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L^p Regularity of some weighted Bergman projections on the unit disc

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Abstract

We show that weighted Bergman projections, corresponding to weights of the form $M(z)(1-|z|^2)^{\alpha}$, where $\alpha > -1$ and $M(z)$ is a radially symmetric, strictly positive and at least C^2 function on $\overline{\mathbb{D}}$, are L^p regular.

Key Words: Weighted Bergman projection, Coefficient multipliers

1. Introduction

Let $\mathbb{D}$ denote the unit disc in $\mathbb{C}^1$ and $dA(z)$ denote the standard Lebesgue measure on $\mathbb{C}^1$. Let $\lambda(r)$ be a strictly positive and continuous function on $[0,1)$. We consider $\lambda(r)$ as a radially symmetric weight on $\mathbb{D}$ by setting $\lambda(z) := \lambda(|z|)$ and denote the space of square integrable functions with respect to the area element $\lambda(z)dA(z)$ by $L^2(\lambda)$. It is clear that $L^2(\lambda)$ is a Hilbert space with the inner product defined by

$$\langle f, g\rangle_\lambda = \int_{\mathbb{D}} f(z)\overline{g(z)}\lambda(z)dA(z)$$

and the norm defined by

$$||f||_\lambda^2 = \int_{\mathbb{D}} |f(z)|^2\lambda(z)dA(z).$$

The closed subspace of holomorphic functions in $L^2(\lambda)$ is denoted by $A^2(\lambda)$. The orthogonal projection operator between these two spaces is called *the weighted Bergman projection* and denoted by $\mathbf{B}_\lambda$, i.e.,

$$\mathbf{B}_\lambda : L^2(\lambda) \to A^2(\lambda).$$

The Riesz representation theorem indicates that $\mathbf{B}_\lambda$ is an integral operator. The kernel of this integral operator is called *the weighted Bergman kernel* and denoted by $B_\lambda(z,w)$, i.e. for any $f \in L^2(\lambda)$,

$$\mathbf{B}_\lambda f(z) = \int_{\mathbb{D}} B_\lambda(z,w)f(w)\lambda(w)dA(w).$$

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The monomials $\{z^n\}_{n=0}^{\infty}$ form an orthogonal basis of $A^2(\lambda)$ and the weighted Bergman kernel is given by the following sum:

$$B_\lambda(z,w) = \sum_{n=0}^{\infty} a_n (z\bar{w})^n, \text{ where } a_n = \frac{1}{\int_{\mathbb{D}} |z|^{2n}\lambda(z)dA(z)}.$$

The coefficients a_n are called *the Bergman coefficients* of weight λ.

For $1 < p < \infty$, we use the standard notation $L^p(\lambda)$ and $A^p(\lambda)$ to denote the respective Banach spaces of p-integrable functions on $\mathbb{D}$ and we use $||.||_{p,\lambda}$ to denote the norm on these spaces.

Let us consider the weights defined by $\lambda_\alpha(r) = (1-r^2)^\alpha$ for $\alpha > -1$, where we set $z = re^{i\theta}$. The Bergman theory for this family of weights are well investigated and can be found in [4].

In particular, the Bergman coefficients of these weights are computed explicitly and the following explicit expression for the weighted kernel is obtained:

$$B_{\lambda_\alpha}(z,w) = \frac{c_\alpha}{(1-z\overline{w})^{2+\alpha}},$$

where c_α is a constant that only depends on α.

Furthermore, this explicit expression for the kernel and Schur's lemma together prove the following theorem.

Theorem 1.1 *For $\alpha > -1$, the weighted Bergman projection $\mathbf{B}_{\lambda_\alpha}$ is bounded from $L^p(\lambda_\alpha)$ to $A^p(\lambda_\alpha)$ for any $1 < p < \infty$.*

Proof. See page 12 of [4] and also [6] and [3].

