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# Oscillation of third-order nonlinear delay difference equations

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**Abstract**

Third-order nonlinear difference equations of the form

$$\Delta(c_n\Delta(d_n\Delta x_n)) + p_n\Delta x_{n+1} + q_n f(x_{n-\sigma}) = 0,\ n \geq n_0$$

are considered. Here, $\{c_n\}$, $\{d_n\}$, $\{p_n\}$, and $\{q_n\}$ are sequences of positive real numbers for $n_0 \in N$, $f$ is a continuous function such that $f(u)/u \geq K > 0$ for $u \neq 0$. By means of a Riccati transformation technique we obtain some new oscillation criteria. Examples are given to illustrate the importance of the results.

**Key words and phrases:** Difference equation, Delay, Third order, Oscillation, Nonoscillation, Riccati transformation

## 1. Introduction

Consider the nonlinear delay difference equation

$$\Delta(c_n\Delta(d_n\Delta x_n)) + p_n\Delta x_{n+1} + q_n f(x_{n-\sigma}) = 0,\ n \geq n_0, \tag{1.1}$$

where $n_0 \in N$ is a fixed integer, $\Delta$ denotes the forward difference operator $\Delta x_n = x_{n+1} - x_n$, and $\sigma$ is a nonnegative integer. The real sequences $\{c_n\}_{n=n_0}^{\infty}$, $\{d_n\}_{n=n_0}^{\infty}$, $\{p_n\}_{n=n_0}^{\infty}$, $\{q_n\}_{n=n_0}^{\infty}$, and the function $f$ satisfy the following conditions:

(h1) $\{d_n\}_{n=n_0}^{\infty}$ is positive, $\lim\limits_{n\to\infty} R_1(n,s) = \infty$, where $R_1(n,s) = \sum\limits_{k=s}^{n} \dfrac{1}{d_k}$ for $n > s \geq n_0$;

(h2) $\{c_n\}_{n=n_0}^{\infty}$ is positive, $\lim\limits_{n\to\infty} R_2(n,s) = \infty$, where $R_2(n,s) = \sum\limits_{k=s}^{n} \dfrac{1}{c_k}$ for $n > s \geq n_0$;

(h3) $p_n \geq 0$, $q_n \geq 0$ and $q_n \not\equiv 0$ for infinitely many values of $n \in N(n_0)$;

(h4) $f \in C(\mathbb{R},\mathbb{R})$, $f(u)/u \geq K$ for some $K > 0$ and for all $u \neq 0$.

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By a solution of eq. (1.1) we mean a nontrivial real sequence $\{x_n\}$ that is defined for $n \geq n_0 - \sigma$ and satisfies eq. (1.1) for all $n \geq n_0$. Clearly if $x_n = A_n$ for $n = n_0 - \sigma,\ n_0 - \sigma + 1, \ldots, n_0 - 1$ are given, then eq. (1.1) has a unique solution satisfying the above initial conditions. A solution $\{x_n\}$ of eq. (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise. eq. (1.1) is called nonoscillatory if all its solutions are nonoscillatory.

The oscillation problem for difference equations has been investigated in recent years; for first-order, second-order, and higher-order equations, respectively; see [15, 20], [8, 9, 11], and [19, 21]. For general theory of oscillation of difference equations, we refer to [1–3, 14], and over 500 references cited therein. Compared to the second-order difference equations, the study of third-order difference equations has received considerably less attention in the literature even though such equations arise in economics, mathematical biology, and other areas of mathematics [5]. Some recent results on third-order difference equations can be found in [10, 24–31]. However, it seems that there is much less known regarding the oscillation of eq. (1.1).

There are many papers dealing with the oscillatory and asymptotic behavior of solutions of difference and differential equations, see for instance Jiang [12], Jiang and Li [13], Li and Yeh [16], Li [17], Luo [18], Philos [22], Saker [26, 27]. The oscillatory behavior of solutions of differential equations and that of their discrete analogs may be quite different. For instance, the differential equation

$$y''' + 8y = 0$$

admits a nonoscillatory solution $y_1(t) = e^{-2t}$ and a pair of oscillatory solutions $y_2(t) = e^t \cos\sqrt{3}t$ and $y_3(t) = e^t \sin\sqrt{3}t$, but the difference equation

$$\Delta^3 x_n + 8x_n = 0,$$

which is a discrete analog of the above differential equation, has there oscillatory solutions $x_n^1 = (-1)^n$, $x_n^2 = \left(\sqrt{7}\right)^n \cos\left[n\left(\arctan\sqrt{3}/2\right)\right]$, and $x_n^3 = \left(\sqrt{7}\right)^n \sin\left[n\left(\arctan\sqrt{3}/2\right)\right]$.

We note that eq. (1.1) may be considered as a discrete analog of the delay differential equation

$$\left(c(t)\left(d(t)x'\right)'\right)' + p(t)x' + q(t)f(x(t-\sigma)) = 0. \tag{1.2}$$

For some work regarding the oscillation of eq. (1.2), we refer to Saker [27] $(p(t) \equiv 0)$ and Tiryaki and Aktas [32] and the references cited therein.

A number of dynamical behaviors of solutions of difference equations are possible; here we will only be concerned with conditions which are sufficient for every solution of eq. (1.1) to be either oscillatory or convergent to zero as $n \to \infty$.

Recently, Saker [26] has established some new conditions which are sufficient for all solutions of

$$\Delta(c_n \Delta(d_n \Delta x_n)^\gamma) + q_n f(x_{n-\sigma}) = 0,\ n \geq n_0, \tag{1.3}$$

where $\gamma \geq 1$ is quotient of odd positive integers, to be either oscillatory or tend to zero as $n \to \infty$.

Our aim in this paper is to present some new oscillation criteria for eq. (1.1) by making use of a Riccati type transformation and arguments developed for differential equations in [32]. It should be noted that the results obtained in this paper extend and improve the related ones in [26].

The paper is organized as follows. In section 2, we will present some lemmas which are useful in establishing our main results. In Section 3, we will state and prove the main results and give examples to illustrate them.

