$ be the Levi-Civita connection of the tangent bundle (TM,g^s) equipped with the Sasaki metric. Then*

$$
\begin{aligned}
(\widehat{\nabla}_{X^H}Y^H)_{(x,u)} &= (\nabla_X Y)^H_{(x,u)} - \frac{1}{2}(R_x(X,Y)u)^V \\
(\widehat{\nabla}_{X^H}Y^V)_{(x,u)} &= (\nabla_X Y)^V_{(x,u)} + \frac{1}{2}(R_x(u,Y)X)^H \\
(\widehat{\nabla}_{X^V}Y^H)_{(x,u)} &= \frac{1}{2}(R_x(u,X)Y)^H \\
(\widehat{\nabla}_{X^V}Y^V)_{(x,u)} &= 0,
\end{aligned}
$$

for all vector fields $X,Y \in \Gamma(TM)$ and $(x,u) \in TM$

Proposition 4 *([10],[4]) Let (M,g) be a Riemannian manifold and $\widehat{R}$ be the Riemann curvature tensor of the tangent bundle (TM,g^s) equipped with the Sasaki metric. Then the following formulae hold.*

$$
\begin{aligned}
1.\quad \widehat{R}_{(x,u)}(X^V,Y^V)Z^V &= 0 \\
2.\quad \widehat{R}_{(x,u)}(X^V,Y^V)Z^H &= [R(X,Y)Z + \frac{1}{4}R(u,X)(R(u,Y)Z) - \frac{1}{4}R(u,Y)(R(u,X)Z)]^H_x \\
3.\quad \widehat{R}_{(x,u)}(X^H,Y^V)Z^V &= -[\frac{1}{2}R(Y,Z)X + \frac{1}{4}R(u,Y)(R(u,Z)X)]^H_x \\
4.\quad \widehat{R}_{(x,u)}(X^H,Y^V)Z^H &= [\frac{1}{4}R(R(u,Y)Z,X)u + \frac{1}{2}R(X,Z)Y]^V_x + \frac{1}{2}[(\nabla_X R)(u,Y)Z]^H_x \\
5.\quad \widehat{R}_{(x,u)}(X^H,Y^H)Z^V &= [R(X,Y)Z + \frac{1}{4}R(R(u,Z)Y,X)u - \frac{1}{4}R(R(u,Z)X,Y)u]^V_x \\
&\quad + \frac{1}{2}[(\nabla_X R)(u,Z)Y - (\nabla_Y R)(u,Z)X]^H_x \\
6.\quad \widehat{R}_{(x,u)}(X^H,Y^H)Z^H &= \frac{1}{2}[(\nabla_Z R)(X,Y)u]^V_x \\
&\quad + [R(X,Y)Z + \frac{1}{4}R(u,R(Z,Y)u)X \\
&\quad + \frac{1}{4}R(u,R(X,Z)u)Y + \frac{1}{2}R(u,R(X,Y)u)Z]^H_x,
\end{aligned}
$$

for all vectors $u,X,Y,Z \in T_xM$.

Definition 3 *Let (M,g) be a Riemannian manifold and $F \in \mathfrak{T}^1_1(M)$ be a tensor field of type $(1,1)$. Then we define a vertical and horizontal vector fields VF, HF on TM by*

$$
\begin{aligned}
VF : TM &\to TTM \\
(x,u) &\mapsto (F(u))^V
\end{aligned}
$$

$$
\begin{aligned}
HF : TM &\to TTM \\
(x,u) &\mapsto (F(u))^H.
\end{aligned}
$$

Locally we have

$$VF = y^i F_i^j \frac{\partial}{\partial y^j} = y^i (F(\frac{\partial}{\partial x^i}))^V \tag{7}$$

$$HF = y^i F_i^j \frac{\partial}{\partial x^j} - y^i y^k F_i^l \Gamma_{lk}^s \frac{\partial}{\partial y^s} = y^i (F(\frac{\partial}{\partial x^i}))^H. \tag{8}$$