□

The purpose of this note is to extend this theorem to more general weights in the following setup. Let $M(r)$ be a strictly positive and at least C^2 function on $[0,1]$. Without loss of generality, we assume that $M(1) = 1$. Consider the radially symmetric weight defined by

$$\mu(z) = M(|z|)(1-|z|^2)^\alpha$$

on $\mathbb{D}$, for some $\alpha > -1$. By the general theory (see [2] and [3]), there exists the weighted Bergman projection operator $\mathbf{B}_\mu : L^2(\mu) \to A^2(\mu)$, which is an integral operator with the weighted Bergman kernel $B_\mu(z,w)$, where

$$B_\mu(z,w) = \sum_{n=0}^{\infty} b_n (z\bar{w})^n, \text{ and } b_n = \frac{1}{\int_{\mathbb{D}} |z|^{2n}\mu(z)dA(z)}.$$

But in this case, it is not easy (unless M is a simple function) to compute the coefficients b_n to get an explicit expression for the weighted kernel and therefore, Schur's lemma is not directly applicable in this case.

Nevertheless, we prove the analog of Theorem 1.1 for $\mathbf{B}_\mu$, without referring to an explicit expression for the kernel or Schur's lemma.

Theorem 1.2 *The weighted Bergman projection $\mathbf{B}_\mu$ is bounded from $L^p(\mu)$ to $A^p(\mu)$ for any $1 < p < \infty$.*

The proof is in two steps; first relating $\mathbf{B}_\mu$ to $\mathbf{B}_{\lambda_\alpha}$ by a coefficient multiplier operator and then showing that this coefficient multiplier operator is bounded.

For the rest of the note, we denote the boundary of $\mathbb{D}$ by $b\mathbb{D}$ and we write $A \lesssim B$ to mean $A \leq cB$ for some constant c that is clear in context. We also use the Szegö projection $\mathbf{T} : L^2(b\mathbb{D}, d\theta) \to H^2$, where $d\theta$ is the arc length on the unit circle and H^p is the Hardy space of order p. We refer to [2] for definitions and standard facts about the Szegö projection and Hardy spaces.

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2. Coefficient multipliers and norm convergence

In this section, before giving the details of the proof of Theorem 1.2, we recall a few facts about coefficient multipliers. See [1] and [2] for general account.

Let X be a Banach space of holomorphic functions on $\mathbb{D}$. Any $f \in X$ has Taylor series expansion

$$f(z) = \sum_{n=0}^{\infty} f_n z^n.$$

Definition 2.1 *A sequence of complex numbers $\{t_n\}$ is called a coefficient multiplier from X to X and denoted by $\{t_n\} \in (X, X)$ if for any function $f \in X$,*

$$t(f)(z) := \sum_{n=0}^{\infty} t_n f_n z^n \quad \textit{is also in } X.$$

It is a fairly general question to characterize the coefficient multipliers on an arbitrary Banach space X and there is no full answer to this question.

Definition 2.2 *For a holomorphic function f on $\mathbb{D}$ and $N \in \mathbb{N}$, let $S_N f$ denote the Taylor polynomial of f of degree N, i.e., $S_N f(z) = \sum_{n=0}^{N} f_n z^n$.*

If X has the property that for any $f \in X$ the sequence of Taylor polynomials $\{S_N f\}$ converges to f, then a sufficient condition for coefficient multipliers can be formulated as follows.

Proposition 2.3 *Let $(X, ||.||)$ be a Banach space of holomorphic functions on $\mathbb{D}$ such that for every $f \in X$ the sequence $\{S_N f\}$ of Taylor polynomials converges to f in the norm of X. Then any sequence of **bounded variation** is a coefficient multiplier from X to X.*

Definition 2.4 *A sequence of complex numbers $\{t_n\}$ is said to be of bounded variation if $|t_0| + \sum_{n=1}^{\infty} |t_n - t_{n-1}|$ is finite.*

Proposition 2.3 appears in [1, Proposition 3.7]. It follows from summation by parts and we repeat its proof for completeness.