## 2. Preparatory lemmas

We begin with the following useful lemma.

**Lemma 2.1** *Suppose that*

*(h5)* $\Delta\left(c_n \Delta z_n\right)+\dfrac{p_n}{d_{n+1}} z_{n+1}=0$ *is nonoscillatory.*

*If* $\{x_n\}$ *is a nonoscillatory solution of eq. (1.1) for* $n \geq n_0$, *then there exists a* $n_1 \geq n_0$ *such that either* $x_n\left(d_n \Delta x_n\right)>0$ *or* $x_n\left(d_n \Delta x_n\right)<0$ *for all* $n \geq n_1$.

**Proof.** Suppose that $\{x_n\}$ is a nonoscillatory solution of eq. (1.1) for $n \geq n_0$. Without loss of generality, we may take $x_n>0$ and $x_{n-\sigma}>0,\ n \geq n_1 \geq n_0$. We see that $y_n=-d_n \Delta x_n$ is a solution of the second order nonhomogeneous difference equation

$$\Delta\left(c_n \Delta y_n\right)+\frac{p_n}{d_{n+1}} y_{n+1}=q_n f(x_{n-\sigma}),\ n \geq n_1. \tag{2.1}$$

Indeed, since $\{x_n\}$ is a nonoscillatory solution of eq. (1.1), we have

$$\Delta\left(c_n \Delta\left(-d_n \Delta x_n\right)\right)+\frac{p_n}{d_{n+1}}\left(-d_{n+1} \Delta x_{n+1}\right)=q_n f(x_{n-\sigma})$$

and hence

$$\Delta\left(c_n \Delta\left(d_n \Delta x_n\right)\right)+p_n \Delta x_{n+1}+q_n f(x_{n-\sigma})=0.$$

We claim that, all solutions of (2.1) are nonoscillatory. We may assume that $z_n>0$ for $n \geq n_1$. Note that $\{-z_n\}$ is also a solution. Let $y_n$ be a oscillatory solution of (2.1). There exist $n_3>n_2>n_1$ such that $y_{n_3} \geq 0,\ y_{n_3+1} \leq 0,\ y_{n_2} \leq 0$, and $y_{n_2+1} \geq 0$. Summing

$$\Delta\left(c_n\left(y_{n+1} z_n-y_n z_{n+1}\right)\right)=z_{n+1} q_n f\left(x_{n-\sigma}\right)$$

from $n_2$ to $n_3-1$, we have

$$c_{n_3}\left(y_{n_3+1} z_{n_3}-z_{n_3+1} y_{n_3}\right)-c_{n_2}\left(y_{n_2+1} z_{n_2}-z_{n_2+1} y_{n_2}\right)=\sum_{k=n_2}^{n_3-1} z_{k+1} q_k f\left(x_{k-\sigma}\right),$$

a contradiction. The proof is complete. □

**Definition 2.1** *Let $\{x_n\}$ be a solution of eq. (1.1). We say that the solution $\{x_n\}$ has property $V_2$ if there exists $n_* \geq n_0$ such that*

$$x_n\Delta x_n > 0, \quad x_n\Delta(d_n\Delta x_n) > 0, \quad x_n\Delta(c_n\Delta(d_n\Delta x_n)) \leq 0$$

*for every $n \geq n_*$.*

**Lemma 2.2** *Let the assumption (h2) hold and $\{x_n\}$ be a nonoscillatory solution of eq. (1.1) such that $x_n\,(d_n\Delta x_n) \geq 0$ for every $n \geq n_1 \geq n_0$. Then $\{x_n\}$ has property $V_2$.*

**Proof.** Let $\{x_n\}$ be an eventually positive solution of *eq.* (1.1). Then there exists an $n_1 \geq n_0$ such that $x_{n-\sigma} > 0$ for $n \geq n_1$. Since $x_n\,(d_n\Delta x_n) > 0$ for every $n \geq n_1 \geq n_0$, we have $\Delta x_n > 0$ for $n \geq n_1$. From eq. (1.1) we have $\Delta(c_n\Delta(d_n\Delta x_n)) \leq 0$ for $n \geq n_1$. Then $\Delta(d_n\Delta x_n)$ is monotone and eventually of one sign. We claim that there is a $n_2 \geq n_1$ such that for $n \geq n_2$, $\Delta(d_n\Delta x_n) > 0$. Suppose to the contrary that $\Delta(d_n\Delta x_n) \leq 0$ for $n \geq n_2$. Since $c_n > 0$ and $c_n\Delta(d_n\Delta x_n)$ is nonincreasing there exists a negative constant $C$ and an $n_3 \geq n_2$ such that $c_n\Delta(d_n\Delta x_n) \leq C$ for $n \geq n_3$. Dividing both sides by $c_n$ and summing from $n_3$ to $n-1$, we obtain

$$d_n\Delta x_n \leq d_{n_3}\Delta x_{n_3} + C\sum_{k=n_3}^{n-1}\frac{1}{c_k}.$$

Letting $n \to \infty$, we see that $d_n\Delta x_n \to -\infty$ by (h2), a contradiction with the fact that $\Delta x_n > 0$. Then $\Delta(d_n\Delta x_n) > 0$. The proof is complete. □

**Lemma 2.3** *Let $\{x_n\}$ be a solution of eq. (1.1) and $\{g_n^*\}_{n=n_0}^{\infty}$ be a sequence of integers which satisfies*

*(h6)* $g_n^* \leq n-\sigma-1$ *and* $\lim_{n\to\infty} g_n^* = \infty$.

