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Simultaneous proximinality of vector valued function spaces

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Abstract

A characterization of best simultaneous approximation of Köthe spaces of vector-valued functions is given. This characterization is a generalization of some analogous theorems for Orlicz Bochner spaces.

Key words and phrases: Simultaneous approximation; Köthe Bochner function space

1. Introduction

Through this paper, let $(T, \sum, \mu)$ be a finite complete measure space and $L^0 = L^0(T)$ denote the space of all (equivalence classes) of Σ-measurable real valued functions. For $f, g \in L^0$, $f \leq g$ means that $f(t) \leq g(t)$ μ-almost every where $t \in T$. A Banach space $(E, \|\cdot\|_E)$ is said to be a Köthe space if

(1) for $f, g \in L^0$, $|f| \leq |g|$ and $g \in E$ imply $f \in E$ and $\|f\|_E \leq \|g\|_E$;

(2) for each $A \in \Sigma$, if $\mu(A)$ is finite then $\chi_A \in E$. See [7, p. 28].

A Köthe space E has absolutely continuous norm if for each $f \in E$ and each decreasing sequence (A_n) converges to 0, then $\|\chi_{A_n} f\|_E \to 0$. A Köthe space E is said to be strictly monotone if $x \geq y \geq 0$ and $\|x\|_E = \|y\|_E$ imply $x = y$. Let E be a Köthe space on the measure space $(T, \sum, \mu)$ and $(X, \|\cdot\|_X)$ be a real Banach space then $E(X)$ is the space (of all equivalence classes) of strongly measurable functions $f : T \to X$ such that $\|f(\cdot)\|_X \in E$ equipped with the norm

$$|||f||| = \|\|f(\cdot)\|_X\|_E .$$

The space $(E(X), |||\cdot|||_E)$ is a Banach space called the Köthe Bochner function space [7]. For a function $F = (f_1, f_2, \ldots, f_n) \in (E(X))^n$, we define the norm of F by

$$|||F||| = \left\| \sum_{i=1}^{n} \|f_i(\cdot)\|_X \right\|_E .$$

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The most important classes of Köthe Bochner function spaces are the Lebesgue Bochner spaces $L^p(X)$, $(1 \leq p < \infty)$ and their generalization the Orlicz-Bochner spaces $L^{\Phi}(X)$. These spaces have been studied by many authors, cf. [2], [4], [5], [6], [8], [10].

Let Y be a closed subspace of X. For a set of elements $x_1, x_2, \ldots, x_n \in X$, define

$$dist\,(x_1, x_2, \ldots, x_n, Y) = \inf_{z \in Y} \left\{ \sum_{i=1}^{n} \|x_i - z\| \right\}.$$

We say that $y_0 \in Y$ is a best simultaneous approximation to the set of elements $x_1, x_2, \ldots, x_n \in X$ if, for every $z \in Y,$ we have

$$\sum_{i=1}^{n} \|x_i - y_0\| \leq \sum_{i=1}^{n} \|x_i - z\|.$$

If every set of elements $x_1, x_2, \ldots, x_n \in X$ admits a best simultaneous approximation in Y, then Y is said to be simultaneously proximinal in X. In case when $n = 1$, we get the usual proximinality.

In this paper, for a given closed subspace Y of X and $F = (f_1, f_2, \ldots, f_n) \in (E(X))^n$, we are interested in the existence of n-tuples $G_0 = (g_0, g_0, \ldots, g_0) \in (E(Y))^n$ such that

$$|||F - G_0||| = \inf_{g \in E(Y)} \|F - (g, g, \ldots, g)\|.$$

If such a function g_0 exists, it is called a best simultaneous approximation of $F = (f_1, f_2, \ldots, f_n)$. The problem of best simultaneous approximation can be viewed as a special case of vector valued approximation. Recent results in this area are due to Pinkus [9]. Results on best simultaneous approximation in general Banach spaces can also be found in [1], [11], [12].

It is the aim of this work to write and prove a formula for the distance $dist_E(f_1, f_2, \ldots, f_n, E(Y))$, where $f_1, f_2, \ldots, f_n \in E(X)$, similar to that of best approximation. This allows us to generalize some recent results in [3].