Proposition 5 *Let (M,g) be a Riemannian manifold and $\widehat{\nabla}$ be the Levi-Civita connection of the tangent bundle (TM, g^s) equipped with the Sasaki metric. If $F \in \mathfrak{T}_1^1(M)$ is a tensor field of type $(1,1)$, then*

$$\begin{aligned}
(\widehat{\nabla}_{X^V} VF)_{(x,u)} &= (F(X))^V_{(x,u)} \\
(\widehat{\nabla}_{X^V} HF)_{(x,u)} &= (F(X))^H_{(x,u)} + \frac{1}{2}(R_x(u, X_x)F(u))^H \\
(\widehat{\nabla}_{X^H} VF)_{(x,u)} &= V(\nabla_X F)(x,u) + \frac{1}{2}(R_x(u, F_x(u))X_x)^H \\
(\widehat{\nabla}_{X^H} HF)_{(x,u)} &= H(\nabla_X F)(x,u) - \frac{1}{2}(R_x(X_x, F_x(u))u)^V,
\end{aligned}$$

where $(x,u) \in TM$ and $X \in \Gamma(TM)$.

Proof. Locally, using formulas (3) and (4), and the Propositions 2 and 3, we have

$$\begin{aligned}
\widehat{\nabla}_{X^V} F^v &= \widehat{\nabla}_{X^V} y^i (F(\frac{\partial}{\partial x^i}))^V = X^V(y^i)(F(\frac{\partial}{\partial x^i}))^V \\
&= X^i (F(\frac{\partial}{\partial x^i}))^V = (F(X))^V
\end{aligned}$$

$$\begin{aligned}
(\widehat{\nabla}_{X^V} HF)_{(x,u)} &= (\widehat{\nabla}_{X^V} y^i (F(\frac{\partial}{\partial x^i}))^H)_{(x,u)} \\
&= (X^V(y^i)(F(\frac{\partial}{\partial x^i}))^H + y^i \widehat{\nabla}_{X^V} F(\frac{\partial}{\partial_i})^H)_{(x,u)} \\
&= X^i (F(\frac{\partial}{\partial x^i}))^H + u^i \frac{1}{2}(R_x(u,X)F_x(\frac{\partial}{\partial x^i}))^H \\
&= (F(X))^H + \frac{1}{2}(R_x(u,X)F_x(u))^H,
\end{aligned}$$

and

$$\begin{aligned}
(\widehat{\nabla}_{X^H} HF)_{(x,u)} &= (\widehat{\nabla}_{X^H} y^k (F(\frac{\partial}{\partial x^k}))^H)_{(x,u)} \\
&= (X^H(y^k)(F(\frac{\partial}{\partial x^k}))^H + y^k \widehat{\nabla}_{X^H} F(\frac{\partial}{\partial_k})^H)_{(x,u)} \\
&= -X^i u^j \Gamma_{ij}^k (F(\frac{\partial}{\partial x^k}))^H + u^k (\nabla_X F(\frac{\partial}{\partial_k}))^H_{(x,u)} \\
&\quad -u^k \frac{1}{2}(R_x(X_x, F_x(\frac{\partial}{\partial x^x}))u)^V.
\end{aligned}$$

Let $U = u^i \frac{\partial}{\partial x^i}$ be a constant vector field, then:

$$
\begin{aligned}
(\widehat{\nabla}_{X^H} HF)_{(x,u)} &= -F(\nabla_X U)^H_{(x,u)} + (\nabla_X F(U))^H_{(x,u)} - \frac{1}{2}(R_x(X_x, F_x(u))u)^V \\
&= ((\nabla_X F)(U))^H_{(x,u)} - \frac{1}{2}(R_x(X_x, F_x(u))u)^V \\
&= (H(\nabla_X F))(x,u) - \frac{1}{2}(R_x(X_x, F_x(u))u)^V.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
(\widehat{\nabla}_{X^H} VF)_{(x,u)} &= (X^H(y^k)(F(\frac{\partial}{\partial x^k}))^V + y^k \widehat{\nabla}_{X^H} F(\frac{\partial}{\partial_k})^V)_{(x,u)} \\
&= V(\nabla_X F)(x,u) + \frac{1}{2}(R_x(u, F_x(u))X_x)^H.
\end{aligned}
$$

