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Value distribution of meromorphic functions and their differences*

Ranran Zhang and Zongxuan Chen

Abstract

Let $f(z)$ be a transcendental meromorphic function. Results are proved concerning the value distribution of the n'th forward difference $\Delta^n f(z)$, in terms of Borel exceptional values of $f(z)$. The results may be partly viewed as discrete analogues of a classical theorem of Hayman dealing with the possible relationships between Picard exceptional values of $f(z)$ and its derivatives.

Key words and phrases: Complex difference; value distribution; Borel exceptional value

1. Introduction and results

Let $f(z)$ be a meromorphic function in the plane. We assume that the reader is familiar with the basic notions of Nevanlinna's theory (see [10]). We use $\sigma(f)$ to denote the order of growth of $f(z)$; and $\lambda(f)$ and $\lambda(1/f)$ to denote, respectively, the exponents of convergence of zero and pole sequences of $f(z)$. Moreover, we use $\delta(a,f)$ to denote the Nevanlinna deficiency of $f(z)$. For a nonzero constant c, the forward differences $\Delta^n f$ are defined (see [1]) by

$$\Delta f(z) = f(z+c) - f(z),\ \Delta^{n+1} f(z) = \Delta^n f(z+c) - \Delta^n f(z),\ n = 1, 2, \ldots.$$

Throughout this paper, we denote by $S(r,f)$ any function satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$, possibly outside a set of r of finite logarithmic measure. A meromorphic function $\alpha(z)$ is said to be a small function of $f(z)$, if $T(r,\alpha) = S(r,f)$.

Recently, there is substantial interest in difference analogues of Nevanlinna's theory, as well as difference equations. The papers [1, 2] investigated the zeros of $\Delta^n f(z)$ under the assumption that $f(z)$ is of small growth order, and obtained many profound results. These results may be viewed as discrete analogues of the following existing theorem on the zeros of $f'(z)$.

Theorem A [4, 11] *Let $f(z)$ be transcendental and meromorphic in the plane with*

$$\liminf_{r\to\infty} \frac{T(r,f)}{r} = 0.$$

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Then $f'(z)$ has infinitely many zeros.

Hayman [9] investigated the possible relationships between Picard exceptional values of $f(z)$ and its derivatives, and obtained the following classical theorem.

Theorem B [9] *If $f(z)$ is transcendental and meromorphic in the plane, then either $f(z)$ assumes every finite value infinitely often, or every derivative of $f(z)$ assumes every finite value except possibly zero infinitely often.*

In this paper, we investigate the value distribution of meromorphic functions and their differences. First we observe that for a general meromorphic function, the difference counterpart of Theorem B doesn't exist, see the following Example 1.1.

Example 1.1 *Let $f(z) = ze^z/(2\pi i(e^z + 1))$ and $c = 2\pi i$. Then*

$$\Delta f(z) = f(z + 2\pi i) - f(z) = \frac{e^z}{e^z + 1}.$$

We see that $f(z)$ assumes 0 finitely often and $\Delta f(z)$ cannot assume 1.

Example 1.1 shows that if $f(z)$ has only one Borel exceptional value, then $\Delta f(z)$ may not assume some finite nonzero value. In this paper, we prove that if $f(z)$ has two Borel exceptional values and if $f(z)$ is not of period c, then $\Delta f(z)$ assumes every finite value except possibly zero infinitely often. Actually, we get the following Theorem, which may be partly viewed as discrete analogues of Theorem B.

Theorem 1.1 *Let $f(z)$ be a finite order transcendental meromorphic function with two Borel exceptional values a, b. Let $c \in C \setminus \{0\}$ and let $s(z)$ be a nonzero small function of $f(z)$. For every positive integer n, set*

$$F_n(z) = \Delta^n f(z) - s(z).$$

Suppose that one of the following two conditions holds:

(i) *$a,\ b \in C$ and $c, 2c, \ldots, nc$ are not periods of $f(z)$;*

(ii) *$a \in C$, $b = \infty$ and $\Delta^n f(z) \not\equiv 0$.*

Then $F_n(z)$ is transcendentally meromorphic and $\delta(0, F_n) \le n/(n + 1)$.