*If $\{x_n\}$ has property $V_2$, then there exists an $n_1 \geq n_0$ such that*

$$d_{n-\sigma}\Delta x_{n-\sigma} \geq R_2\,(n-\sigma-1, g_n^*)\,c_n\Delta(d_n\Delta x_n) \text{ for } n \geq n_1.$$

**Proof.** Let $\{x_n\}$ be a solution of eq. (1.1) which has property $V_2$. Without loss of generality, we may also assume that $x_n > 0$ and $x_{n-\sigma} > 0$, $n \geq n_1 \geq n_0$. Since $\lim_{n\to\infty} g_n^* = \infty$ and $\Delta x_n > 0$, $\Delta(d_n\Delta x_n) > 0$, and $\Delta(c_n\Delta(d_n\Delta x_n)) \leq 0$ for every $n \geq n_2 \geq n_1$,

$$d_{n-\sigma}\Delta x_{n-\sigma} = d_{g_n^*}\Delta x_{g_n^*} + \sum_{k=g_n^*}^{n-\sigma-1}\frac{c_k\Delta(d_k\Delta x_k)}{c_k} \geq R_2\,(n-\sigma-1, g_n^*)\,c_n\Delta(d_n\Delta x_n)$$

and then we have

$$d_{n-\sigma}\Delta x_{n-\sigma} \geq R_2\,(n-\sigma-1, g_n^*)\,c_n\Delta(d_n\Delta x_n).$$

The proof is complete. □

**Lemma 2.4** *Let $\mu_n$ be a positive sequence defined for $n \geq n_0$ and set*

$$\phi_n = d_{n+2}\Delta\left(c_{n+1}\Delta\mu_n\right) + \mu_n p_n.$$

*Furthermore assume that the following conditions are satisfied:*

*(h7)* $\Delta\mu_n \geq 0,\ \phi_n \geq 0,\ \Delta\left(d_{n+2}\Delta\left(c_{n+1}\Delta\mu_n\right)\right) \geq 0$ *(or* $\Delta\left(\mu_n p_n\right) \leq 0$*) for* $n \geq n_0$*;*

*(h8)* $\displaystyle\sum_{n=n_0}^{\infty}\left(K\mu_n q_n - \Delta\phi_n\right) = \infty,$ *where* $K\mu_n q_n - \Delta\phi_n \geq 0$ *for* $n \geq n_0$*.*

*If (h1) holds and $\{x_n\}$ is a nonoscillatory solution of eq. (1.1) which satisfies $x_n\left(c_n\Delta x_n\right) \leq 0$ for $n$ sufficiently large, then $\lim_{n\to\infty} x_n = 0$.*

**Proof.** Let $\{x_n\}$ be a nonoscillatory solution of eq. (1.1). Without loss of generality, we may assume that $x_n > 0$ and $x_{n-\sigma} > 0$ for $n \geq n_1 \geq n_0$ for some $n_1$ sufficiently large. The proof when $\{x_n\}$ is eventually negative is similar, as the substitution $y_n = -x_n$ transforms eq. (1.1) into an equation of the same form. Since $x_n\left(c_n\Delta x_n\right) \leq 0$ for $n$ sufficiently large, $\Delta x_n$ becomes nonpositive for all $n \geq n_2$ for some $n_2 \geq n_1$. Let $\lim_{n\to\infty} x_n = \lambda \geq 0$. Assume that $\lambda \neq 0$. There exists an $n_3 \geq n_2$ such that $x_n \geq \lambda$ for $n \geq n_3$. Summing eq. (1.1) from $n_3$ to $n-1$, we obtain from (h7) that

$$\mu_n c_n \Delta(d_n\Delta x_n) \leq C_1 - \lambda\sum_{k=n_3}^{n-1}\left(K\mu_k q_k - \Delta\phi_k\right), \tag{2.2}$$

where $C_1$ is a constant. Employing (h8) we see from (2.2) that $\mu_n c_n\Delta(d_n\Delta x_n)$ must take on negative values for $n$ sufficiently large. By using (h1) we see that $x_n$ must be eventually negative, a contradiction. Hence $\lambda = 0$. This completes the proof. □

## 3. Oscillation criteria

In this section we give the main results of our paper.

**Theorem 3.1** *Assume that $(h1)$-$(h8)$ hold, and that there exists a positive sequence $\{\rho_n\}_{n=n_0}^{\infty}$ such that*

$$\limsup_{n\to\infty}\sum_{k=n_0}^{n}\left[K\rho_k q_k - \frac{d_{k-\sigma}\left(\Delta\rho_k d_{k+1} - \rho_k p_k R_2\left(k-\sigma-1, g_k^*\right)\right)^2}{4\rho_k R_2\left(k-\sigma-1, g_k^*\right)d_{k+1}^2}\right] = \infty. \tag{3.1}$$

*Then every solution $\{x_n\}$ of eq. (1.1) is either oscillatory or satisfies $x_n \to 0$ as $n \to \infty$.*

**Proof.** Let $\{x_n\}$ be a nonoscillatory solution of eq. (1.1). Without loss of generality, we may assume that $x_n > 0$ and $x_{n-\sigma} > 0$ eventually. From Lemma 2.1 it follows that $\Delta x_n > 0$ or $\Delta x_n < 0$ for $n \geq n_1 \geq n_0$. If

$\Delta x_n > 0$ for $n \geq N \geq n_1$, then $\{x_n\}$ has property $V_2$ by Lemma 2.2. We define

$$w_n = \rho_n \frac{c_n \Delta \left(d_n \Delta x_n\right)}{x_{n-\sigma}}, \quad n \geq N.$$

Then, $w_n > 0$ and in view of eq. (1.1) by employing Lemma 2.3 we have

$$\begin{aligned} \Delta \omega_n &= -\frac{f(x_{n-\sigma})}{x_{n-\sigma}} \rho_n q_n - \frac{\rho_n}{x_{n-\sigma}} p_n \Delta x_{n+1} + \frac{c_{n+1} \Delta \left(d_{n+1} \Delta x_{n+1}\right) x_{n-\sigma} \Delta \rho_n}{x_{n-\sigma} x_{n-\sigma+1}} \\ &\quad - \frac{c_{n+1} \Delta \left(d_{n+1} \Delta x_{n+1}\right) \rho_n \Delta x_{n-\sigma}}{x_{n-\sigma} x_{n-\sigma+1}} \\ &\leq -K \rho_n q_n - \left( \omega_{n+1}^2 \left( \frac{\rho_n R_2 \left(n - \sigma - 1, g_n^*\right)}{\left(\rho_{n+1}\right)^2 d_{n-\sigma}} \right) \right. \\ &\quad \left. - \omega_{n+1} \left( \frac{\Delta \rho_n}{\rho_{n+1}} - \frac{p_n \rho_n R_2 \left(n - \sigma - 1, g_n^*\right)}{d_{n+1} \rho_{n+1}} \right) \right) \\ &= -K \rho_n q_n - A_n w_{n+1}^2 + w_{n+1} B_n, \end{aligned} \tag{3.2}$$

where

$$A_n = \frac{\rho_n R_2 \left(n - \sigma - 1, g_n^*\right)}{\rho_{n+1}^2 d_{n-\sigma}}, \; B_n = \frac{\Delta \rho_n}{\rho_{n+1}} - \frac{p_n \rho_n R_2 \left(n - \sigma - 1, g_n^*\right)}{d_{n+1} \rho_{n+1}}.$$