□

3. Harmonicity of a vector field $X : (M,g) \longrightarrow (TM, g^S)$

Lemma 1 *Let (M,g) be a Riemannian manifold and (TM, g^s) be the tangent bundle equipped with the Sasaki metric. If $X, Y \in \Gamma(TM)$ are a vector fields and $(x,u) \in TM$ such that $X_x = u$, then we have*

$$d_x X(Y_x) = Y^h_{(x,u)} + (\nabla_Y X)^v_{(x,u)}.$$

Proof. Let (U, x^i) be a local chart on M in $x \in M$ and $(\pi^{-1}(U), x^i, y^j)$ be the induced chart on TM, if $X_x = X^i(x)\frac{\partial}{\partial x^i}|_x$ and $Y_x = Y^i(x)\frac{\partial}{\partial x^i}|_x$, then

$$d_x X(Y_x) = Y^i(x)\frac{\partial}{\partial x^i}|_{(x,X_x)} + Y^i(x)\frac{\partial X^k}{\partial x^i}(x)\frac{\partial}{\partial y^k}|_{(x,X_x)},$$

thus the horizontal part is given by

$$
\begin{aligned}
(d_x X(Y_x))^h &= Y^i(x)\frac{\partial}{\partial x^i}|_{(x,X_x)} - Y^i(x)X^j(x)\Gamma^k_{ij}(x)\frac{\partial}{\partial y^k}|_{(x,X_x)} \\
&= Y^h_{(x,X_x)},
\end{aligned}
$$

and the vertical part is given by

$$
\begin{aligned}
(d_x X(Y_x))^v &= \{Y^i(x)\frac{\partial X^k}{\partial x^i}(x) + Y^i(x)X^j(x)\Gamma^k_{ij}(x)\}\frac{\partial}{\partial y^k}|_{(x,X_x)} \\
&= (\nabla_Y X)^v_{(x,X_x)}.
\end{aligned}
$$

□

Using the Lemma 1 and Proposition 3, we obtain the following proposition.

Proposition 6 *Let (M,g) be a Riemannian manifold and (TM,g^s) be the tangent bundle equipped with the Sasaki metric, if $X : M \longrightarrow TM$ is a smooth vector field then its tension field is given by*

$$\tau(X) = (tr_g R(X, \nabla_* X)*)^H + (tr_g \nabla^2 X)^V.$$

Note that if X is parallel (i.e., $\nabla X = 0$) then X is harmonic. Conversely we have the following theorem proved by Ishihara [5], [7].

Theorem 1 *Let (M,g) be a compact Riemannian manifold and $X \in \Gamma(TM)$, then X is harmonic with respect to Sasaki metric on TM if and only if X is parallel.*

4. Biharmonicity of a vector field $X : (M,g) \longrightarrow (TM, g^S)$

For a vector field $X \in \Gamma(TM)$ we denote

$$\tau^h(X) = tr_g R(X, \nabla_* X)* \quad (9)$$

$$\tau^v(X) = tr_g \nabla^2 X. \quad (10)$$

Theorem 2 *Let (M,g) be a compact Riemannian manifold and $X \in \Gamma(TM)$, then X is biharmonic with respect to Sasaki metric on TM if and only if X is harmonic.*