Remark *The following Examples, 1.2–1.4, show that Theorem 1.1 is false, if $f(z)$ has at most one Borel exceptional value. So the requirement "$f(z)$ has two Borel exceptional values" in Theorem 1.1 cannot be weakened.*

Example 1.2 *Let $f(z)$ and c be as in Example 1.1, and let $s(z) = 1$. Then $f(z)$ has only one Borel exceptional value 0, and*

$$F_1(z) = \Delta f(z) - s(z) = \frac{e^z}{e^z + 1} - 1 = \frac{-1}{e^z + 1}$$

has no zeros.

Example 1.3 *Let $f(z) = e^z + z$, $c = 2\pi i$ and $s(z) = \pi i$. Then $f(z)$ has only one Borel exceptional value ∞, and*

$$F_1(z) = \Delta f(z) - s(z) = f(z + 2\pi i) - f(z) - \pi i = \pi i$$

has no zeros.

Example 1.4 *Let $f(z)=\Gamma'(z)/\Gamma(z)$, $c=1$ and $s(z)=1$. Then $f(z)$ has no Borel exceptional values, and*

$$F_1(z)=\Delta f(z)-s(z)=f(z+1)-f(z)-1=\frac{1}{z}-1$$

has only one zero.

We give the following two corollaries. Corollary 1.1 is obtained directly from Theorem 1.1. Corollary 1.2 cannot be obtained directly from Theorem 1.1, since the condition "$N\big(r,1/(f-a)\big)+N(r,f)=S(r,f)$" in Corollary 1.2 dose not imply that a and ∞ are Borel exceptional values. However, using the same method as in Part II of proof of Theorem 1.1, we can easily prove Corollary 1.2.

Corollary 1.1 *Let $f(z)$ be a finite order transcendental meromorphic function with two Borel exceptional values, and let $c\in C\setminus\{0\}$. For every positive integer n, if $\Delta^n f(z)\not\equiv 0$ and $c,2c,\ldots,nc$ are not periods of $f(z)$, then $\Delta^n f(z)$ assumes every finite value except possibly zero infinitely often.*

Corollary 1.2 *Let $c\in C\setminus\{0\}$ and $a\in C$. Let $f(z)$ be a transcendental meromorphic function of finite order such that*

$$N\big(r,1/(f-a)\big)+N(r,f)=S(r,f).$$

For every positive integer n, if $\Delta^n f(z)\not\equiv 0$, then $\Delta^n f(z)$ assumes every finite value except possibly zero infinitely often.

Next we give the conditions under which $\Delta^n f(z)$ assumes every finite value (including zero) infinitely often.

Theorem 1.2 *Let $f(z)$ be a transcendental meromorphic function with $1<\sigma(f)<\infty$. Let $c\in C\setminus\{0\}$ and $a\in C$. Suppose that*

$$\max\{\lambda(f-a),\lambda(1/f)\}<\sigma(f)-1.$$

Then for every positive integer n, $\Delta^n f(z)$ assumes every finite value infinitely often.

2. Lemmas for the proofs of theorems

Lemma 2.1 [6] *Let $f(z)$ be a nonconstant meromorphic function of finite order, and let η_1, η_2 be two arbitrary complex numbers. Then*

$$m\left(r,\frac{f(z+\eta_1)}{f(z+\eta_2)}\right)=S(r,f).$$

Lemma 2.2 [7, 8] *Let $f(z)$ be a nonconstant finite order meromorphic function and let $c\neq 0$ be an arbitrary complex number. Then*

$$T(r+|c|,f)=T(r,f)+S(r,f),$$

$$N(r+|c|,f)=N(r,f)+S(r,f).$$

It is shown in [5, p. 66], that for an arbitrary $c\neq 0$, the following inequalities

$$\big(1+o(1)\big)T\big(r-|c|,f(z)\big)\leq T\big(r,f(z+c)\big)\leq\big(1+o(1)\big)T\big(r+|c|,f(z)\big)$$

hold as $r \to \infty$ for a general meromorphic function. From the proof we see that the above relations are also true for $N(r, f(z+c))$. So by these relations and Lemma 2.2, we get the following lemma.