Completing the square in (3.2) we obtain

$$\Delta w_n < -\left[ K \rho_n q_n - \frac{B_n^2}{4A_n} \right]. \tag{3.3}$$

Summing (3.3) from $N$ to $n$, we obtain

$$-w_N < w_{n+1} - w_N < -\sum_{k=N}^{n} \left[ K \rho_k q_k - \frac{B_k^2}{4A_k} \right]$$

which yields

$$\sum_{k=N}^{n} \left[ K \rho_k q_k - \frac{B_k^2}{4A_k} \right] < w_N$$

for all large $n$ and this is contrary to (3.1).

If $\Delta x_n < 0$ for $n \geq N$, then by Lemma 2.4 we have $\lim_{n \to \infty} x_n = 0$. The proof is complete. □

**Example 3.1** *Consider the third order delay difference equation*

$$\Delta^3 x_n + \frac{1}{5n^2} \Delta x_{n+1} + \left(1 - \frac{1}{5n^2}\right) x_{n-3} = 0, \quad n \geq 1. \tag{3.4}$$

*Note that* $\Delta^2 z_n + \frac{1}{5n^2} z_{n+1} = 0$ *is nonoscillatory by [1, Theorem 1.14.2]. Taking* $\mu_n = \rho_n = 1$ *and* $g_n^* = n-4$, *we have*

$$\limsup_{n\to\infty} \sum_{k=n_0}^{n} \left[ K\rho_k q_k - \frac{d_{k-\sigma}\left(\Delta\rho_k d_{k+1} - \rho_k p_k R_2\left(k-\sigma-1, g_k^*\right)\right)^2}{4\rho_k R_2\left(k-\sigma-1, g_k^*\right) d_{k+1}^2} \right] = \sum_{k=1}^{\infty} \left(1 - \frac{1}{5k^2} - \frac{1}{100k^4}\right) = \infty.$$

*Thus, condition (3.1) is satisfied. The other conditions of Theorem 3.1 are also satisfied. Hence every solution* $\{x_n\}$ *of eq. (3.4) is either oscillatory or satisfies* $x_n \to 0$ *as* $n \to \infty$. *We note that* $\{\cos\frac{n\pi}{3}\}$ *is an oscillatory solution of eq. (3.4).*

**Example 3.2** *Consider the third order difference equation*

$$\Delta^3 x_n + \frac{1}{2^{n+1}} \Delta x_{n+1} + \frac{1}{8}\left(1 + \frac{1}{2^n}\right) x_n = 0, \quad n \geq 1. \tag{3.5}$$

*Note that* $\Delta^2 z_n + \frac{1}{2^{n+1}} z_{n+1} = 0$ *is nonoscillatory ([1, Theorem 1.14.2]). Taking* $\mu_n = \rho_n = 1$ *and* $g_n^* = n-1$, *condition (3.1) is satisfied. The other conditions of Theorem 3.1 are also satisfied. Hence, every solution* $\{x_n\}$ *of eq. (3.5) is either oscillatory or satisfies* $x_n \to 0$ *as* $n \to \infty$. *Indeed, the sequence* $\{2^{-n}\}$ *is such a solution of eq. (3.5).*

**Theorem 3.2** *Assume that (h1)-(h8) hold. Let* $\{\rho_n\}_{n=n_0}^{\infty}$ *be a positive sequence and* $\{H_{m,n}\}$, $m \geq n \geq n_0$, *be a double sequence such that*

*(i)* $H_{m,m} = 0$ *for* $m \geq n_0$;

*(ii)* $H_{m,n} > 0$ *for* $m > n \geq n_0$;

*(iii)* $\Delta_2 H_{m,n} = H_{m,n+1} - H_{m,n} \leq 0$ *and* $-\Delta_2 H_{m,n} = h_{m,n}\sqrt{H_{m,n}}$ *for* $m \geq n \geq n_0$.

*If*

$$\limsup_{m\to\infty} \frac{1}{H_{m,m_0}} \sum_{n=m_0}^{m-1} \left[ K H_{m,n}\rho_n q_n - \frac{Q_{m,n}^2}{4A_n} \right] = \infty \text{ for every } m_0 \geq n_0, \tag{3.6}$$

*where*

$$Q_{m,n} = h_{m,n} - \left(\frac{\Delta\rho_n}{\rho_{n+1}} - \frac{p_n\rho_n R_2\left(n-\sigma-1, g_n^*\right)}{d_{n+1}\rho_{n+1}}\right)\sqrt{H_{m,n}}, \;\; A_n = \frac{\rho_n R_2\left(n-\sigma-1, g_n^*\right)}{\rho_{n+1}^2 d_{n-\sigma}},$$

*then every solution* $\{x_n\}$ *of eq. (1.1) is either oscillatory or* $\lim_{n\to\infty} x_n = 0$.

