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Hilbert functions and Betti numbers of reverse lexicographic ideals in the exterior algebra

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Abstract

Let K be a field, V a K-vector space with basis $e_1, \dots, e_n$ and let E be the exterior algebra of V. We study the class of reverse lexicographic ideals in E. We analyze the behaviour of their Hilbert functions and Betti numbers.

Key Words: Graded ring, exterior algebra, monomial ideal, minimal resolution

1. Introduction

Let K be a field, V a K-vector space with basis $e_1, \dots, e_n$ and let E be the exterior algebra of V. In [1, Theorem 4.4] Aramova, Herzog and Hibi proved that in the exterior algebra the lexicographic ideals give the maximal Betti numbers among all graded ideals with a given Hilbert function. The result holds in any characteristic. Such a result is the analogue of a result proved for graded ideals in a polynomial ring by Bigatti [3] and Hulett [6] in characteristic zero, and by Pardue [7] in any characteristic. Afterwards, Deery [5] has proved an analogue of Bigatti, Hulett and Pardue's result about minimal Betti numbers. He showed that revlex segment ideals give the lowest Betti numbers among all stable ideals with the same Hilbert function in a polynomial ring. Our aim is to prove a similar result for graded ideals in the exterior algebra. We introduce the reverse lexicographic ideals in the exterior algebra E, study their Hilbert functions and prove that the reverse lexicographic ideals have minimal Betti numbers for given Hilbert functions. In this context the stable ideals (see [1] and [2]) play an important role. In [1], Aramova, Herzog and Hibi gave an explicit minimal free resolution for this class of monomial ideals and obtained a formula for their Betti numbers (Theorem 2.2). This formula will be a fundamental tool in the article.

The paper is organized as follows.

Section 2 contains preliminary notions and results.

In Section 3 we introduce the reverse lexicographic segment and analyze its shadow (Definition 3.5). As in the polynomial case the shadow of a reverse lexicographic segment is not necessarily a reverse lexicographic segment. Therefore we determine the conditions under which this property holds (Theorem 3.6 and Corollary 3.8). We characterize sets of monomials of E that generates a reverse lexicographic ideal.

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Section 4 is devoted to the study of the behaviour of the Hilbert functions of reverse lexicographic ideals (Propositions 4.2 and 4.3). In Section 5 we study the graded Betti numbers of reverse lexicographic ideals in the exterior algebra E. The main statement is the following:

Theorem 1.1 *Let $I \subsetneq E$ be a strongly stable ideal and $J \subsetneq E$ a reverse lexicographic ideal. Suppose that $H_{E/J}(d) = H_{E/I}(d)$, for all d. Then $\beta_{i,j}(E/I) \geq \beta_{i,j}(E/J)$, for all i and j.*

The key result is Theorem 5.3, which is obtained by a certain decomposition of a subset of monomials of the same degree in the exterior algebra.

2. Preliminaries and notation

Let K be a field. We denote by $E = K\langle e_1, ..., e_n\rangle$ the exterior algebra of a K-vector space V with basis $e_1, \dots, e_n$. For any subset $\sigma = \{i_1, \dots, i_d\}$ of $\{1, \dots, n\}$ with $1 \leq i_1 < i_2 < \dots < i_d \leq n$ we write $e_\sigma = e_{i_1} \wedge \dots \wedge e_{i_d}$ and call e_σ a monomial of degree d. The set of monomials in E forms a K-basis of E of cardinality 2^n.

In order to simplify the notation we put $fg = f \wedge g$ for any two elements f and g in E. An element $f \in E$ is called *homogeneous* of degree j if $f \in E_j$, where $E_j = \bigwedge^j V$. An ideal I is called *graded* if I is generated by homogeneous elements. If I is graded, then $I = \oplus_{j \geq 0} I_j$, where I_j is the K-vector space of all homogeneous elements $f \in I$ of degree j.