Proof. Since the Taylor polynomials converge, for any given $f \in X$ and $\epsilon > 0$ there exists an N such that for any $k > N$

$$\left|\left|\sum_{n=k}^{\infty} f_n z^n\right|\right| < \epsilon.$$

Let $\{t_n\}$ be the sequence of bounded variation and $|t_0| + \sum_{n=1}^{\infty} |t_n - t_{n-1}| \leq K$. Summation by parts and bounded variation hypothesis give

$$\begin{aligned}
\left|\left|\sum_{n=k}^{\infty} t_n f_n z^n\right|\right| &= \left|\left|\sum_{n=k}^{\infty}(t_{n+1} - t_n) \sum_{j=n+1}^{\infty} f_j z^j + t_k \sum_{n=k}^{\infty} f_n z^n\right|\right| \\
&\leq \left[|t_k| + \sum_{n=k}^{\infty} |t_{n+1} - t_n|\right] \epsilon \\
&\leq K\epsilon.
\end{aligned}$$

This shows that $t(f)(z) = \sum_{n=0}^{\infty} t_n f_n z^n$ is in X and finishes the proof. □

In order to use this proposition in the proof of Theorem 1.2, we have to check whether Taylor polynomials converge in $A^p(\mu)$. This turns out to be true even in a more general form.

Proposition 2.5 *For $1 < p < \infty$ and any integrable radial weight $\lambda(r)$, the Taylor series of every function in $A^p(\lambda)$ converges in norm.*

In particular, the claim is true for $A^p(\lambda_\alpha)$ and $A^p(\mu)$. The statement for $A^p(\lambda_\alpha)$ is in [5]. The general case is obtained by just imitating the proof in [5].

Proof. This is done in three steps.

Step One. The holomorphic polynomials are dense in $A^p(\lambda)$.

For any $f \in A^p(\lambda)$ and for any $0 < \rho < 1$, define $f_\rho(z) = f(\rho z)$. Each f_ρ is holomorphic in a larger disc and the Taylor polynomials of each f_ρ converges uniformly on $\mathbb{D}$ and hence in $A^p(\lambda)$. Therefore it is enough to show that

$$\lim_{\rho \to 1^-} ||f - f_\rho||_{p,\lambda} = 0.$$

For any holomorphic f, the averages

$$M_p^p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta$$

are well defined and non-decreasing functions of r (see [2, page 26]). Moreover

$$M_p^p(r, f_\rho) = M_p^p(\rho r, f) \leq M_p^p(r, f).$$

Since $f \in A^p(\lambda)$ and

$$||f||_{p,\lambda}^p = \int_0^1 r\lambda(r) M_p^p(r, f) dr.$$

$M_p^p(r,f)$ is integrable with respect to the weight $r\lambda(r)dr$.

On the other hand, $f_\rho \to f$ pointwise on $\mathbb{D}$ as $\rho \to 1^-$ so by the Lebesgue dominated convergence theorem $\lim_{\rho\to 1^-} M_p^p(r, f - f_\rho) = 0$. We also have

$$M_p^p(r, f - f_\rho) \leq 2^p \left(M_p^p(r, f) + M_p^p(r, f_\rho) \right) \leq 2^{p+1} M_p^p(r, f).$$

Therefore again the Lebesgue dominated convergence theorem implies

$$\begin{aligned}
\lim_{\rho\to 1^-} ||f - f_\rho||_{p,\lambda}^p &= \lim_{\rho\to 1^-} \int_0^1 r\lambda(r) M_p^p(r, f - f_\rho) dr \\
&= \int_0^1 r\lambda(r) \lim_{\rho\to 1^-} M_p^p(r, f - f_\rho) dr \\
&= 0.
\end{aligned}$$

This finishes the first step.