**Proof.** Let $\{x_n\}$ be a nonoscillatory solution of eq. (1.1), which may assume to be eventually positive. Proceeding as in the proof of Theorem 3.1 we arrive at the inequality (3.2). Then we see that

$$\sum_{n=N}^{m-1} K H_{m,n}\rho_n q_n \leq \sum_{n=N}^{m-1} H_{m,n}\left(-\Delta w_n + w_{n+1}B_n - A_n w_{n+1}^2\right)$$

$$
\begin{aligned}
&= H_{m,N} w_N + \sum_{n=N}^{m-1} \left\{ w_{n+1} \Delta_2 H_{m,n} + H_{m,n} \left( B_n w_{n+1} - A_n w_{n+1}^2 \right) \right\} \\
&= H_{m,N} w_N - \sum_{n=N}^{m-1} \left\{ w_{n+1}^2 A_n H_{m,n} + w_{n+1} \left( h_{m,n} \sqrt{H_{m,n}} - H_{m,n} B_n \right) \right\} \qquad (3.7) \\
&\leq H_{m,N} w_N + \sum_{n=N}^{m-1} \frac{\left( h_{m,n} - B_n \sqrt{H_{m,n}} \right)^2}{4A_n},
\end{aligned}
$$

where $B_n$ is as defined in the proof of Theorem 3.1. Thus we obtain

$$
\frac{1}{H_{m,N}} \sum_{n=N}^{m-1} \left[ K H_{m,n} \rho_n q_n - \frac{Q_{m,n}^2}{4A_n} \right] \leq w_N,
$$

which clearly contradicts (3.6).

If $\Delta x_n < 0$ for $n \geq N$, then by Lemma 2.4 we have $\lim_{n\to\infty} x_n = 0$. The proof is complete. □

As an immediate consequence of Theorem 3.2, we get the following corollary.

**Corollary 3.1** *Assume that all the assumptions of Theorem 3.2 hold, except that the condition (3.6) is replaced by*

$$
\limsup_{m\to\infty} \frac{1}{H_{m,m_0}} \sum_{n=m_0}^{m-1} H_{m,n} \rho_n q_n = \infty \text{ for every } m_0 \geq n_0,
$$

$$
\limsup_{m\to\infty} \frac{1}{H_{m,m_0}} \sum_{n=m_0}^{m-1} \frac{Q_{m,n}^2}{A_n} < \infty \text{ for every } m_0 \geq n_0.
$$

*Then, every solution $\{x_n\}$ of eq. (1.1) is either oscillatory or $\lim_{n\to\infty} x_n = 0$.*

**Example 3.3** *Consider the third order delay difference equation*

$$
\Delta^3 x_n + \frac{9}{2^{n+1}} \Delta x_{n+1} + \frac{27}{32} \left( 1 - \frac{1}{2^n} \right) x_{n-2} = 0. \qquad (3.8)
$$

*Taking $\mu_n = \rho_n = 1$, $g_n^* = n - 3$ and $H_{m,n} = m - n$ condition (3.6) is satisfied. The other conditions of Theorem 3.2 are also satisfied. Hence every solution $\{x_n\}$ of eq. (3.8) is either oscillatory or satisfies $x_n \to 0$ as $n \to \infty$. The sequence $\{(-\frac{1}{2})^n\}$ is such a solution of eq. (3.8).*

**Example 3.4** *Consider the third order delay difference equation*

$$
\Delta^3 x_n + \frac{27}{32} x_{n-2} = 0, \quad n \geq 1. \qquad (3.9)
$$

*Taking* $\mu_n = \rho_n = 1,\ g_n^* = n-3$ *and* $H_{m,n} = m-n,$ *we have*

$$\limsup_{m\to\infty}\frac{1}{H_{m,n_0}}\sum_{n=n_0}^{m-1}\left[KH_{m,n}\rho_n q_n - \frac{Q_{m,n}^2}{4A_n}\right] = \limsup_{m\to\infty}\frac{1}{m-1}\sum_{n=1}^{m-1}\left[\frac{27}{32}(m-n) - \frac{1}{4(m-n)}\right] = \infty.$$

*Thus, condition (3.6) is satisfied. The other conditions of Theorem 3.2 are also satisfied. Hence every solution* $\{x_n\}$ *of eq. (3.9) is either oscillatory or satisfies* $x_n \to 0$ *as* $n\to\infty$. *The sequence* $\{2^{-n}\}$ *is a solution of eq. (3.9).*

**Remark 1** *One may choose* $\{H_{m,n}\}$ *in appropriate manners, to derive several special oscillation criteria for eq. (1.1). Some choices are*

$$H_{m,n} = (m-n)^{\lambda}, \quad \lambda \geq 1,\ m\geq n\geq n_0,$$

$$H_{m,n} = \left(\log\frac{m+1}{n+1}\right)^{\lambda}, \quad \lambda\geq 1,\ m\geq n\geq n_0,$$

$$H_{m,n} = (m-n)^{(\lambda)}, \quad \lambda > 2,\ m\geq n\geq n_0,$$

*where* $(m-n)^{(\lambda)} = (m-n)(m-n+1)...(m-n+\lambda-1)$.

**Theorem 3.3** *Let* $\{H_{m,n}\}$ *and* $\{h_{m,n}\}$ *be as in Theorem 3.2 and let*

$$0 < \inf_{n\geq n_0}\left[\liminf_{m\to\infty}\frac{H_{m,n}}{H_{m,n_0}}\right] \leq \infty, \tag{3.10}$$

$$\limsup_{m\to\infty}\frac{1}{H_{m,n_0}}\sum_{n=n_0}^{m-1}\frac{Q_{m,n}^2}{A_n} < \infty. \tag{3.11}$$

*If there is a sequence* $\{\psi_N\}$ *such that*

$$\limsup_{m\to\infty}\frac{1}{H_{m,N}}\sum_{n=N}^{m-1}\left[KH_{m,n}\rho_n q_n - \frac{Q_{m,n}^2}{4A_n}\right] \geq \psi_N \text{ for every } N\geq n_0, \tag{3.12}$$

*and*

$$\sum_{n=n_0}^{\infty} A_n\left[\psi_{n+1}^{+}\right]^2 = \infty, \text{where } \psi_{n+1}^{+} = \max\{\psi_{n+1}, 0\}, \tag{3.13}$$

*then every solution* $\{x_n\}$ *of eq. (1.1) is either oscillatory or* $\lim_{n\to\infty} x_n = 0$.