Let $e_\sigma = e_{i_1} e_{i_2} \cdots e_{i_d}$ be a monomial of degree d. We define

$$\mathrm{supp}(e_\sigma) = \{i_1, i_2, \dots, i_d\} = \{i : e_i \text{ divides } e_\sigma\},$$

and we write

$$\mathrm{m}(e_\sigma) = \max\{i : i \in \mathrm{supp}(e_\sigma)\}.$$

If $I \subsetneq E$ is a monomial ideal, we denote by $G(I)$ the unique minimal set of monomial generators of I.

The function $H_{E/I}(j) = \dim_K(E/I)_j$, $j = 0, 1, \dots$ is called the *Hilbert function* of E/I and the polynomial $H_{E/I} = \sum_{j \geq 0} H_{E/I}(j) t^i$ is called the *Hilbert series* of E/I.

The possible Hilbert functions of graded algebras of the form E/I are described by the Kruskal-Katona Theorem [1, Theorem 4.1], which is the precise analogue to Macaulay's theorem (see, e.g., [4]).

For later use we recall the next definitions, which are quoted from [1], [2].

Definition 2.1 *Let $I \subsetneq E$ be a monomial ideal. I is called stable if for each monomial $e_\sigma \in I$ and each $j < \mathrm{m}(e_\sigma)$ one has $e_j e_{\sigma \setminus \{\mathrm{m}(e_\sigma)\}} \in I$.*

I is called strongly stable if for each monomial $e_\sigma \in I$ and each $j \in \sigma$ one has that $e_i e_{\sigma \setminus \{j\}} \in I$ for all $i < j$.

Now let I be a graded ideal of E. Consider the minimal graded free resolution of E/I over E

$$F : \dots \to F_2 \overset{d_2}{\to} F_1 \overset{d_1}{\to} F_0 \to E/I \to 0.$$

Each of the modules F_i is of the form $\oplus_j E(-j)^{\beta_{i,j}(E/I)}$ and the maps d_i are described by matrices with homogeneous coefficients in E. Moreover, the resolution of E/I is always infinite, unless $I = (0)$. Indeed $\ker d_i \neq 0$ for all i since the kernel of d_i contains the submodule $(e_1 e_2 \cdots e_n)F_i$. Viewing K as a left E-module via the canonical epimorphism, we have that

$$\beta_{i,j}(E/I) = \dim_K \ \mathrm{Tor}_i^E(E/I, K)_j.$$

The numbers $\beta_{i,j}(E/I)$ are called the *graded Betti numbers.*

In [1], Aramova, Herzog and Hibi studied the minimal graded free resolution of E/I when I is a stable ideal of E and proved a formula for computing the graded Betti numbers of E/I. More precisely [1, Corollary 3.3]:

Theorem 2.2 *Let $I \subsetneq E$ be a stable ideal. Then*

$$\beta_{i,j+i}(E/I) = \sum_{u \in G(I)_{j+1}} \binom{\mathrm{m}(u)+i-2}{\mathrm{m}(u)-1}, \quad \text{for all} \quad i \geq 1.$$

3. Revlex ideals in the exterior algebra

In this section we introduce the reverse lexicographic ideals in the exterior algebra $E = K\langle e_1, ..., e_n\rangle$ and study some of their properties.

Let M_d denote the set of all monomials of degree $d \geq 1$ in E.

We write $>_{\mathrm{revlex}}$ for the *reverse lexicographic order* (revlex for short) on the finite set M_d, i.e., if $u = e_{i_1} e_{i_2} \cdots e_{i_d}$ and $v = e_{j_1} e_{j_2} \cdots e_{j_d}$ are monomials belonging to M_d with $1 \leq i_1 < i_2 < \ldots < i_d \leq n$ and $1 \leq j_1 < j_2 < \ldots < j_d \leq n$, then

$$u >_{\mathrm{revlex}} v \ \text{ if } \ i_d = j_d, i_{d-1} = j_{d-1}, \ldots, i_{s+1} = j_{s+1} \text{ and } i_s < j_s \text{ for some } 1 \leq s \leq d.$$

From now on, in order to simplify the notation, we will write $>$ instead of $>_{\mathrm{revlex}}$.