Lemma 2.3 *Let $f(z)$ be a nonconstant finite order meromorphic function and let $c \neq 0$ be an arbitrary complex number. Then*

$$T\big(r, f(z+c)\big) = T(r,f) + S(r,f),$$

$$N\big(r, f(z+c)\big) = N(r,f) + S(r,f).$$

Remark *Chiang and Feng [3] have obtained some results similar to the above Lemmas 2.1–2.3, and their work is independent from [6, 7, 8].*

Lemma 2.4 [13] *Let $f(z)$ be a transcendental meromorphic function. Let $P(f)$ be a polynomial in $f(z)$ of the form*

$$P(f) = a_n(z)f(z)^n + a_{n-1}(z)f(z)^{n-1} + \cdots + a_0(z),$$

where all coefficients $a_j(z)$ are small functions of $f(z)$ and $a_n(z) \not\equiv 0$. Then

$$T\big(r, P(f)\big) = nT(r,f) + S(r,f).$$

Lemma 2.5 [12] *Let $f(z)$ be a nonconstant meromorphic function, and suppose that*

$$\Psi(z) = a_n(z)f(z)^n + a_{n-1}(z)f(z)^{n-1} + \cdots + a_0(z)$$

has small meromorphic coefficients $a_j(z)$, $a_n(z) \not\equiv 0$. Then either

$$T(r,f) \le \overline{N}(r, 1/\Psi) + \overline{N}(r,f) + S(r,f)$$

or

$$\Psi(z) = a_n \left(f + \frac{a_{n-1}}{na_n} \right)^n.$$

Lemma 2.5 is a version of Tumura-Clunie type theorems. Next we will establish a difference analogue of Lemma 2.5. To this end, we introduce some notations. The difference polynomial $H(z,f)$ is defined by

$$H(z,f) = \sum_{\lambda \in J} a_\lambda(z) \prod_{j=1}^{\tau_\lambda} f(z+\delta_{\lambda,j})^{\mu_{\lambda,j}}, \tag{2.1}$$

where J is an index set, $\delta_{\lambda,j}$ are complex constants, $\mu_{\lambda,j}$ are nonnegative integers, and $a_\lambda(z) (\not\equiv 0)$ are small meromorphic functions of $f(z)$. The maximal total degree of $H(z,f)$ in $f(z)$ and the shifts of $f(z)$ is defined by

$$\deg_f H = \max_{\lambda \in J} \sum_{j=1}^{\tau_\lambda} \mu_{\lambda,j}.$$

For $l = 0, 1, \ldots, \deg_f H$, we define

$$J_l = \left\{ \lambda \in J \Big| \sum_{j=1}^{\tau_\lambda} \mu_{\lambda,j} = l \right\}. \tag{2.2}$$

Lemma 2.6 *Let $f(z)$ be a transcendental meromorphic function of finite order such that*

$$N(r, 1/f) + N(r, f) = S(r, f). \tag{2.3}$$

Suppose that the difference polynomial (2.1) in $f(z)$ with small meromorphic coefficients is of maximal total degree $\deg_f H \geq 1$. If there exist two different integers $m, k \in \{0, 1, \ldots, \deg_f H\}$ such that

$$\sum_{\lambda \in J_m} a_\lambda(z) \prod_{j=1}^{\tau_\lambda} f(z + \delta_{\lambda,j})^{\mu_{\lambda,j}} \not\equiv 0, \quad \sum_{\lambda \in J_k} a_\lambda(z) \prod_{j=1}^{\tau_\lambda} f(z + \delta_{\lambda,j})^{\mu_{\lambda,j}} \not\equiv 0, \tag{2.4}$$

where J_m, J_k are defined by (2.2), then $H(z, f)$ is transcendentally meromorphic and

$$T(r, f) \leq \overline{N}(r, 1/H) + S(r, f).$$

Proof. Since there exist two different integers $m,\ k \in \{0, \ldots, \deg_f H\}$ satisfying (2.4), we may assume, without losing generality, that $m > k$ and

$$\sum_{\lambda \in J_s} a_\lambda(z) \prod_{j=1}^{\tau_\lambda} f(z + \delta_{\lambda,j})^{\mu_{\lambda,j}} \equiv 0$$

for $s = m + 1, \ldots, \deg_f H$, where J_s are defined by (2.2). Thus, $H(z, f)$ takes the form

$$H(z, f) = \sum_{i=0}^{m} b_i(z) f(z)^i, \tag{2.5}$$

where for $i = 0, \ldots, m$,

$$b_i(z) = \sum_{\lambda \in J_i} a_\lambda(z) \prod_{j=1}^{\tau_\lambda} \left(\frac{f(z + \delta_{\lambda,j})}{f(z)} \right)^{\mu_{\lambda,j}}, \quad J_i = \left\{ \lambda \in J \Big| \sum_{j=1}^{\tau_\lambda} \mu_{\lambda,j} = i \right\}.$$