**Proof.** As in the proof of Theorem 3.2, we have (3.7). Then, we have

$$\sum_{n=N}^{m-1} KH_{m,n}\rho_n q_n \leq H_{m,N}w_N + \sum_{n=N}^{m-1}\frac{Q_{m,n}^2}{4A_n} - \sum_{n=N}^{m-1}\left[w_{n+1}\sqrt{A_nH_{m,n}} + \frac{Q_{m,n}}{2\sqrt{A_n}}\right]^2. \tag{3.14}$$

The remainder of the proof of this case is similar to ones given in [16, 18] and hence is omitted. □

**Example 3.5** *Consider the third order difference equation*

$$\Delta^2\left(\frac{1}{n^3}\Delta x_n\right)+\frac{2n+1}{\left[n\left(n+1\right)\right]^2}\left(x_n+x_n^3\right)=0,\quad n\geq 1. \tag{3.15}$$

*We take* $\mu_n=n^2,\ \rho_n=1,\ g_n^*=n-1,\ H_{m,n}=m-n,$ *and* $\psi_n=\frac{1}{n^2}$. *In view of* $Q_{m,n}=\frac{1}{\sqrt{m-n}}$ *and* $A_n=n^3,$ *we see that*

$$\limsup_{m\to\infty}\frac{1}{H_{m,n_0}}\sum_{n=n_0}^{m-1}\frac{Q_{m,n}^2}{A_n}=\limsup_{m\to\infty}\frac{1}{m-1}\sum_{n=1}^{m-1}\frac{1}{(m-n)\,n^3}=0<\infty,$$

$$\sum_{n=n_0}^{\infty}A_n\left[\psi_{n+1}^{+}\right]^2=\sum_{n=1}^{\infty}\frac{n^3}{\left(n+1\right)^4}=\infty,$$

*and*

$$\limsup_{m\to\infty}\frac{1}{H_{m,N}}\sum_{n=N}^{m-1}\left[KH_{m,n}\rho_nq_n-\frac{Q_{m,n}^2}{4A_n}\right]$$

$$=\limsup_{m\to\infty}\frac{1}{m-N}\sum_{n=N}^{m-1}\left[(m-n)\left(\frac{1}{n^2}-\frac{1}{\left(n+1\right)^2}-\frac{1}{4\left(m-n\right)n^3}\right)\right]=\frac{1}{N^2}=\psi_N.$$

*Since the conditions of Theorem 3.3 hold, every solution* $\{x_n\}$ *of eq. (3.15) is either oscillatory or satisfies* $x_n\to 0$ *as* $n\to\infty$.

**Theorem 3.4** *Let* $\{H_{m,n}\}$ *and* $\{h_{m,n}\}$ *be as in Theorem 3.2, and let (3.10) hold. Suppose that*

$$\liminf_{m\to\infty}\frac{1}{H_{m,n_0}}\sum_{n=n_0}^{m-1}KH_{m,n}\rho_nq_n<\infty, \tag{3.16}$$

*and there is a sequence* $\{\psi_N\}$ *satisfying (3.13) and*

$$\liminf_{m\to\infty}\frac{1}{H_{m,N}}\sum_{n=N}^{m-1}\left[KH_{m,n}\rho_nq_n-\frac{Q_{m,n}^2}{4A_n}\right]\geq\psi_N\ \text{for every}\ N\geq n_0. \tag{3.17}$$

*Then every solution* $\{x_n\}$ *of eq. (1.1) either oscillatory or* $\lim_{n\to\infty}x_n=0$.

**Proof.** The proof of Theorem 3.4 is similar to that of Theorem 3.3 and hence is omitted. □

**Theorem 3.5** *Assume that* $(h1)$*-*$(h8)$ *hold. Suppose there exist a positive sequence* $\{\rho_n\}_{n=n_0}^{\infty}$ *and a sequence* $\{F_{m,n}\}_{m,n=n_0}^{\infty}$ *such that*

$$1+\frac{F_{m,n}}{\rho_{n+1}}+\frac{p_n\rho_nR_2}{d_{n+1}\rho_{n+1}}-\frac{\Delta\rho_n}{\rho_{n+1}}\geq 0$$

*and*

$$\limsup_{m\to\infty} \sum_{n=n_0}^{m} \left[ \prod_{k=n_0}^{n-1} \left(1 + \frac{F_{m,k}}{\rho_{k+1}} - B_k\right)\right] \left(K\rho_n q_n - \frac{1}{4A_n}\left(\frac{F_{m,n}}{\rho_{n+1}}\right)^2\right) = \infty, \tag{3.18}$$

*then every solution* $\{x_n\}$ *of eq. (1.1) is either oscillatory or* $\lim_{n\to\infty} x_n = 0$.

**Proof.** Let $\{x_n\}$ be a nonoscillatory solution of eq. (1.1). Without loss of generality, we may assume that $x_n > 0$ and $x_{n-\sigma} > 0$ for all $n \ge N$ for some $N \ge n_0$. Proceeding as in the proof of Theorem 3.1 we arrive at

$$\Delta\omega_n \le -K\rho_n q_n + \omega_{n+1}B_n - \omega_{n+1}^2 \left(\frac{\rho_n R_2}{\rho_{n+1}^2 d_{n-\sigma}}\right), \quad n \ge N. \tag{3.19}$$

From (3.19) and Young's inequality, we have

$$\Delta\omega_n \le -K\rho_n q_n + \omega_{n+1}B_n - \omega_{n+1}^2 \frac{\rho_n R_2}{\rho_{n+1}^2 d_{n-\sigma}} - \frac{1}{4A_n}\left(\frac{F_{m,n}}{\rho_{n+1}}\right)^2 + \frac{1}{4A_n}\left(\frac{F_{m,n}}{\rho_{n+1}}\right)^2, \quad n \ge N$$

or

$$\omega_{n+1} - \omega_n \le -K\rho_n q_n + \omega_{n+1}\left(B_n - \frac{F_{m,n}}{\rho_{n+1}}\right) + \frac{1}{4A_n}\left(\frac{F_{m,n}}{\rho_{n+1}}\right)^2, \quad n \ge N.$$

It follows that

$$\sum_{n=N}^{m} \left[ \prod_{k=N}^{n-1} \left(1 + \frac{F_{m,k}}{\rho_{k+1}} - B_k\right)\right] \left(K\rho_n q_n - \frac{1}{4A_n}\left(\frac{F_{m,n}}{\rho_{n+1}}\right)^2\right) \le \omega_N.$$

Hence

$$\limsup_{m\to\infty} \sum_{n=N}^{m} \left[ \prod_{k=N}^{n-1} \left(1 + \frac{F_{m,k}}{\rho_{k+1}} - B_k\right)\right] \left(K\rho_n q_n - \frac{1}{4A_n}\left(\frac{F_{m,n}}{\rho_{n+1}}\right)^2\right) \le \omega_N,$$

which contradicts (3.18).