2. Distance formula

Through this section, X is a real Banach space and $E(X)$ is a Köthe Bochner function space. For $f_1, f_2, \ldots, f_n \in E(X)$, we define $dist_E(f_1, \ldots, f_n, E(Y))$ by

$$\begin{aligned} dist_E(f_1, f_2, \ldots, f_n, E(Y)) &= \inf_{g \in E(Y)} |||(f_1, f_2, \ldots, f_n) - (g, g, \ldots, g)||| \\ &= \inf_{g \in E(Y)} \left\| \sum_{i=1}^{n} \|f_i(\cdot) - g(\cdot)\|_X \right\|_E. \end{aligned}$$

We also define $B(f_1, f_2, \ldots, f_n, E(Y))$ by the set

$$\{g \in E(Y) : \left\| \sum_{i=1}^{n} \|f_i(\cdot) - g(\cdot)\|_X \right\|_E = dist_E(f_1, f_2, \ldots, f_n, E(Y))\}.$$

Lemma 1 *Let* $f_1, f_2, \ldots, f_n \in E(X)$*,* Y *a closed subspace of* X *and* $g : T \to Y$ *be a strongly measurable function with* $g(t) \in B(f_1(t), f_2(t), \ldots, f_n(t), Y)$ *for almost all* $t \in T$*. Then* $g \in E(Y) \cap B(f_1, f_2, \ldots, f_n, E(Y))$*.*

Proof. Since $g(t) \in B(f_1(t), f_2(t), \ldots, f_n(t), Y)$, for almost all $t \in T$, we have

$$|||g||| \leq \frac{2}{n} \sum_{i=1}^{n} |||f_i|||,$$

which shows that $g \in E(Y)$. Also, for any $h \in E(Y)$, we have

$$\left\| \sum_{i=1}^{n} \|f_i(\cdot) - g(\cdot)\|_X \right\|_E \leq \left\| \sum_{i=1}^{n} \|f_i(\cdot) - h(\cdot)\|_X \right\|_E,$$

thus $g \in B(f_1, f_2, \ldots, f_n, E(Y))$. □

We can now state and prove the main result.

Theorem 2 *Let* Y *be a closed subspace of the real Banach space* X *and* $E(X)$ *be a Köthe Bochner function space with absolutely continuous norm. If* $f_1, f_2, \ldots, f_n \in E(X)$*, then the distance function* $dist(f_1(\cdot), f_2(\cdot), \ldots, f_n(\cdot))$ *belongs to* E *and*

$$\|dist(f_1(\cdot), f_2(\cdot), \ldots, f_n(\cdot), Y)\|_E = dist_E(f_1, f_2, \ldots, f_n, E(Y)).$$

Proof. Let $f_1, f_2, \ldots, f_n \in E(X)$, then there exist sequences $(f_{n,i})$, $1 \leq i \leq n$, of simple functions in $E(X)$ such that

$$\|f_{n,i}(t) - f_i(t)\| \to 0, \quad i = 1, 2, \ldots, n, \text{ for almost all } t \text{ in } T.$$

The continuity of the distance function implies that

$$|dist(f_{n,1}(t), f_{n,2}(t), \ldots, f_{n,n}(t), Y) - dist(f_1(t), f_2(t), \ldots, f_n(t), Y)| \to 0.$$

Set

$$H_n(t) = dist(f_{n,1}(t), f_{n,2}(t), \ldots, f_{n,n}(t), Y),$$

then H_n is a measurable function. Therefore the $dist(f_1(\cdot), \ldots, f_n(\cdot), Y)$ is measurable and

$$dist(f_1(t), f_2(t), \ldots, f_n(t), Y) \leq \sum_{i=1}^{n} \|f_i(t) - z\|_X, \text{ for all } z \in Y.$$

As a consequence we can write

$$dist(f_1(t), f_2(t), \ldots, f_n(t), Y) \leq \sum_{i=1}^{n} \|f_i(t) - g(t)\|_X, \quad \text{for all } g \in E(Y),$$

and we obtain, for all $g \in E(Y)$,

$$\|dist(f_1(\cdot), f_2(\cdot), \ldots, f_n(\cdot), Y)\|_E \leq \left\| \sum_{i=1}^{n} \|f_i(\cdot) - g(\cdot)\|_X \right\|_E.$$

Thus, $dist\left(f_1(\cdot), f_2(\cdot), \ldots, f_n(\cdot), Y\right) \in E$ and

$$\left\| dist\left(f_1(\cdot), f_2(\cdot), \ldots, f_n(\cdot), Y\right)\right\|_E \leq dist_E\left(f_1, f_2, \ldots, f_n, E(Y)\right). \tag{1}$$

Fix $\varepsilon > 0$. Since $E(X)$ is a Köthe Bochner function space with absolutely continuous norm, the simple functions are dense in $E(X)$, [7]. That is, there exist simple functions f_i^* in $E(X)$ such that

$$|||f_i - f_i^*||| < \frac{\varepsilon}{n}, \text{ for } i = 1, \ldots, n.$$

Assume that

$$f_i^*(t) = \sum_{k=1}^{m} x_k^i \ \ \chi_{A_k}(t),$$

where the A_k's are pairwise disjoint measurable sets of T with $\bigcup_{k=1}^{m} A_k = T$, χ_{A_k}'s are the characteristic functions related to the A_k's and $x_k^i \in X$, for $k = 1, 2, ..., m$ and $i = 1, \ldots, n$. Since $\mu(T)$ is finite, we can put $\alpha = |||\chi_T|||$. For each $k = 1, 2, ..., m$, let $y_k \in Y$ satisfy

$$\sum_{i=1}^{n} \left\|x_k^i - y_k\right\|_X \leq \ dist\left(x_k^1, x_k^2, ..., x_k^n, Y\right) + \frac{\varepsilon}{\alpha}.$$