Proof. Let X_t be a compactly supported variation of X defined by $X_t = (1+t)X$. From the formulas (9) and (10) we have

$$\begin{aligned} \tau^h(X_t) &= (1+t)^2 \tau^h(X) \\ \tau^v(X_t) &= (1+t)\tau^v(X) \end{aligned}$$

$$\begin{aligned} E_2(X_t) &= \frac{1}{2}\int |\tau(X_t)|^2_{g^s} v_g \\ &= \frac{1}{2}\int |\tau^h(X_t)|^2_g v_g + \frac{1}{2}\int |\tau^v(X_t)|^2_g v_g \\ &= \frac{(1+t)^4}{2}\int |\tau^h(X)|^2_g v_g + \frac{(1+t)^2}{2}\int |\tau^v(X)|^2_g v_g \end{aligned}$$

then

$$\begin{aligned} \frac{d}{dt}E_2(X_t)|_{t=0} &= \frac{1}{2}\int |\tau^h(X)|^2_g v_g + \frac{1}{2}\int |\tau^v(X)|^2_g v_g \\ &= \frac{1}{2}\int |\tau(X)|^2_{g^s} v_g. \end{aligned}$$

Hence

$$\frac{d}{dt}E_2(X_t)|_{t=0} = 0 \Leftrightarrow \tau(X) = 0.$$

□

As a consequence of Theorems 1 and 2, we get the following corollary.

Corollary 1 *Let (M,g) be a compact Riemannian manifold and $X \in \Gamma(TM)$, then X is biharmonic with respect to Sasaki metric on TM if and only if X is parallel.*

Remark 2 *If $X \in \Gamma(TM)$ is a compactly supported vector field then X is biharmonic with respect to Sasaki metric on TM if and only if X is harmonic.*

Lemma 2 *Let (M,g) be a Riemannian manifold and (TM,g^s) be the tangent bundle equipped with the Sasaki metric. If $X : M \longrightarrow TM$ is a smooth vector field then the Jacobi tensor $J_X(\tau^v(X)^V)$ is given by*

$$\begin{aligned} J_X(\tau^v(X)^V)_{(x,u)} &= \Big\{tr_g\nabla^2(\tau^v(X))\Big\}^V_{(x,u)} + \Big\{tr_g\Big(R(u,\nabla_*\tau^v(X))* \\ &+ R(\tau^v(X),\nabla_*X)* + \frac{1}{2}R(u,\tau^v(X))R(u,\nabla_*X)*\Big)\Big\}^H_{(x,u)}, \end{aligned}$$

for all $(x,u) \in TM$.

Proof. Let $(x,u) \in TM$ and $\{e_i\}_{i=1}^m$ be a local orthonormal frame on M such that $(\nabla_{e_i}e_i)_x = 0$, denote by $F_i = \frac{1}{2}R(*,\tau^v(X))e_i)$, we have:

$$\begin{aligned} \nabla^X_{e_i}(\tau^v(X))^V|_{(x,u)} &= \widehat{\nabla}_{e_i^H+(\nabla_{e_i}X)^V}\tau^v(X)^V|_{(x,u)} \\ &= (\nabla_{e_i}\tau^v(X))^V|_{(x,u)} + \frac{1}{2}(R(u,\tau^v(X))e_i)^H \\ &= (\nabla_{e_i}\tau^v(X))^V|_{(x,u)} + HF_i(x,u). \end{aligned}$$

Then

$$\begin{aligned} tr_g\nabla^2(\tau^v(X))^V_{(x,u)} &= \sum_{i=1}^m \Big\{\nabla^X_{e_i}\nabla^X_{e_i}(\tau^v(X))^V\Big\}(x,u) \\ &= \sum_{i=1}^m \Big\{\widehat{\nabla}_{e_i^H+(\nabla_{e_i}X)^V}\Big((\nabla_{e_i}\tau^v(X))^V + HF_i\Big)\Big\}_{(x,u)} \\ &= \sum_{i=1}^m \Big\{\widehat{\nabla}_{e_i^H}(\nabla_{e_i}\tau^v(X))^V + \widehat{\nabla}_{e_i^H}HF_i + \widehat{\nabla}_{(\nabla_{e_i}X)^V}HF_i\Big\}_{(x,u)}. \end{aligned}$$