Definition 3.1 *A nonempty set $M \subseteq M_d$ is called a reverse lexicographic segment of degree d (revlex segment of degree d, for short) if for all $v \in M$ and all $u \in M_d$ such that $u > v$, we have that $u \in M$.*

Example 3.2 *Let $E = K\langle e_1, e_2, e_3, e_4, e_5\rangle$. The subset of M_2*

$$W = \{e_1e_2, e_1e_3, e_2e_3, e_1e_4\}$$

is a revlex segment of degree 2, whereas the subset of M_4

$$W' = \{e_1e_2e_3e_4, e_1e_2e_4e_5, e_1e_3e_4e_5\}$$

is not a revlex segment, because $e_1e_2e_3e_5 > e_1e_2e_4e_5$ and $e_1e_2e_3e_5 \notin W'$.

Definition 3.3 *Let $I = \oplus_{j\geq 0} I_j$ be a monomial ideal of E. We say that I is a reverse lexicographic ideal of E if, for every j, I_j is spanned by a revlex segment (as K-vector space).*

Example 3.4 *Let $E = K\langle e_1, e_2, e_3, e_4\rangle$. The ideal $I = (e_1e_2, e_2e_3e_4) \subsetneq E$ is not a revlex ideal since $I_3 = \langle e_1e_2e_3, e_1e_2e_4, e_2e_3e_4\rangle$ is not spanned as K-vector space by a revlex segment. In fact, $e_1e_3e_4 > e_2e_3e_4$ and $e_1e_3e_4 \notin I_3$.*
The ideal $I = (e_1e_2, e_1e_3e_4) \subsetneq E$ is a revlex ideal. In fact

$$I_2 = \langle e_1e_2\rangle$$

is spanned as K-vector space by a revlex segment of degree 2;

$$I_3 = \langle e_1e_2e_3, e_1e_2e_4, e_1e_3e_4\rangle$$

is spanned as K-vector space by a revlex segment of degree 3; and

$$I_4 = \langle e_1e_2e_3e_4\rangle$$

is spanned as K-vector space by a revlex segment of degree 4.

It is clear that reverse lexicographic $\Rightarrow$ strongly stable $\Rightarrow$ stable.

From now on, for the sake of simplicity, given a monomial ideal $I = \oplus_{j\geq 0} I_j$ of E we will say that I_j is *a reverse lexicographic segment of degree j* if I_j is spanned as K-vector space by a reverse lexicographic segment of degree j.

Definition 3.5 *Let M be a subset of monomials of E. Set $\mathbf{e}_i = \{e_1, \ldots, e_i\}$. We define the set*

$$\mathbf{e}_i M = \{ue_j : u \in M, \quad j \notin \operatorname{supp}(u), \quad j = 1, \ldots, i\}.$$

Note that $\mathbf{e}_i M = \emptyset$ if, for every monomial $u \in M$ and for every $j = 1, \ldots, i$, one has $j \in \operatorname{supp}(u)$.

If M is a set of monomial of degree $d < n$, $\mathbf{e}_n M$ is called the *shadow* of M and is denoted by $\operatorname{Shad}(M)$:

$$\operatorname{Shad}(M) = \{ue_j : u \in M, \ j \notin \operatorname{supp}(u), \quad j = 1, \ldots, n\}.$$

Note that, if M is a revlex segment of degree d, then $\operatorname{Shad}(M)$ needs not be a revlex segment of degree $d+1$.

For example, if $E = K\langle e_1, e_2, e_3, e_4, e_5\rangle$ and $M = \{e_1e_2, e_1e_3\}$, then

$$\operatorname{Shad}(M) = \{e_1e_2e_3, e_1e_2e_4, e_1e_3e_4, e_1e_2e_5, e_1e_3e_5\}$$

is not a revlex segment of degree 3. Infact $e_2e_3e_4 > e_1e_2e_5$ but $e_2e_3e_4 \notin \operatorname{Shad}(M)$.

