

On similarity of powers of shift operators

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Abstract

Let M_z and B denote, respectively, the multiplication operator and the backward shift operator on a weighted Hardy space. We present sufficient conditions so that M_{z^n} is similar to $\bigoplus_1^n M_z$, and B^n is similar to $\bigoplus_1^n B$. The first part generalizes a result obtained by Yucheng Li.

Key Words: Weighted Hardy space, multiplication operator, similarity

1. Introduction

Let \mathbb{D} be the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$, and $\{\beta(k)\}_k$ be a sequence of positive numbers with $\beta(0) = 1$. The weighted Hardy space $H^2(\beta)$ consists of all formal power series $f(z) = \sum_{k=0}^{\infty} \hat{f}(k)z^k$ for which

$$\|f\|_{\beta}^2 = \sum_{k=0}^{\infty} |\hat{f}(k)|^2 \beta(k)^2 < \infty.$$

It is well known that $H^2(\beta)$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \sum_{k=0}^{\infty} \hat{f}(k) \overline{\hat{g}(k)} \beta(k)^2.$$

The classical Hardy space $H^2(\mathbb{D})$, the classical Bergman space $A^2(\mathbb{D})$, and the classical Dirichlet space \mathcal{D} are weighted Hardy spaces with $\beta(k) = 1$, $\beta(k) = (k+1)^{-\frac{1}{2}}$, and $\beta(k) = (k+1)^{\frac{1}{2}}$, respectively. For $\alpha > -1$ the weighted Bergman space, denoted by $A_{\alpha}^2(\mathbb{D})$, is defined to be the space of all analytic functions on \mathbb{D} for which

$$\|f\|^2 = \int_{\mathbb{D}} |f(z)|^2 (\alpha+1)(1-|z|^2)^{\alpha} dA(z) < \infty.$$

We remark that $A_{\alpha}^2(\mathbb{D})$ is a weighted Hardy space.

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Similarity of operators is a weaker concept than unitary equivalence. Much of the study of similarity has been driven by a desire to characterize operators. It is easy to see that similarity preserves the spectrum and various parts of the spectrum. Some other concepts related to operators may be studied via similarity. For instance, when we focus on the reflexivity of operators, we observe that reflexivity is the same for two similar operators; invariant subspaces of an operator also can be identified in terms of invariant subspaces of a similar operator.

Similarity of some operators is discussed in the literature. R. K. Campbell-Wright [1] has considered the problem of similarity of compact composition operators. For similarity of subnormal operators, one can see [2]. A. L. Shields [4] has presented some conditions equivalent to the similarity of the multiplication operator M_z on two weighted Hardy spaces. The statement of the main theorem of [3] states that the multiplication operator M_{z^n} is similar to $\bigoplus_1^n M_z$ on the weighted Bergman space $A_\alpha^2(\mathbb{D})$ ($\alpha > -1$). In this paper, at first, we generalize this result. We establish the above result for a large class of weighted Hardy spaces with a much easier proof. Then, we consider the backward shift denoted by B , and give sufficient conditions for similarity of B^n and $\bigoplus_1^n B$.

2. Main result

It is known that the multiplication operator M_z is bounded on $H^2(\beta)$ if and only if

$$\sup_k \frac{\beta(k+1)}{\beta(k)} < \infty.$$

Indeed, $\|M_z\| = \sup_k \frac{\beta(k+1)}{\beta(k)}$.

Theorem 1 *Let n be an arbitrary positive integer. Suppose that $\{\beta(k)\}_{k=0}^\infty$, the weight sequence of $H^2(\beta)$, satisfies the following two conditions:*

- (i) $\{\beta(k)\}_k$ is a non-increasing sequence;
- (ii) there exists $c > 0$ so that for every non-negative integer k ,

$$\beta(k) \leq c\beta(n(k+1) - 1).$$

Then M_{z^n} is similar to $\bigoplus_1^n M_z$ on $H^2(\beta)$.