Using Proposition (5), we obtain

$$\begin{aligned} tr_g\nabla^2(\tau^v(X))^V_{(x,u)} &= \sum_{i=1}^m \Big\{(\nabla_{e_i}\nabla_{e_i}\tau^v(X))_{(x,u)} - \frac{1}{4}R_x(e_i,R_x(u,\tau^v(X))e_i)u\Big\}^V \\ &+ \sum_{i=1}^m \Big\{\frac{1}{2}(R_x(u,\nabla_{e_i}\tau^v(X))e_i + \frac{1}{2}(\nabla_{e_i}R_x(u,\tau^v(X))e_i) + \frac{1}{2}R_x(\tau^v(X),\nabla_*u)* \\ &+ \frac{1}{4}R_x(u,\nabla_{e_i}X)R_x(u,\tau^v(X))e_i + \frac{1}{2}R_x(\nabla_{e_i}X,\tau^v(X))e_i\Big\}^H. \end{aligned}$$

From Proposition 2 and lemma 1, we have

$$\begin{aligned} tr_g(\widehat{R}(\tau^v(X)^V, dX)dX) &= \sum_{i=1}^m \Big\{ \widehat{R}((\tau^v(X))^V, e_i^H)e_i^H + \widehat{R}((\tau^v(X))^V, (\nabla_{e_i}X)^V)e_i^H \\ &+ \widehat{R}((\tau^v(X))^V, e_i^H)(\nabla_{e_i}X)^V + \widehat{R}((\tau^v(X))^V, (\nabla_{e_i}X)^V)(\nabla_{e_i}X)^V \Big\}. \end{aligned}$$

By calculating at (x,u), we obtain

$$\begin{aligned} tr_g(\widehat{R}(\tau^v(X)^V, dX)dX)_{(x,u)} &= \sum_{i=1}^m \Big\{ -\frac{1}{4}R(R(u,\tau^v(X))e_i, e_i)u + \frac{1}{2}R(e_i,e_i)\tau^v(X) \Big\}_x^V \\ &+ \sum_{i=1}^m \Big\{ R(\tau^v(X), \nabla_{e_i}X)e_i + \frac{1}{4}R(u,\tau^v(X))R(u,\nabla_{e_i}X)e_i \\ &- \frac{1}{4}R(u,\nabla_{e_i}X)R(u,\tau^v(X))e_i + \frac{1}{2}R(\tau^v(X),\nabla_{e_i}X)e_i \\ &+ \frac{1}{4}R(u,\tau^v(X))R(u,\nabla_{e_i}X)e_i - \frac{1}{2}(\nabla_{e_i}R)(u,\tau^v(X))e_i \Big\}_x^H. \end{aligned}$$

Considering the formula (2), we deduce

$$\begin{aligned} J_X(\tau^v(X)^V)_{(x,u)} &= \Big\{ tr_g\nabla^2(\tau^v(X)) \Big\}_{(x,u)}^V + \Big\{ \frac{1}{2}R(u,\nabla_{e_i}\tau^v(X))e_i \\ &+ \frac{1}{2}\nabla_{e_i}R(u,\tau^v(X))e_i + R(u,\tau^v(X))R(u,\nabla_{e_i}X)e_i - (\nabla_{e_i}R)(u,\tau^v(X))e_i \Big\}_x^H. \end{aligned}$$

From the following equality

$$\nabla_{e_i}R(u,\tau^v(X))e_i = (\nabla_{e_i}R)(u,\tau^v(X))e_i + R(\nabla_{e_i}u,\tau^v(X))e_i + R(u,\nabla_{e_i}\tau^v(X))e_i.$$

The proof of Lemma 2 is completed. □

Lemma 3 *Let (M,g) be a Riemannian manifold and (TM,g^s) be the tangent bundle equipped with the Sasaki metric, if $X: M \longrightarrow TM$ is a smooth vector field then the Jacobi tensor $J_X(\tau^h(X)^H)$ is given by*

$$\begin{aligned} J_X(\tau^h(X)^H)_{(x,u)} &= tr_g\Big\{ 2R(\tau^h(X),*)\nabla_*X - R(*,\nabla_*\tau^h(X))u + \frac{1}{2}R(R(u,\nabla_*)*,\tau^h(X))u \Big\}_{(x,u)}^V \\ &+ tr_g\Big\{ \nabla_*\nabla_*\tau^h(X) + R(u,\nabla_*X)\nabla_*\tau^h(X) + \frac{1}{2}R(u,\nabla_*\nabla_*X)\tau^h(X) \\ &+ R(u,R(\tau^h(X),*)u)* + R(\tau^h(X),*)* + (\nabla_{\tau^h(X)}R)(u,\nabla_*X)* \Big\}_{(x,u)}^H \qquad (11) \end{aligned}$$

for all $(x,u) \in TM$.