Theorem 3.6 *Let M be a revlex segment of degree d of E and let i be an integer such that $n > i > d+1$. Suppose $\mathbf{e}_{i+1} M \neq \emptyset$. The following conditions are equivalent:*

(1) $\mathbf{e}_{i+1} M$ *is a revlex segment of degree $d+1$.*

(2) $e_{i-d}e_{i-(d-1)} \cdots e_{i-2}e_{i-1} \in M$.

Proof. Set $M' = \mathbf{e}_{i+1}M$.

(1)$\Rightarrow$(2). Suppose that $e_{i-d}e_{i-(d-1)}\cdots e_{i-2}e_{i-1} \notin M$.

Let u be the smallest monomial that belongs to M. Then $u > e_{i-d}e_{i-(d-1)}\cdots e_{i-2}e_{i-1}$ and $\mathrm{m}(u) \leq i-1$. Let w be the greatest monomial of degree d such that $u > w$. It follows that $w \geq e_{i-d}e_{i-(d-1)}\cdots e_{i-2}e_{i-1}$ and $\mathrm{m}(w) \leq i-1$. Therefore $i \notin \mathrm{supp}(w)$ and so $we_i \neq 0$.

Clearly, $we_i > ue_{i+1}$. Hence, since $ue_{i+1} \in M'$ and since M' is a revlex segment by assumption, we have that $we_i \in M'$. Therefore there exist $v \geq u$ and $j \in \{1,\ldots,i+1\}$, $j \notin \mathrm{supp}(v)$, such that

$$we_i = ve_j. \tag{1}$$

Since $v \geq u$, it follows that $\mathrm{m}(v) \leq i-1$ and consequently $i \notin \mathrm{supp}(v)$. Hence, from (1) we have $i = j$ and $w = v$. Then $w = v \geq u$, which contradicts the choice of w.

Thus $e_{i-d}e_{i-(d-1)}\cdots e_{i-2}e_{i-1} \in M$.

(2)$\Rightarrow$(1). Let u be the smallest monomial that belongs to M'. We will show that every monomial w of degree $d+1$ such that $w > u$ is an element of M'.

We have that $u = e_{i_1}\cdots e_{i_d}e_{i+1}$, with $e_{i_1}\cdots e_{i_d} \in M$. Let $w = e_{j_1}\cdots e_{j_{d+1}}$ with $w > u$. Then $\mathrm{m}(w) = j_{d+1} \leq \mathrm{m}(u) = i+1$.

(Case 1). Suppose $\mathrm{m}(w) = j_{d+1} = i+1$. We have

$$w = e_{j_1}\cdots e_{j_d}e_{i+1} > u = e_{i_1}\cdots e_{i_d}e_{i+1} \Rightarrow e_{j_1}\cdots e_{j_d} > e_{i_1}\cdots e_{i_d}.$$

Since M is a revlex segment, it follows that $e_{j_1}\cdots e_{j_d} \in M$. Hence $w = e_{j_1}\cdots e_{j_d}e_{j_{d+1}} \in M'$ and M' is a revlex segment of degree $d+1$.

(Case 2). Suppose $\mathrm{m}(w) = j_{d+1} < i+1$.

In this case $e_{j_1}\cdots e_{j_d} \geq e_{i-d}e_{i-(d-1)}\cdots e_{i-2}e_{i-1}$. In fact $j_d < j_{d+1} \leq i$. Therefore, since M is a revlex segment, $e_{j_1}\cdots e_{j_d} \in M$ and $w \in M'$. $\square$

Remark 3.7 In Theorem 3.6 we may assume $i > d+1$, since otherwise the problem is trivial.

As consequences of Theorem 3.6 we obtain the following corollaries.

Corollary 3.8 *Let M be a revlex segment of degree d of E such that $d < n-2$. The following conditions are equivalent:*

(1) $\mathrm{Shad}(M)$ *is a revlex segment of degree $d+1$.*

(2) $e_{n-(d+1)}\cdots e_{n-3}e_{n-2} \in M$.