By setting $g(t) = \sum_{k=1}^{m} y_k \ \chi_{A_k}(t)$, we obtain the inequalities

$$\begin{aligned}
\left\| \sum_{i=1}^{n} \|f_i^*(\cdot) - g(\cdot)\|_X \right\|_E &= \left\| \sum_{k=1}^{m} \chi_{A_k}(\cdot) \left[\sum_{i=1}^{n} \left\|x_k^i - y_k\right\|_X\right]\right\|_E \\
&\leq \left\| \sum_{k=1}^{m} \chi_{A_k}(\cdot) \left[dist\left(x_k^1, x_k^2, ..., x_k^n, Y\right) + \frac{\varepsilon}{\alpha}\right]\right\|_E \\
&\leq \|dist\left(f_1^*(\cdot), f_2^*(\cdot), ..., f_n^*(\cdot), Y\right)\|_E + \left|\left|\left| \sum_{k=1}^{m} \chi_{A_k}\right|\right|\right| \frac{\varepsilon}{\alpha} \\
&\leq \|dist\left(f_1^*(\cdot), f_2^*(\cdot), ..., f_n^*(\cdot), Y\right)\|_E + \frac{\varepsilon}{\alpha} \quad |||\chi_T||| \\
&= \|dist\left(f_1^*(\cdot), f_2^*(\cdot), ..., f_n^*(\cdot), Y\right)\|_E \quad + \varepsilon.
\end{aligned}$$

This also gives the following inequalities:

$$\begin{aligned} dist_E\left(f_1,\ldots,f_n,E\left(Y\right)\right) &\leq dist_E\left(f_1^*,\ldots,f_n^*,E\left(Y\right)\right)+\sum_{i=1}^{n}|||f_i-f_i^*||| \\ &< \left\|\sum_{i=1}^{n}\left\|f_i^*\left(\cdot\right)-g\left(\cdot\right)\right\|_X\right\|_E+\varepsilon \\ &\leq \left\|dist\left(f_1^*\left(\cdot\right),...,f_n^*\left(\cdot\right),Y\right)\right\|_E+2\varepsilon \\ &\leq \left\|dist\left(f_1\left(\cdot\right),...,f_n\left(\cdot\right),Y\right)\right\|_E+\sum_{i=1}^{n}|||f_i-f_i^*|||+2\varepsilon \\ &< \left\|dist\left(f_1\left(\cdot\right),\ldots,f_n\left(\cdot\right),Y\right)\right\|_E+3\varepsilon. \end{aligned}$$

Thus, we have

$$dist_E\left(f_1,f_2,\ldots,f_n\ ,E\left(Y\right)\right)<\left\|dist\left(f_1\left(\cdot\right),f_2\left(\cdot\right),\ldots,f_n\left(\cdot\right),Y\right)\right\|_E+3\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, it holds that

$$dist_E\left(f_1,f_2,\ldots,f_n,E\left(Y\right)\right)\leq\left\|dist\left(f_1\left(\cdot\right),f_2\left(\cdot\right),\ldots,f_n\left(\cdot\right),Y\right)\right\|_E. \tag{2}$$

Using inequalities (1) and (2) we get the required results. □

A direct consequence of the previous is the following result

Corollary 3 *Let Y be a closed subspace of the real Banach space X and $E\left(X\right)$ be a Köthe Bochner function space with absolutely continuous and strictly monotone norm. For $f_1,\ldots,f_n\in E\left(X\right)$ and $g\in B\left(f_1,\ldots,f_n,E\left(Y\right)\right)$ it is necessary and sufficient that $g\left(t\right)\in B\left(f_1\left(t\right),\ldots,f_n\left(t\right),Y\right)$ for almost all $t\in T$.*

Next, we give the simultaneous proximinality of $E(Y)$ in $E(X)$:

Theorem 4 *If Y is simultaneously proximinal in the real Banach space X, then $B\left(f_1,f_2,\ldots,f_n,E\left(Y\right)\right)\neq\emptyset$, for every set of simple functions $f_1,f_2,\ldots,f_n$ in $E(X)$.*