Proof. Let $(x,u) \in TM$ and $\{e_i\}_{i=1}^m$ be a local orthonormal frame on M such that $(\nabla_{e_i}e_i)_x = 0$, if we denote by

$$F_i = \frac{1}{2}R(e_i,\tau^h(X))* \qquad (12)$$

and

$$G_i = \frac{1}{2}R(*, \nabla_{e_i}X)\tau^h(X)). \tag{13}$$

In the first, using Proposition 3, we calculate

$$\begin{aligned} tr_g\nabla^2(\tau^h(X))^H_{(x,u)} &= \sum_{i=1}^m \Big\{\nabla^X_{e_i}\nabla^X_{e_i}(\tau^h(X))^H\Big\}_{(x,u)} \\ &= \sum_{i=1}^m \Big\{\widehat{\nabla}_{e_i^H+(\nabla_{e_i}X)^V}\Big((\nabla_{e_i}\tau^h(X))^H - VF_i + HG_i\Big)\Big\}_{(x,u)}. \end{aligned}$$

From Proposition 5, we have

$$\begin{aligned} tr_g\nabla^2(\tau^h(X))^H_{(x,u)} &= \sum_{i=1}^m \Big\{(\nabla_{e_i}\nabla_{e_i}\tau^h(X))^H + (\frac{1}{2}R(u, \nabla_{e_i}X)\nabla_{e_i}\tau^h(X))^H - V(\nabla_{e_i}F_i). \\ &- (\frac{1}{2}R(e_i, \nabla_{e_i}\tau^h(X))u)^V - \frac{1}{2}(R(u, F_i(u))e_i)^H - (F_i(\nabla_{e_i}X))^V + H(\nabla_{e_i}G_i) \\ &- \frac{1}{2}(R(e_i, G(u))u)^V + (G_i(\nabla_{e_i}X))^H + \frac{1}{2}(R(u, \nabla_{e_i}X)G_i(u))^H\Big\}_{(x,u)}. \end{aligned} \tag{14}$$

On substituting (12) and (13) in (14), we arrive at

$$\begin{aligned} tr_g\nabla^2(\tau^h(X))^H_{(x,u)} &= \sum_{i=1}^m \Big\{\nabla_{e_i}\nabla_{e_i}\tau^h(X) + R(u, \nabla_{e_i}X)\nabla_{e_i}\tau^h(X) + \frac{1}{2}R(u, \nabla_{e_i}\nabla_{e_i}X)\tau^h(X) \\ &+ \frac{1}{2}(\nabla_{e_i}R)(u, \nabla_{e_i}X)\tau^h(X) + \frac{1}{4}R(u, \nabla_{e_i}X)(R(u, \nabla_{e_i}X)\tau^h(X)) \\ &- \frac{1}{4}R(u, R(e_i, \tau^h(X)u)e_i\Big\}^H_{(x,u)} \\ &- \sum_{i=1}^m \Big\{\frac{1}{2}R(e_i, \tau^h(X))\nabla_{e_i}X + R(e_i, \nabla_{e_i}\tau^h(X))u + \frac{1}{2}(\nabla_{e_i}R)(e_i, \tau^h(X))u \\ &+ \frac{1}{4}R(e_i, R(u, \nabla_{e_i}X)\tau^h(X))u\Big\}^V_{(x,u)}. \end{aligned} \tag{15}$$