If $\Delta x_n < 0$ for $n \ge N$, then by Lemma 2.4 we have $\lim_{n\to\infty} x_n = 0$. The proof is complete. □

**Example 3.6** *Consider the third order difference equation*

$$\Delta^2\left(\frac{1}{n^5}\Delta x_n\right) + \frac{1}{4n^3}x_n\left(\beta + e^{x_n}\right) = 0, \quad n \ge 1, \tag{3.20}$$

*where* $\beta > 1$ *and* $f(u) = u(\beta + e^u)$ *with* $K = \beta$. *Taking* $\mu_n = n^2$, $\rho_n = n$, $g_n^* = n-1$, *and* $F_{m,n} = n^2$, *we have*

$$\limsup_{m\to\infty} \sum_{n=1}^{m} \left[ \prod_{k=1}^{n-1} \left(1 + \frac{F_{m,k}}{\rho_{k+1}} - B_k\right)\right] \left(K\rho_n q_n - \frac{1}{4A_n}\left(\frac{F_{m,n}}{\rho_{n+1}}\right)^2\right)$$

$$= \frac{\beta - 1}{4}\limsup_{m\to\infty} \sum_{n=1}^{\infty} \frac{(n-1)!}{n^2} = \infty$$

*Thus, condition (3.18) is satisfied. The other conditions of Theorem 3.6 are also satisfied. Hence every solution* $\{x_n\}$ *of eq. (3.20) is either oscillatory or satisfies* $x_n \to 0$ *as* $n \to \infty$.

In the proof of following theorem we use a generalized Riccati transformation technique.

**Theorem 3.6** *Assume that* (h1)-(h8) *hold. Let* $\{\rho_n\}_{n=n_0}^{\infty}$ *be a positive sequence. Furthermore, we assume that there exists a double sequence* $\{H_{m,n} : m \geq n \geq n_0\}$ *such that* (i)-(iii). *If*

$$\limsup_{m\to\infty} \frac{1}{H_{m,m_0}} \sum_{n=m_0}^{m-1} \left[ H_{m,n}\Psi_n - \frac{h_{m,n}^2}{4A_n} \right] = \infty \text{ for every } m_0 \geq n_0, \tag{3.21}$$

*where*

$$\Psi_n = \rho_n \left( Kq_n - \frac{p_n^2 d_{n-\sigma} R_2}{4d_{n+1}^2} - \Delta\left(c_{n-\sigma}\alpha_{n-1}\right) \right), \quad \alpha_n = -\frac{\left(\Delta\rho_n d_{n+1} - p_n\rho_n R_2\right) d_{n-\sigma}}{2d_{n+1}\rho_n R_2 c_{n-\sigma+1}}.$$

*Then every solution* $\{x_n\}$ *of eq. (1.1) is either oscillatory or* $\lim_{n\to\infty} x_n = 0$.

**Proof.** We proceed as in Theorem 3.1, take $x_{n-\sigma} > 0$ for all $n \geq N$ for some $N$ sufficiently large. Define

$$w_n = \rho_n \left[ \frac{c_n \Delta\left(d_n \Delta x_n\right)}{x_{n-\sigma}} + c_{n-\sigma}\alpha_{n-1} \right], \quad n \geq N.$$

Then follows the proof of Theorem 3.1, we obtain

$$\Delta\omega_n \leq -K\rho_n q_n + \frac{\omega_{n+1}}{\rho_{n+1}} \left( \Delta\rho_n - \frac{p_n\rho_n R_2}{d_{n+1}} \right) - \left( \frac{\rho_n R_2}{d_{n-\sigma}} \right) \left[ \frac{\omega_{n+1}}{\rho_{n+1}} - c_{n-\sigma+1}\alpha_n \right]^2$$

$$+\rho_n \Delta\left(c_{n-\sigma}\alpha_{n-1}\right) + \frac{p_n\rho_n R_2}{d_{n+1}} c_{n-\sigma+1}\alpha_n = -\Psi_n - A_n\omega_{n+1}^2, \quad n \geq N.$$

The remainder of proof is similar to that of the proof of Theorem 3.2 and hence is omitted.

If $\Delta x_n < 0$ for $n \geq N$, then by Lemma 2.4 we have $\lim_{n\to\infty} x_n = 0$. The proof is complete. □

**Example 3.7** *Consider the third order difference equation*

$$\Delta^3 x_n + 2c^{1-1/2^n} \left(\sqrt{c} - 1\right)^3 x_n \left(4 - \cos x_n\right) = 0, \quad c > 1, \quad n \geq 1. \tag{3.22}$$

*Taking* $\mu_n = \rho_n = c^{1/2^n}$, $g_n^* = n - 1$ *and* $H_{m,n} = m - n$, *we have*

$$\limsup_{m\to\infty} \frac{1}{H_{m,m_0}} \sum_{n=m_0}^{m-1} \left[ H_{m,n}\Psi_n - \frac{h_{m,n}^2}{4A_n} \right]$$

$$= \limsup_{m\to\infty} \frac{1}{m - m_0} \sum_{n=m_0}^{m-1} \left[ 6c\left(\sqrt{c} - 1\right)^3 (m - n) - \frac{1}{4\left(m - n\right)} \right] = \infty.$$

*Thus, condition (3.21) is satisfied. The other conditions of Theorem 3.6 are also satisfied. Hence every solution* $\{x_n\}$ *of eq. (3.22) is either oscillatory or* $\lim_{n\to\infty} x_n = 0$.