Step Two. We show that the operator norms of S_N's (defined in Definition 2.2) are uniformly bounded. For this we need a well-known result about the Szegö projection. Let $\mathbf{T} : L^2(b\mathbb{D}, d\theta) \to H^2$ denote the Szegö projection. By using the fact that $\mathbf{T}$ is also bounded from $L^p(b\mathbb{D}, d\theta)$ to H^p for any $1 < p < \infty$, one can prove (see [2, page 27]) that there exists $C > 0$, independent of N and h, such that

$$\int_0^{2\pi} |S_N h(e^{i\theta})|^p d\theta \leq C \int_0^{2\pi} |h(e^{i\theta})|^p d\theta \tag{2.6}$$

for any $h \in H^p$. The proof is only to note that $\overline{S_N f(e^{i\theta})} = e^{-iN\theta} T\left(e^{iN\theta}\overline{f(e^{i\theta})}\right)$ which is clear for f a polynomial, and follows in general since polynomials are dense in H^p.

Now we calculate the operator norms of S_N's. For given $f \in A^p(\lambda)$,

$$\begin{aligned}
||S_N f||_{p,\lambda}^p &= \int_{\mathbb{D}} |S_N f(z)|^p \lambda(z) dA(z) \\
&= \int_0^1 r\lambda(r) dr \int_0^{2\pi} |S_N f(re^{i\theta})|^p d\theta \\
&\leq C \int_0^1 r\lambda(r) dr \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \quad \text{since } f_r \in H^p \\
&= C \int_{\mathbb{D}} |f(z)|^p \lambda(z) dA(z) \\
&= C ||f||_{p,\lambda}^p.
\end{aligned}$$

This implies that $\sup_N ||S_N||_{op} \leq C$ and finishes the second step.

Step Three. Next, we show that $\lim_{N\to\infty} ||S_N f - f||_{p,\lambda} = 0$ for any $f \in A^p(\lambda)$.
Given f and $\epsilon > 0$, by the first step there exists a polynomial Q such that $||Q - f||_{p,\lambda}^p < \epsilon$. Then

$$\begin{aligned} ||S_N f - f||_{p,\lambda}^p &\leq ||S_N f - S_N Q||_{p,\lambda}^p + ||S_N Q - Q||_{p,\lambda}^p + ||Q - f||_{p,\lambda}^p \\ &\leq (C+1)\epsilon + ||S_N Q - Q||_{p,\lambda}^p. \end{aligned}$$

Note that $S_N Q = Q$ for large enough N and therefore for sufficiently large N,

$$||S_N f - f||_{p,\lambda}^p \leq (C+1)\epsilon.$$

Since this is true for any $\epsilon > 0$ we get $\lim_{N\to\infty} ||S_N f - f||_{p,\lambda} = 0$. This finishes the last step and the proof of the proposition.

□

3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 by using Propositions 2.3 and 2.5. Recall that a_n's are the Bergman coefficients of $(1-|z|^2)^\alpha$ and b_n's are the Bergman coefficients of μ. Let $\mathcal{R}$ denote the coefficient multiplier operator for the sequence $\left\{\frac{b_n}{a_n}\right\}$. The following identity relates the two Bergman projections:

$$\mathbf{B}_\mu f(z) = \mathcal{R}\left[\mathbf{B}_{\lambda_\alpha}(fM)\right](z). \tag{3.1}$$

Indeed, for any $f \in L^2(\mu)$,

$$\begin{aligned} \mathbf{B}_\mu f(z) &= \int_{\mathbb{D}} \sum_{n=0}^{\infty} b_n (z\bar{w})^n f(w)\mu(w)dA(w) = \sum_{n=0}^{\infty} b_n z^n \int_{\mathbb{D}} \bar{w}^n f(w)\mu(w)dA(w) \\ &= \sum_{n=0}^{\infty} a_n z^n \frac{b_n}{a_n} \int_{\mathbb{D}} \bar{w}^n f(w)\mu(w)dA(w) \\ &= \mathcal{R}\left[\sum_{n=0}^{\infty} a_n z^n \int_{\mathbb{D}} \bar{w}^n f(w)\mu(w)dA(w)\right] \\ &= \mathcal{R}\left[\mathbf{B}_{\lambda_\alpha}(fM)\right](z). \end{aligned}$$