In particular, $b_m(z) \not\equiv 0$ and $b_k(z) \not\equiv 0$.

Since the coefficients $a_\lambda(z)$ of $H(z, f)$ are small functions of $f(z)$, we have $T(r, a_\lambda) = S(r, f)$. So by Lemma 2.1, we get

$$m(r, b_i) = S(r, f)$$

for $i = 0, 1, \ldots, m$. Moreover, by (2.3) and Lemma 2.3, we have

$$N(r, b_i) \leq \sum_{\lambda \in J_i} \left(N(r, a_\lambda) + \sum_{j=1}^{\tau_\lambda} \mu_{\lambda,j} \Big(N\big(r, f(z + \delta_{\lambda,j})\big) + N(r, 1/f) \Big) \right) + O(1) = S(r, f).$$

So

$$T(r, b_i) = S(r, f) \tag{2.6}$$

for $i = 0, \ldots, m$. By (2.5), (2.6), $b_m(z) \not\equiv 0$ and Lemma 2.4, we see that $H(z, f)$ is transcendentally meromorphic.

Applying Lemma 2.5 to (2.5), we get either

$$T(r, f) \le \overline{N}(r, 1/H) + S(r, f) \tag{2.7}$$

or

$$H(z, f) = b_m \left(f + \frac{b_{m-1}}{m b_m} \right)^m. \tag{2.8}$$

If (2.7) holds, there is nothing to prove. So in the following discussion, we assume that (2.8) holds. First we affirm that $b_{m-1} \not\equiv 0$. Otherwise, (2.8) yields

$$H(z, f) = b_m(z) f(z)^m,$$

and so by (2.5), we have

$$\sum_{i=0}^{m-1} b_i(z) f(z)^i \equiv 0.$$

By this equality and Lemma 2.4, we get $b_i(z) \equiv 0$ for $i = 0, \ldots, m-1$. This contradicts $b_k(z) \not\equiv 0$, $k < m$. Thus, $b_{m-1} \not\equiv 0$, and by (2.6) we have

$$T\left(r, \frac{b_{m-1}}{m b_m}\right) = S(r, f).$$

Applying the second main theorem for small target functions and noting (2.3), we get

$$\begin{aligned} T(r, f) &\le \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f + \frac{b_{m-1}}{m b_m}}\right) + S(r, f) \\ &= \overline{N}\left(r, \frac{1}{f + \frac{b_{m-1}}{m b_m}}\right) + S(r, f). \end{aligned}$$

Moreover, by (2.8) and $T(r, b_m) = S(r, f)$, we have

$$\overline{N}\left(r, \frac{1}{f + \frac{b_{m-1}}{m b_m}}\right) = \overline{N}\left(r, \frac{1}{H}\right) + S(r, f).$$

Therefore

$$T(r, f) \le \overline{N}(r, 1/H) + S(r, f).$$

□

3. Proof of Theorem 1.1

Part I We assume that the condition (i) in Theorem 1.1 holds. Set $g(z)=1/(f(z)-b)$. Then $g(z)$ has two Borel exceptional values $1/(a-b)$, ∞. Let $1/(a-b)=d$. By Hadamard's factorization theory, $g(z)$ takes the form

$$g(z)=h(z)e^{p(z)}+d, \tag{3.1}$$

where $p(z)$ is a polynomial and $h(z)$ is a meromorphic function satisfying $\sigma(h)<\sigma(g)$. So $\sigma(g)=\deg p\geq 1$, and $g(z)$ is of regular growth, i.e.,

$$\limsup_{r\to\infty}\frac{\log T(r,g)}{\log r}=\liminf_{r\to\infty}\frac{\log T(r,g)}{\log r}=\sigma(g). \tag{3.2}$$