**Corollary 3.2** *Assume that all the assumptions of Theorem 3.5 hold, except that the condition (3.21) is replaced by*

$$\limsup_{m\to\infty}\frac{1}{H_{m,m_0}}\sum_{n=m_0}^{m-1}H_{m,n}\rho_n\left(Kq_n-\frac{p_n^2\rho_n d_{n-\sigma}R_2}{4d_{n+1}^2}-\Delta\left(c_{n-\sigma}\alpha_{n-1}\right)\right)=\infty \text{ for every } m_0\geq n_0,$$

$$\limsup_{m\to\infty}\frac{1}{H_{m,m_0}}\sum_{n=m_0}^{m-1}\frac{h_{m,n}^2}{A_n}<\infty \text{ for every } m_0\geq n_0.$$

*Then, every solution* $\{x_n\}$ *of eq. (1.1) is either oscillatory or* $\lim_{n\to\infty}x_n=0$.

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## References


[1] Agarwal, R.P., Bohner, M., Grace, S.R., O'Regan, D.: Discrete Oscillation Theory, Hindawi Publishing Corporation. New York. 2005.

[2] Agarwal, R.P.: Difference Equations and Inequalities, Theory, Methods and Applications, Second Edition, Revised and Expanded. New York. Marcel Dekker 2000.

[3] Agarwal, R.P., Wong, P.J.Y.: Advanced Topics in Difference Equations, Kluwer Academic Publishers, Drodrecht. 1997.

[4] Andruch-Sobiło, A., Migda, M.: On the oscillation of solutions of third order linear difference equations of neutral type. Math. Bohem. 130, 19–33 (2005).

[5] Artzroumi, M.: Generalized stable population theory. J. Math. Biology. 21, 363-381 (1985).

[6] Došlá, Z., Kobza, A.: Global asymptotic properties of third-order difference equations. Comput. Math. Appl. 48, 191–200 (2004).

[7] Došlá, Z., Kobza, A.: On third-order linear difference equations involving quasi-differences. Adv. Difference Equ. 1–13 (2006).

[8] Grace, S.R., Lalli, B.S.: Oscillation theorems for forced neutral difference equations. J. Math. Anal. Appl. 187, 91–106 (1994).

[9] Grace, S.R., Lalli, B.S.: Oscillation theorems for second order delay and neutral difference equations. Utilitas Math. 45, 197–211 (1994).

[10] Graef, J., Thandapani, E.: Oscillatory and asymptotic behavior of solutions of third order delay difference equations. Funk. Ekvac. 42, 355–369 (1999).

[11] Hooker, J.W., Patula, W.T.: A second-order nonlinear difference equation: Oscillation and asymptotic behavior. J. Math. Anal. Appl. 91, 9–29 (1983).

[12] Jiang, J.: Oscillation of second order nonlinear neutral delay difference equations. Appl. Math. Comput. 146, 791–801 (2003).

[13] Jiang, J., Li, X.: Oscillation of second order nonlinear neutral differential equations. Appl. Math. Comput. 135, 531–540 (2003).

[14] Kelley, W.G., Peterson, A.C.: Difference Equations; An Introduction with Applications. New York. Academic Press 1991.

[15] Ladas, G.: Explicit conditions for oscillations of difference equations. J. Math. Anal. Appl. 153, 276-287 (1990).

[16] Li, H.-J., Yeh, C.-C.: Oscillation criteria for second-order neutral delay difference equations. Advances in difference equations, II. Comput. Math. Appl. 36, 123–132 (1998).

[17] Li, H.-J.: Oscillation criteria for second order linear differential equations. J. Math. Anal. Appl. 194, 217–234 (1995).

[18] Luo, J.: Oscillation criteria for second-order quasi-linear neutral difference equations. Comput. Math. Appl. 43, 1549–1557 (2002).

[19] Philos, Ch.G.: Oscillations in a class of difference equations. Appl. Math. Comput. 48, 45–57 (1992).

[20] Philos, Ch.G.: On oscillations of some difference equations. Funk. Ekvac. 34, 157–172 (1991).

[21] Philos, Ch.G., Sficas, Y.G.: Positive solutions of difference equations. Proc. Amer. Math. Soc. 108, 107–115 (1990).

[22] Philos, Ch.G.: Oscillation theorems for linear differential equations of second order. Arch. Math. 53, 482–492 (1989).

[23] Popenda, J., Schmeidel, E.: Nonoscillatory solutions of third order difference equations. Portugal. Math. 49, 233–239 (1992).

[24] Saker, S.H.: Oscillation and asymptotic behavior of third-order nonlinear neutral delay difference equations. Dynam. Systems Appl. 15, 549–567 (2006).

[25] Saker, S.H.: Oscillation of third-order difference equations. Port. Math. 61, 249–257 (2004).

[26] Saker, S.H.: Oscillation and asymptotic behavior of third-order nonlinear delay difference equations. J. Contemp. Math. to appear.

[27] Saker, S.H.: Oscillation criteria of third-order nonlinear delay differential equations. Math. Slovaca. 56, 433–450 (2006).

[28] Smith, B.: Linear third-order difference equations: Oscillatory and asymptotic behavior. Rocky Mountain J. Math. 22, 1559–1564 (1992).

[29] Smith, B., Taylor, Jr.W.E.: Nonlinear third order difference equations: Oscillatory and asymptotic behavior. Tamakang J. Math. 19, 91–95 (1988).

[30] Smith, B.: Oscillation and nonoscillation theorems for third order quasi adjoint difference equations. Port. Math. 45, 229–243 (1988).

[31] Smith, B., Taylor, Jr.W.E.: Asymptotic behavior of solutions of a third order difference equation. Port. Math. 44, 113–117 (1987).

[32] Tiryaki, A., Aktas, M.F.: Oscillation criteria of a certain class of third order nonlinear delay differential equations with damping. J. Math. Anal. Appl. 325, 54–68 (2007).

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