Corollary 3.9 *Let $M = \{e_{\sigma_1},\ldots,e_{\sigma_t}\}$ be a set of monomials of E and let $d_1 = \min\{\deg(e_{\sigma_i}) : i = 1,\ldots,t\}$ and $d_2 = \max\{\deg(e_{\sigma_i}) : i = 1,\ldots,t\}$, with $d_2 < n-2$. Then $I = (M)$ is a revlex ideal if and only if*

(1) I_j *is a revlex segment for* $d_1 \leq j \leq d_2$*;*

(2) $e_{n-(d_2+1)}\cdots e_{n-3}e_{n-2} \in M$.

4. Hilbert functions of revlex ideals

In this section we put our attention on the behaviour of the Hilbert functions of revlex ideals in the exterior algebra $E = K\langle e_1, e_2, \ldots, e_n \rangle$.

For a finite subset S of E, we denote by $M(S)$ the set of all monomials in S and we denote by $|S|$ its cardinality.

Proposition 4.1 *Let M be a revlex segment of degree $d < n-2$ of E. Then $Shad(M)$ is a revlex segment if and only if $|M| \geq \binom{n-2}{d}$.*

Proof. From Corollary 3.8, $\mathrm{Shad}(M)$ is a revlex segment if and only if the monomial $e_{n-(d+1)} \cdots e_{n-3}e_{n-2}$ belongs to M. Set $u = e_{n-(d+1)} \cdots e_{n-3}e_{n-2}$. It follows that $u \in M$ if and only if all the monomials $e_{i_1}e_{i_2} \cdots e_{i_d}$ with $1 \leq i_1 < i_2 < \ldots < i_d \leq n-2$ are in M.

Hence $\mathrm{Shad}(M)$ is a revlex segment if and only if $|M| \geq \binom{n-2}{d}$. □

Proposition 4.2 *Let $I \subsetneq E$ be a graded ideal. Let d be a positive integer such that $d < n-2$. Suppose*

(1) I_d is a revlex segment,

(2) $H_{E/I}(d) \leq \binom{n}{d} - \binom{n-2}{d}$.

Then $H_{E/I}(d+1) \leq H_{E/I}(d)$.

Proof. Since $H_{E/I}(d) = \dim_K E_d - \dim_K I_d = \binom{n}{d} - \dim_K I_d$, we have $\dim_K I_d \geq \binom{n-2}{d}$ by the assumption. It follows that $\mathrm{Shad}(I_d)$ is a revlex segment (Proposition 4.1) and $e_{n-(d+1)} \cdots e_{n-3}e_{n-2} \in M(I_d)$.

Let $u = e_{i_1} \cdots e_{i_d}$, $1 \leq i_1 < \ldots < i_d \leq n$, be the smallest monomial in $M(I_d)$. Then $u \leq e_{n-(d+1)} \cdots e_{n-3}e_{n-2}$ and therefore $\mathrm{m}(u) \in \{n-2, n-1, n\}$.

(Case 1). Suppose $\mathrm{m}(u) \in \{n-2, n-1\}$. Then ue_n is the smallest monomial in $M(I_{d+1})$ and all larger monomials are in $M(I_{d+1})$.

Set
$$A' = \{w \in M_{d+1} : w < ue_n\}$$
and
$$A = \{v \in M_d : v < u\}.$$
Note that $|A| = H_{E/I}(d)$.

If $w = e_\tau < ue_n$, then $n \in \mathrm{supp}(w)$ and consequently $e_{\tau \setminus \{n\}} < u$. Therefore there is an injection from A' to A i.e. $|A'| \leq |A|$. It follows that $H_{E/I}(d+1) \leq |A'| \leq H_{E/I}(d)$.

(Case 2). Suppose $\mathrm{m}(u) = n$. Then $I_{d+1} = \langle M_{d+1} \rangle$, $H_{E/I}(d+1) = 0$ and the claim is trivially true. □

For a graded ideal $I = \oplus_{j \geq 0} I_j$ in E, the *initial degree* of I, denoted by $\mathrm{indeg}(I)$, is the minimum s such that $I_s \neq 0$.