By (3.2) and the fact that ∞ is a Borel exceptional value of $g(z)$, we get

$$N(r,g)=S(r,g). \tag{3.3}$$

Observe that

$$\Delta^n f(z)=\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}f(z+jc), \tag{3.4}$$

$$\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}=(1-1)^n=0, \tag{3.5}$$

where $\binom{n}{j}$ are the binomial coefficients. Substituting $f(z)=1/g(z)+b$ into (3.4) and noting (3.5), we get

$$\Delta^n f(z)=\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}\left(\frac{1}{g(z+jc)}+b\right)=\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}\frac{1}{g(z+jc)}.$$

So

$$F_n(z)=\Delta^n f(z)-s(z)=\frac{E_1(z)-s(z)\prod_{j=0}^{n}g(z+jc)}{\prod_{j=0}^{n}g(z+jc)}, \tag{3.6}$$

where

$$E_1(z)=\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}\prod_{\substack{i=0\\ i\neq j}}^{n}g(z+ic). \tag{3.7}$$

Let $g_1(z)=g(z)-d$. Then $g_1(z)$ has two Borel exceptional values 0, ∞, and $g_1(z)$ is of regular growth. So

$$N(r,g_1)=S(r,g_1),\quad N(r,1/g_1)=S(r,g_1).$$

Set

$$E_2(z)=E_1(z)-s(z)\prod_{j=0}^{n}g(z+jc).$$

Substituting (3.7) into $E_2(z)$ and then replacing $g(z)$ by $g(z) = g_1(z) + d$, we get

$$E_2(z) = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} \prod_{\substack{i=0 \\ i\neq j}}^{n} \big(g_1(z+ic)+d\big) - s(z) \prod_{j=0}^{n} \big(g_1(z+jc)+d\big). \tag{3.8}$$

By calculation, we obtain

$$-s(z)\prod_{j=0}^{n}\big(g_1(z+jc)+d\big) = -s(z)\prod_{j=0}^{n} g_1(z+jc) + P(z,g_1) - s(z)d^{n+1}, \tag{3.9}$$

and for $j = 0,\dots,n$,

$$\prod_{\substack{i=0 \\ i\neq j}}^{n} \big(g_1(z+ic)+d\big) = P_j(z,g_1) + d^n, \tag{3.10}$$

where $P(z,g_1)$ and $P_j(z,g_1)$ are difference polynomials in $g_1(z)$ and its shifts such that the degree of every term in $P(z,g_1)$ and $P_j(z,g_1)$ is at most n and at least 1. By (3.8)–(3.10) and noting (3.5), we get

$$E_2(z) = -s(z)\prod_{j=0}^{n} g_1(z+jc) + P(z,g_1) + \sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}P_j(z,g_1) - s(z)d^{n+1}.$$

Since $s(z) \not\equiv 0$ and $d = 1/(a-b) \neq 0$, we have $-s(z)\prod_{j=0}^{n} g_1(z+jc) \not\equiv 0$ and $-s(z)d^{n+1} \not\equiv 0$. So by Lemma 2.6, we get $E_2(z) \not\equiv 0$ and

$$T(r,g_1) \le \overline{N}(r,1/E_2) + S(r,g_1).$$

By the above results and noting that $F_n(z) = E_2(z)/\prod_{j=0}^{n} g(z+jc)$ and $g_1(z) = g(z) - d$, we obtain

$$F_n(z) \not\equiv 0$$

and

$$T(r,g) \le \overline{N}(r,1/E_2) + S(r,g). \tag{3.11}$$

In order to estimate the zeros of $F_n(z)$, we proceed to discuss the common zeros of $E_2(z)$ and $\prod_{j=0}^{n} g(z+jc)$. Let z_0 be such a common zero. Then z_0 is a zero of $E_1(z)$ or a pole of $s(z)$. Assume that z_0 is a zero of $E_1(z)$ and that

$$g(z_0+jc) \neq \infty \tag{3.12}$$

for $j = 0,\dots,n$. Since $\prod_{j=0}^{n} g(z_0+jc) = 0$, there exists an integer $l \in \{0,\dots,n\}$ such that $g(z_0+lc) = 0$. By (3.7), (3.12) and the fact that $g(z_0+lc) = 0$, $E_1(z_0) = 0$, we get