Proof. Let $f_i\left(1\leq i\leq n\right)$ be simple functions in $E\left(X\right)$. Then $f_i\ \left(1\leq i\leq n\right)$ can be written as

$$f_i(t)=\sum_{k=1}^{m}u_k^i\ \chi_{A_k}(t),\quad i=1,\ldots,n,$$

where A_k's are pairwise disjoint measurable sets of T with $\bigcup_{k=1}^{m}A_k=T$. Also, we may take $\mu\left(A_k\right)>0$, for each $k=1,2,...,m$. By assumption we know that for each $k=1,2,...,m$, there exists a best simultaneous approximation w_k in Y of the n-tuples $\left(u_k^1,u_k^2,...,u_k^n\right)\in X^n$ such that

$$dist\left(x_k^1,x_k^2,..,x_k^n,Y\right)=\sum_{i=1}^{n}\left\|u_k^i-w_k\right\|_X.$$

Set

$$g(t)=\sum_{k=1}^{n} w_k \ \chi_{A_k}(t),$$

then for any $\alpha>0$ and $h\in E(Y)$, we obtain that

$$\begin{aligned} & \left\| \ \sum_{i=1}^{n} \left\| f_i(\cdot)-h(\cdot)\right\|_X \ \right\|_E \\ \geq & \left\| \sum_{k=1}^{m} \chi_{A_k}(\cdot) \left[\sum_{i=1}^{n} \left\| u_k^i - w_k \right\|_X \right] \right\|_E \\ = & \left\| \ \sum_{i=1}^{m} \left\| f_i(\cdot)-g(\cdot)\right\|_X \ \right\|_E . \end{aligned}$$

By taking infimum over all $h\in E(Y)$, it results that

$$dist_E(f_1,f_2,\ldots,f_n,E(Y))=\left\| \ \sum_{i=1}^{n} \left\| f_i(\cdot)-g(\cdot)\right\|_X \ \right\|_E .$$

This implies that the set of simple functions $f_1,f_2,\ldots,f_n$ in $E(X)$ admits a best simultaneous approximation. □

Theorem 5 *Let Y be a closed subspace of the real Banach space X and $E(X)$ be a Köthe Bochner function space with absolutely continuous and strictly monotone norm. If $E(Y)$ is simultaneous proximinal in $E(X)$, then Y is simultaneous proximinal in X.*

Proof. Let $x_1,x_2,\ldots,x_n\in X$. Set $f_i(t)=x_i \quad (1\leq i\leq n)$ for almost all $t\in T$. Since

$$\begin{aligned} |||f_i||| &= \left\| \left\| f_i(\cdot)\right\|_X \right\|_E = \left\| \left\| x_i\chi_T(\cdot)\right\|_X \right\|_E \\ &= \left\| x_i\right\|_X \ |||\chi_T||| \, , \text{ for } 1\leq i\leq n, \end{aligned}$$

which is finite, then $f_i\in E(X),(1\leq i\leq n)$. By assumption there exists $g\in E(Y)$ such that

$$\left\| \sum_{i=1}^{n} \left\| f_i(\cdot)-g(\cdot)\right\|_X \right\|_E < \left\| \sum_{i=1}^{n} \left\| f_i(\cdot)-h(\cdot)\right\|_X \ \right\|_E , \quad \text{for all } h\in E(Y)$$

Because $E(X)$ is a Köthe Bochner function space with a strictly monotone norm, for almost $t\in T$, we thus have

$$\sum_{i=1}^{n} \left\| f_i(t)-g(t)\right\|_X \ \leq \sum_{i=1}^{n} \left\| f_i(t)-h(t)\right\|_X .$$

Fix $t_0\in T$ and $y=g(t_0)$, then $y\in Y$ and

$$\sum_{i=1}^{n} \left\| x_i-y\right\|_X \leq \sum_{i=1}^{n} \left\| x_i-h(t)\right\|_X , \text{ for all } h\in E(Y).$$

Since Y is embedded isometrically into $E(Y)$, we obtain

$$\sum_{i=1}^{n} \|x_i - y\|_X \leq \sum_{i=1}^{n} \|x_i - z\|_X \text{, for all } z \in Y.$$

□

Theorem 6 *Let Y be a closed separable subspace of the real Banach space X and $E(X)$ be a Köthe Bochner function space with absolutely continuous and strictly monotone norm. Then $E(Y)$ is simultaneous proximinal in $E(X)$ if and only if Y is simultaneous proximinal in X.*

Proof. The necessity is included in Theorem 5. Now for sufficiency suppose that Y is simultaneous proximinal in X and let $f_1, f_2, \ldots, f_n$ be a set of functions in $E(X)$, then Theorem 3.4 in [1] guarantees that there exists a measurable function $g : T \to X$ such that $g(t) \in B(f_1(t), f_2(t), \ldots, f_n(t), Y)$ for almost all $t \in T$. We conclude that $E(Y)$ is simultaneous proximinal in $E(X)$ on account of Lemma 1. Thus

$$g \in B(f_1, f_2, \ldots, f_n, E(Y)), \text{ for almost all } t \in T,$$

thereby completing the proof of the theorem. □

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