Proposition 4.3 *Let $I \subsetneq E$ be a revlex ideal with initial degree $d' < n-2$. Then*

$$H_{E/I}(d+1) \leq H_{E/I}(d), \text{ for all } d \geq d'.$$

Proof. If $d > d'$, then the claim follows from Proposition 4.1 and Proposition 4.2. So suppose $d = d'$.

If $H_{E/I}(d) \leq \binom{n}{d} - \binom{n-2}{d}$, then $H_{E/I}(d+1) \leq H_{E/I}(d)$ by Proposition 4.2. So assume $H_{E/I}(d) > \binom{n}{d} - \binom{n-2}{d}$. It follows that $u = e_{n-(d+1)} \cdots e_{n-3}e_{n-2} \notin M(I_d)$. Let $v = e_{i_1} \cdots e_{i_d}$ be the smallest monomial in $G(I)$ of degree d, then $v > u$ implies $\mathrm{m}(v) \leq \mathrm{m}(u)$ and therefore $\mathrm{m}(v) \notin \{n-1, n\}$.

Consider $w \geq ve_n$. Since I_{d+1} is a revlex segment then $w \in M(I_{d+1})$. Set

$$A' = \{u' \in M_{d+1} : u' < ve_n\}.$$

We have that $H_{E/I}(d+1) \leq |A'|$ and if $u' \in A'$, then $\mathrm{m}(u') = n$. Set $u' = v'e_n$. From $u' = v'e_n < ve_n$ it follows that $v' < v$ and $v' \in M_d \setminus M(I_d)$. This shows that there is an injection from A' to $A = M_d \setminus M(I_d)$. Hence $H_{E/I}(d+1) \leq |A'| \leq |A| = H_{E/I}(d)$. □

5. Graded Betti numbers

In this section we study the graded Betti numbers of revlex ideals in the exterior algebra $E = K\langle e_1, ..., e_n\rangle$. Let $I \subsetneq E$ be a monomial ideal and $1 \leq i \leq n$. We define the following sets:

$$G(I;i) = \{u \in G(I) : \mathrm{m}(u) = i\}, \ m_i(I) = |G(I;i)|, \ m_{\leq i}(I) = \sum_{j \leq i} m_j(I).$$

With every subset of monomials of E we can associate a *decomposition* as follows.

Let M be a set of monomials of degree d of E. We have

$$M = \mathcal{M}_0 \cup \mathcal{M}_1 e_n,$$

where $\mathcal{M}_0$ is the set of all monomials $u \in M$ such that $\mathrm{m}(u) \leq n-1$ and $\mathcal{M}_1$ is the set of all monomials $w \in E$ of degree $d-1$ with $\mathrm{m}(w) \leq n-1$ such that $we_n \in M$. Such a decomposition will be called the e_n-*decomposition* of M, and will be denoted by $\{\mathcal{M}_0, \mathcal{M}_1\}$.

Example 5.1 *Let $E = K\langle e_1, e_2, e_3, e_4, e_5\rangle$. Let $M = \{e_1e_2e_3, e_1e_2e_4, e_1e_3e_4, e_2e_3e_4, e_1e_2e_5, e_1e_3e_5, e_2e_3e_5\}$. Then the e_5-decomposition of M is*

$$M = \mathcal{M}_0 \cup \mathcal{M}_1 e_5$$

where $\mathcal{M}_0 = \{e_1e_2e_3, e_1e_2e_4, e_1e_3e_4, e_2e_3e_4\}$ and $\mathcal{M}_1 = \{e_1e_2, e_1e_3, e_2e_3\}$.