$$\binom{n}{l}(-1)^{n-l}\prod_{\substack{i=0 \\ i\neq l}}^{n} g(z_0+ic) = 0.$$

This equality shows that there exists an integer $s \in \{0,\dots,n\}\setminus\{l\}$ such that $g(z_0+sc) = 0$. So we have

$$g(z_0+lc) - g(z_0+sc) = 0.$$

Since $g(z)=1/(f(z)-b)$ and $c,2c,\ldots,nc$ are not periods of $f(z)$, we have

$$g(z+lc)-g(z+sc)\not\equiv 0.$$

Thus, the integrated counting function of the common zeros of $E_2(z)$ and $\prod_{j=0}^{n}g(z+jc)$, denoted by $N_1(r)$, satisfies

$$N_1(r)\le N(r,s)+\sum_{j=0}^{n}N\big(r,g(z+jc)\big)+\sum_{\substack{l\neq s\\ l,s\in\{0,\ldots,n\}}}N\left(r,\frac{1}{g(z+lc)-g(z+sc)}\right).$$

By (3.3), Lemma 2.3 and $T(r,s)=S(r,g)$, the above inequality becomes

$$N_1(r)\le S(r,g)+\sum_{\substack{l\neq s\\ l,s\in\{0,\ldots,n\}}}N\left(r,\frac{1}{g(z+lc)-g(z+sc)}\right). \tag{3.13}$$

Since $p(z)$ in (3.1) is a polynomial of degree $\deg p=\sigma(g)\ge 1$, we have

$$p(z)=a_m z^m+p_1(z),$$

where $a_m(\neq 0)$ is a constant, $m=\sigma(g)\ge 1$, and $p_1(z)$ is a polynomial of degree at most $m-1$. For $l\neq s$,

$$p(z+lc)-p(z+sc)=cm(l-s)a_m z^{m-1}+\ldots=p_{l,s}(z), \tag{3.14}$$

where $p_{l,s}(z)$ are polynomials of degree $m-1$. By (3.1) and (3.14), we have

$$g(z+lc)-g(z+sc)=\big(h(z+lc)e^{p_{l,s}(z)}-h(z+sc)\big)e^{p(z+sc)}.$$

Since $\sigma(h)<\sigma(g)$ and $\sigma(e^{p_{l,s}(z)})<\sigma(g)$, it follows by (3.2) and Lemma 2.3 that

$$T\big(r,h(z+lc)e^{p_{l,s}(z)}-h(z+sc)\big)=S(r,g).$$

So for $l,s\in\{0,\ldots,n\}$, $l\neq s$, we have

$$N\left(r,\frac{1}{g(z+lc)-g(z+sc)}\right)=S(r,g). \tag{3.15}$$

By (3.13) and (3.15), we get

$$N_1(r)\le S(r,g). \tag{3.16}$$

Since $N_1(r)$ denotes the common zeros of $E_2(z)$ and $\prod_{j=0}^{n}g(z+jc)$, it follows from (3.6) that

$$N(r,1/F_n)\ge N(r,1/E_2)-N_1(r).$$

Combining this inequality with (3.11) and (3.16), we get

$$T(r,g)\le N(r,1/F_n)+S(r,g).$$

Moreover, $g(z)=1/(f(z)-b)$. So

$$T(r,f)\le N(r,1/F_n)+S(r,f). \tag{3.17}$$

By (3.17), $F_n(z)\not\equiv 0$ and noting that $f(z)$ is a transcendental meromorphic function, we see that $F_n(z)$ is transcendentally meromorphic.