Here we change the order of integration and summation but this doesn't cause any problems. We can truncate the summation, which is equivalent to looking at the Taylor polynomials of $\mathbf{B}_\mu f$ and $\mathbf{B}_{\lambda_\alpha}(fM)$, and take limit by using Proposition 2.5. Now, it suffices to prove that the multiplier operator $\mathcal{R}$ is bounded from $A^p(\mu)$ to $A^p(\mu)$ (actually, we have to show that $\mathcal{R}$ is bounded from $A^p(\lambda_\alpha)$ to $A^p(\mu)$ but since M is of class C^2 and thus bounded; the inclusion map $i : A^p(\lambda_\alpha) \to A^p(\mu)$ is bounded). By the closed graph theorem it is enough to show that $\mathcal{R}(f) \in A^p(\mu)$ for any $f \in A^p(\mu)$. Moreover, Proposition 2.3 implies that it is enough to show that the sequence $\left\{\frac{b_n}{a_n}\right\}$ is of bounded variation.

It is immediate that the sequence $\left\{\frac{b_n}{a_n}\right\}$ is bounded from below and above. Moreover, a direct computation gives that

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \lim_{n\to\infty} \frac{\int_0^1 r^{2n+1}(1-r^2)^\alpha dr}{\int_0^1 r^{2n+1}\mu(r)dr} = M(1)^{-1}.$$

We quantify this computation to get that the sequence $\left\{\frac{b_n}{a_n}\right\}$ is indeed of bounded variation.

Lemma 3.2 $\left|\frac{b_n}{a_n} - \frac{b_{n-1}}{a_{n-1}}\right| \lesssim \frac{1}{n^2}$, *i.e., the sequence* $\left\{\frac{b_n}{a_n}\right\}$ *is of bounded variation and therefore* $\mathcal{R}$ *is bounded from* $A^p(\mu)$ *to* $A^p(\mu)$.

Proof.

First, we consider the difference between elements of the sequence $\left\{\frac{b_n}{a_n}\right\}$. Here, all the integrals are taken with respect to r and from 0 to 1.

$$\begin{aligned}
\frac{b_n}{a_n} - \frac{b_{n-1}}{a_{n-1}} &= \frac{\int r^{2n+1}(1-r^2)^\alpha}{\int r^{2n+1}\mu(r)} - \frac{\int r^{2n-1}(1-r^2)^\alpha}{\int r^{2n-1}\mu(r)} \\
&= \frac{\int r^{2n+1}(1-r^2)^\alpha \int r^{2n-1}\mu(r) - \int r^{2n-1}(1-r^2)^\alpha \int r^{2n+1}\mu(r)}{\int r^{2n+1}\mu(r) \int r^{2n-1}\mu(r)} \\
&=: \frac{B(n)}{A(n)}.
\end{aligned}$$

We can rewrite the numerator as

$$\begin{aligned}
B(n) &= \int r^{2n+1}(1-r^2)^\alpha \int r^{2n-1}(1-r^2)\mu(r) - \int r^{2n-1}(1-r^2)^{\alpha+1} \int r^{2n+1}\mu(r) \\
&= \int r^{2n+1}\left[(1-M(r))\,(1-r^2)^\alpha\right] \int r^{2n-1}(1-r^2)\mu(r) \\
&\qquad - \int r^{2n-1}\left[(1-M(r))\,(1-r^2)^{\alpha+1}\right] \int r^{2n+1}\mu(r) \\
&=: B_1(n) - B_2(n).
\end{aligned}$$