Remark 5.2 *Let M be a set of monomials of degree d of E and $\{\mathcal{M}_0, \mathcal{M}_1\}$ be the e_n-decomposition of M. We can observe that if $i < n$, then $m_{\leq i}(M) = m_{\leq i}(\mathcal{M}_0)$. In particular, $m_{\leq n-1}(M) = |\mathcal{M}_0|$. Moreover, $m_{\leq n}(M) = |M|$. We also have $m_{\leq d}(M) \in \{0, 1\}$, since the only monomial u of degree d such that $\mathrm{m}(u) = d$ is $u = e_1 \cdots e_d$. Moreover $m_{\leq i}(M) = 0$, for $i < d$.*

Theorem 5.3 *Let $J \subsetneq E$ be a revlex ideal generated in degree d and let $I \subsetneq E$ be a strongly stable ideal generated in the same degree, such that $\dim_K J_d \geq \dim_K I_d$. Then*

$$m_{\leq i}(J) \geq m_{\leq i}(I), \quad \text{for} \quad 1 \leq i \leq n.$$

Proof. The proof will proceed by induction on n.
Suppose $n > 3$ and assume that the assertion is true for $n-1$. We have

$$m_{\leq n}(J) = \dim_K J_d \geq \dim_K I_d = m_{\leq n}(I).$$

So suppose $i = n-1$ and let $\{\mathcal{M}_0, \mathcal{M}_1\}$ be the e_n-decomposition of $G(J)$ and $\{\mathcal{N}_0, \mathcal{N}_1\}$ the e_n-decomposition of $G(I)$.

If $\mathcal{M}_1 = \emptyset$, then we have

$$|\mathcal{M}_0| = \dim_K J_d \geq \dim_K I_d \geq |\mathcal{N}_0|$$

and so $m_{\leq n-1}(J) = |\mathcal{M}_0| \geq |\mathcal{N}_0| = m_{\leq n-1}(I)$.

If $\mathcal{M}_1 \neq \emptyset$, since J is a revlex ideal generated in degree d, then $\mathcal{M}_0$ contains all monomials w of degree d such that $\mathrm{m}(w) \leq n-1$. Hence $|\mathcal{M}_0| \geq |\mathcal{N}_0|$ and so $m_{\leq n-1}(J) = |\mathcal{M}_0| \geq |\mathcal{N}_0| = m_{\leq n-1}(I)$.

Suppose $i < n-1$. From the $n-1$ case we have $|\mathcal{M}_0| \geq |\mathcal{N}_0|$. Since $\mathcal{M}_0$ is a revlex segment and $\mathcal{N}_0$ is strongly stable, by the induction hypothesis

$$m_{\leq i}(\mathcal{M}_0) \geq m_{\leq i}(\mathcal{N}_0), \text{ for } 1 \leq i \leq n-1,$$

and therefore $m_{\leq i}(J) = m_{\leq i}(\mathcal{M}_0) \geq m_{\leq i}(\mathcal{N}_0) = m_{\leq i}(I)$. □

Example 5.4 *Let $E = K\langle e_1, e_2, e_3, e_4, e_5 \rangle$.*
Let $J = (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4, e_2e_3e_4, e_1e_2e_5, e_1e_3e_5, e_2e_3e_5)$ be a revlex ideal of degree 3, and
$I = (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4, e_1e_2e_5, e_1e_3e_5)$ a strongly stable ideal generated in degree 3.

We have $m_{\leq i}(J) = m_{\leq i}(I) = 0$, for $i = 1, 2$; $m_{\leq 3}(J) = 1 = m_{\leq 3}(I)$; $m_{\leq 4}(J) = 4 \geq m_{\leq 4}(I) = 3$ and $m_{\leq 5}(J) = 7 \geq m_{\leq 5}(I) = 5$.

Lemma 5.5 *Let $I \subsetneq E$ be a strongly stable ideal generated in degree d and let $I_{\langle d+1 \rangle}$ be the ideal generated by the elements of I_{d+1}. Then*

$$m_i(I_{\langle d+1 \rangle}) = m_{\leq i-1}(I), \qquad \text{for all } i.$$

Proof. See [1, Theorem 4.3]. □

Hence we can state the following theorem.