On the other hand we have

$$\begin{aligned} tr_g\Big\{(\widehat{R}(\tau^h(X)^H, dX)dX\Big\}_{(x,u)} &= \sum_{i=1}^{m}\Big\{R(\tau^h(X),e_i)e_i + \frac{3}{4}R(u,R(\tau^h(X),e_i)u)e_i \\ &+ (\nabla_{\tau^h(X)}R)(u,\nabla_{e_i}X)e_i - \frac{1}{2}(\nabla_{e_i}R)(u,\nabla_{e_i}X)\tau^h(X) \\ &- \frac{1}{4}R(u,\nabla_{e_i}X)R(u,\nabla_{e_i}X)\tau^h(X)\Big\}^H_{(x,u)} \\ &+ \sum_{i=1}^{m}\Big\{\frac{1}{2}(\nabla_{e_i}R)(\tau^h(X),e_i)u + \frac{1}{2}R(R(u,\nabla_{e_i}X)e_i,\tau^h(X))u \\ &+ \frac{3}{2}R(\tau^h(X),e_i)\nabla_{e_i}X - \frac{1}{4}R(R(u,\nabla_{e_i}X)\tau^h(X),e_i)u\Big\}^V_{(x,u)}. \qquad (16) \end{aligned}$$

By summing (15) and (16), we obtain the formula (11). □

From Lemma 2 and Lemma 3, we deduce the next theorem.

Theorem 3 *Let (M,g) be a Riemannian manifold and (TM,g^s) be the tangent bundle equipped with the Sasaki metric, if $X : M \longrightarrow TM$ is a smooth vector field then the bitension field of X is given by*

$$\begin{aligned} \tau_2(X)_{(x,u)} &= tr_g\Big\{\nabla^2(\tau^v(X)) + 2R(\tau^h(X),*)\nabla_*X - R(*,\nabla_*\tau^h(X))u \\ &+ \frac{1}{2}R(R(u,\nabla_*)*,\tau^h(X))u\Big\}^V_{(x,u)} \\ &+ tr_g\Big\{R(u,\nabla_*\tau^v(X))* + R(\tau^v(X),\nabla_*X)* + \frac{1}{2}R(u,\tau^v(X))R(u,\nabla_*X)* \\ &+ \nabla_*\nabla_*\tau^h(X) + R(u,\nabla_*X)\nabla_*\tau^h(X) + \frac{1}{2}R(u,\nabla_*\nabla_*X)\tau^h(X) \\ &+ R(u,R(\tau^h(X),*)u)* + R(\tau^h(X),*)* + (\nabla_{\tau^h(X)}R)(u,\nabla_*X)*\Big\}^H_{(x,u)}, \end{aligned}$$

for all $(x,u) \in TM$.

By Theorem 3 we have the following theorem.

Theorem 4 *Let (M,g) be a Riemannian manifold and (TM,g^s) its tangent bundle equipped with the Sasaki metric. A vector field $X : M \longrightarrow TM$ is biharmonic if and only if the following conditions are verified*

$$0 = tr_g\Big\{\nabla^2(\tau^v(X)) + 2R(\tau^h(X),*)\nabla_*X - R(*,\nabla_*\tau^h(X))u + \frac{1}{2}R(R(u,\nabla_*)*,\tau^h(X))u\Big\}_x$$

and

$$
\begin{aligned}
0 &= tr_g\Big\{R(u,\nabla_*\tau^v(X))*+R(\tau^v(X),\nabla_*X)*+\frac{1}{2}R(u,\tau^v(X))R(u,\nabla_*X)*+\nabla_*\nabla_*\tau^h(X)\\
&+ R(u,\nabla_*X)\nabla_*\tau^h(X)+\frac{1}{2}R(u,\nabla_*\nabla_*X)\tau^h(X)+R(u,R(\tau^h(X),*)u)*+R(\tau^h(X),*)*\\
&+ (\nabla_{\tau^h(X)}R)(u,\nabla_*X)*\Big\}_x
\end{aligned}
$$

for all $(x,u)\in TM$.

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Received: 15.03.2010