Next, we integrate $B_1(n)$ and $B_2(n)$ by parts twice to obtain

$$
\begin{aligned}
B_1(n) &= \frac{1}{(2n+2)2n}\int r^{2n+2}\left[(1-M(r))\,(1-r^2)^{\alpha}\right]'\int r^{2n}\left[M(r)(1-r^2)^{\alpha+1}\right]' \\
&=: \frac{1}{(2n+2)2n}C_1(n)C_2(n)
\end{aligned}
$$

$$
\begin{aligned}
B_2(n) &= \frac{1}{2n(2n+1)}\int r^{2n+1}\left[(1-M(r))\,(1-r^2)^{\alpha+1}\right]''\int r^{2n+1}\left[M(r)(1-r^2)^{\alpha}\right] \\
&=: \frac{1}{2n(2n+1)}C_3(n)C_4(n).
\end{aligned}
$$

Here, C_1, C_2, C_3, C_4 denote the respective integrals. Note that we don't get any boundary terms after integration by parts since M is of class C^2 on $[0,1]$ and $M(1)=1$.

In order to finish the proof, it suffices to show that

$$
\sup_n\left\{n^2\left|\frac{B_1(n)}{A(n)}\right|\right\} \quad \text{and} \quad \sup_n\left\{n^2\left|\frac{B_2(n)}{A(n)}\right|\right\} \quad \text{are finite.}
$$

Thus, it is enough to show that

$$
\sup_n\left\{\left|\frac{C_1(n)C_2(n)}{A(n)}\right|\right\} \quad \text{and} \quad \sup_n\left\{\left|\frac{C_3(n)C_4(n)}{A(n)}\right|\right\} \quad \text{are finite.}
$$

We start with the first one.

$$
\begin{aligned}
\frac{C_1(n)C_2(n)}{A(n)} &= \frac{\int r^{2n+2}\left[(1-M(r))\,(1-r^2)^{\alpha}\right]'\int r^{2n}\left[M(r)(1-r^2)^{\alpha+1}\right]'}{\int r^{2n+1}M(r)(1-r^2)^{\alpha}\int r^{2n-1}M(r)(1-r^2)^{\alpha}} \\
&\to \frac{\left[(1-M(r))\,(1-r^2)^{\alpha}\right]'\left[M(r)(1-r^2)^{\alpha+1}\right]'}{M(r)(1-r^2)^{\alpha}M(r)(1-r^2)^{\alpha}}\Big|_{r=1} \quad \text{as } n\to\infty \\
&= 2(\alpha+1)^2M'(1).
\end{aligned}
$$

This shows that the first supremum is indeed finite. Note that the condition $M(1)=1$ is used here.

We argue the same way for the second one.

$$
\begin{aligned}
\frac{C_3(n)C_4(n)}{A(n)} &= \frac{\int r^{2n+1}\left[(1-M(r))\,(1-r^2)^{\alpha+1}\right]''\int r^{2n+1}\left[M(r)(1-r^2)^{\alpha}\right]}{\int r^{2n+1}M(r)(1-r^2)^{\alpha}\int r^{2n-1}M(r)(1-r^2)^{\alpha}} \\
&\to \frac{\left[(1-M(r))\,(1-r^2)^{\alpha+1}\right]''\left[M(r)(1-r^2)^{\alpha}\right]}{M(r)(1-r^2)^{\alpha}M(r)(1-r^2)^{\alpha}}\Big|_{r=1} \quad \text{as } n\to\infty \\
&= 2(\alpha+2)(\alpha+1)M'(1) - 2(1+\alpha)\,(1-M(1))\,.
\end{aligned}
$$

This shows that the second supremum is indeed finite. Again, note that the condition $M(1)=1$ is used here. This finishes the proof Lemma 3.2.

□

Since Lemma 3.2 is established, we conclude the proof of Theorem 1.2.

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