By $F_n(z)=\Delta^n f(z)-s(z)$ and (3.4), we get

$$T(r,F_n)\le \sum_{j=0}^{n} T\big(r,f(z+jc)\big)+S(r,f).$$

So by Lemma 2.3, $T(r,F_n)$ satisfies

$$T(r,F_n)\le (n+1)T(r,f)+S(r,f). \tag{3.18}$$

Combining (3.17) and (3.18), we get

$$\delta(0,F_n)=1-\limsup_{r\to\infty}\frac{N(r,1/F_n)}{T(r,F_n)}\le n/(n+1).$$

Part II We assume that the condition (ii) in Theorem 1.1 holds. Set $g_2(z)=f(z)-a$. Then 0, ∞ are Borel exceptional values of $g_2(z)$, and

$$N(r,g_2)=S(r,g_2),\quad N(r,1/g_2)=S(r,g_2).$$

Substituting $f(z)=g_2(z)+a$ into $F_n(z)$, we get

$$F_n(z)=\Delta^n\big(g_2(z)+a\big)-s(z)=\Delta^n g_2(z)-s(z).$$

Since $\Delta^n g_2(z)=\Delta^n f(z)\not\equiv 0$ and $s(z)\not\equiv 0$, by Lemma 2.6, it follows that $F_n(z)$ is transcendentally meromorphic and

$$T(r,g_2)\le \overline{N}(r,1/F_n)+S(r,g_2),$$

and so

$$T(r,f)\le \overline{N}(r,1/F_n)+S(r,f). \tag{3.19}$$

Moreover, we still have (3.18). By (3.18) and (3.19), we get

$$\delta(0,F_n)\le n/(n+1).$$

4. Proof of Theorem 1.2

Since $\max\{\lambda(f-a),\lambda(1/f)\}<\sigma(f)-1$ and $1<\sigma(f)<\infty$, we have

$$f(z)=a+h(z)e^{q(z)}, \tag{4.1}$$

where $q(z)$ is a polynomial of degree $\deg q=\sigma(f)>1$, and $h(z)$ is a nonzero meromorphic function satisfying $\sigma(h)<\sigma(f)-1$. Let

$$q(z)=d_k z^k+\widetilde{q}(z),$$

where $d_k(\neq 0)$ is a constant, $k=\sigma(f)>1$, and $\widetilde{q}(z)$ is a polynomial of degree at most $k-1$. For $j=1,\ldots,n$,

$$q(z+jc)-q(z)=jkd_kcz^{k-1}+q_j(z), \tag{4.2}$$

where $q_j(z)$ are polynomials of degree at most $k-2$. Let $q_0(z)\equiv 0$. By (3.4), (3.5), (4.1) and (4.2), we have

$$\begin{aligned}\Delta^n f(z) &= \sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}f(z+jc)\\ &= \sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}\left(a+h(z+jc)e^{q(z+jc)}\right)\\ &= e^{q(z)}\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}h(z+jc)e^{q_j(z)}e^{jkd_kcz^{k-1}}.\end{aligned}$$

Set

$$T(z)=\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}h(z+jc)e^{q_j(z)}e^{jkd_kcz^{k-1}},\quad t(z)=e^{kd_kcz^{k-1}}.$$

Then we have

$$T(z)=\sum_{j=0}^{n}\alpha_j(z)t^j(z), \tag{4.3}$$

where for $j=0,\ldots,n$,

$$\alpha_j(z)=\binom{n}{j}(-1)^{n-j}h(z+jc)e^{q_j(z)}\not\equiv 0. \tag{4.4}$$

Since $t(z)$ is of regular growth $\sigma(t)=k-1>0$ and noting that $\sigma(h)<k-1$ and $\sigma(e^{q_j(z)})\leq k-2$, we get

$$T(r,\alpha_j)=S(r,t) \tag{4.5}$$

for $j=0,\ldots,n$. By Lemma 2.4 and (4.3)–(4.5), we get $T(z)\not\equiv 0$. So $\Delta^n f(z)=e^{q(z)}T(z)\not\equiv 0$ and the condition (ii) in Theorem 1.1 holds. Thus, $\Delta^n f(z)$ assumes every nonzero finite value infinitely often. Moreover, applying Lemma 2.6 to (4.3), we get

$$T(r,t)\leq \overline{N}(r,1/T)+S(r,t)=\overline{N}(r,1/\Delta^n f)+S(r,t).$$

Therefore, $\Delta^n f(z)$ assumes every finite value infinitely often.

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Ranran ZHANG, Zongxuan CHEN
School of Mathematical Sciences,
South China Normal University,
Guangzhou 510631, People's Republic of CHINA
e-mails: zhrr19820315@163.com, chzx@vip.sina.com

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