Theorem 5.6 *Let $I \subsetneq E$ be a strongly stable ideal and $J \subsetneq E$ a revlex ideal. Suppose that $H_{E/J}(d) = H_{E/I}(d)$, for all d. Then*

$$\beta_{i,j}(E/I) \geq \beta_{i,j}(E/J), \qquad \text{for all } i \text{ and } j.$$

Proof. The proof is similar to that of [1, Theorem 4.4].
Set $G(I)_d = \{u \in G(I) : \deg u = d\}$. From Theorem 2.2, we have

$$\beta_{i,j+i}(E/I) = \sum_{u \in G(I)_{j+1}} \binom{\mathrm{m}(u)+i-2}{\mathrm{m}(u)-1}, \quad \text{for all} \quad i \geq 1.$$

Since $G(I)_{j+1} = G(I_{\langle j+1\rangle}) - G(I_{\langle j\rangle})\{e_1, \ldots, e_n\}$ we may write the above sum as a difference $\beta_{i,j+i}(E/I) = C - D$, with

$$\begin{aligned}
C &= \sum_{u \in G(I_{\langle j+1\rangle})} \binom{\mathrm{m}(u)+i-2}{\mathrm{m}(u)-1} \\
&= \sum_{t=1}^{n} \sum_{u \in G(I_{\langle j+1\rangle}:t)} \binom{t+i-2}{t-1} = \sum_{t=1}^{n} m_t(I_{\langle j+1\rangle}) \binom{t+i-2}{t-1} \\
&= \sum_{t=1}^{n} (m_{\leq t}(I_{\langle j+1\rangle}) - m_{\leq t-1}(I_{\langle j+1\rangle})) \binom{t+i-2}{t-1} \\
&= m_{\leq n}(I_{\langle j+1\rangle}) \binom{n+i-2}{n-1} + \\
&+ \sum_{t=1}^{n-1} (m_{\leq t}(I_{\langle j+1\rangle})) \left[\binom{t+i-2}{t-1} - \binom{(t+1)+i-2}{t} \right] \\
&= m_{\leq n}(I_{\langle j+1\rangle}) \binom{n+i-2}{n-1} - \sum_{t=1}^{n-1} (m_{\leq t}(I_{\langle j+1\rangle})) \binom{t+i-2}{t}
\end{aligned}$$

and

$$D = \sum_{u \in G(I_{\langle j\rangle})\{e_1, \ldots, e_n\}} \binom{\mathrm{m}(u)+i-2}{\mathrm{m}(u)-1}$$

$$= \sum_{t=2}^{n-1} (m_{\leq t-1}(I_{\langle j+1\rangle})) \binom{t+i-2}{t-1},$$

from Lemma 5.5. The number of generators of $I_{\langle j+1\rangle}$ and $J_{\langle j+1\rangle}$ are equal for all j, then $m_{\leq n}(I_{\langle j+1\rangle}) = m_{\leq n}(J_{\langle j+1\rangle})$, and it follows from Theorem 5.3 that $m_{\leq i}(J_{\langle j+1\rangle}) \geq m_{\leq i}(I_{\langle j+1\rangle})$ for all i. Therefore if we compare the above expressions we have:

$$
\begin{aligned}
\beta_{i,j+i}(E/I) = {} & m_{\leq n}(I_{\langle j+1\rangle})\binom{n+i-2}{n-1} - \sum_{t=1}^{n-1}(m_{\leq t}(I_{\langle j+1\rangle}))\binom{t+i-2}{t} + \\
& - \sum_{t=2}^{n-1}(m_{\leq t-1}(I_{\langle j+1\rangle}))\binom{t+i-2}{t-1} \\
& \geq m_{\leq n}(J_{\langle j+1\rangle})\binom{n+i-2}{n-1} - \sum_{t=1}^{n-1}(m_{\leq t}(J_{\langle j+1\rangle}))\binom{t+i-2}{t} + \\
& - \sum_{t=2}^{n-1}(m_{\leq t-1}(J_{\langle j+1\rangle}))\binom{t+i-2}{t-1} = \beta_{i,j+i}(E/J).
\end{aligned}
$$

The required inequality follows. $\square$

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