Proof. Define the mapping S_1 on $\bigoplus_1^n H^2(\beta)$ by

$$S_1(\bigoplus_1^n f_i) = \sum_{k=0}^\infty \sum_{i=1}^n \hat{f}_i(k) z^{nk+(i-1)},$$

where each f_i is represented as $f_i = \sum_{k=0}^{\infty} \hat{f}_i(k)z^k$. It is obvious that S_1 is linear and $\ker S_1 = \{0\}$. Furthermore, (i) implies that

$$\begin{aligned} \|S_1(\oplus_1^n f_i)\|^2 &= \sum_{k=0}^{\infty} \sum_{i=1}^n |\hat{f}_i(k)|^2 [\beta(nk + (i - 1))]^2 \\ &\leq \sum_{i=1}^n \sum_{k=0}^{\infty} |\hat{f}_i(k)|^2 \beta(k)^2 = \|\oplus_1^n f_i\|^2. \end{aligned}$$

This shows that $S_1(\oplus_1^n f_i) \in H^2(\beta)$, whenever $f_i \in H^2(\beta)$ for $i = 1, \dots, n$; besides, S_1 is a bounded operator.

To show that S_1 is surjective, take $f \in H^2(\beta)$. For $i = 1, \dots, n$ define $f_i = \sum_{k=0}^{\infty} \hat{f}(kn + i - 1)z^k$. Then, using (ii), we see that

$$\begin{aligned} \|f_i\|^2 &= \sum_{k=0}^{\infty} |\hat{f}(kn + i - 1)|^2 \beta(k)^2 \\ &\leq c^2 \sum_{k=0}^{\infty} |\hat{f}(kn + i - 1)|^2 [\beta(n(k + 1) - 1)]^2 \\ &\leq c^2 \sum_{k=0}^{\infty} |\hat{f}(kn + i - 1)|^2 [\beta(kn + i - 1)]^2 \\ &\leq c^2 \|f\|^2. \end{aligned}$$

Thus, $f_i \in H^2(\beta)$, for $i = 1, \dots, n$, and it can be easily verified that $S_1(\oplus_1^n f_i) = f$.

At last, if $\oplus_1^n f_i \in \bigoplus_1^n H^2(\beta)$ then a direct computation shows that

$$(S_1 \circ \left(\bigoplus_1^n M_z\right))(\oplus_1^n f_i) = M_{z^n} S_1(\oplus_1^n f_i).$$

Hence, the mapping S_1 gives the similarity of M_{z^n} and $\bigoplus_1^n M_z$. □

As a consequence of Theorem 1, we bring the following result due to Yucheng Li [3].

Corollary 1 *The multiplication operator M_{z^n} is similar to $\bigoplus_1^n M_z$ on $A_\alpha^2(\mathbb{D})$, $\alpha > -1$.*

Proof. Consider the space $H^2(\beta)$ with a non-increasing weight sequence $\{\beta(k)\}_k$ defined by

$$\beta(k) = (\alpha + 1)(k + 1)^{\frac{-1-\alpha}{2}}.$$

If $c = n^{\frac{1+\alpha}{2}}$ then

$$\beta(k) \leq c\beta(n(k + 1) - 1)$$

for all $k \geq 0$. Thus, the hypotheses of Theorem 1 hold. On the other hand, we observe that $H^2(\beta) = A_\alpha^2(\mathbb{D})$ with equivalent norms. Hence, the result holds, using Theorem 1. □

Recall that the multiplication operator M_z on $H^2(\beta)$ is, indeed, unitary equivalent to an injective unilateral weighted shift operator, with the weight sequence $\{w_k\}_k$ defined by $w_k = \frac{\beta(k+1)}{\beta(k)}$, $k \geq 0$; conversely,

every injective unilateral weighted shift operator can be represented as M_z on $H^2(\beta)$, for a suitable choice of β . With this in mind, the preceding theorem and its corollary are, in fact, results about injective unilateral weighted shift operators.

The backward shift operator B on $H^2(\beta)$ is given by

$$(Bf)(z) = \frac{f(z) - f(0)}{z};$$

it is bounded if and only if $\sup_k \frac{\beta(k-1)}{\beta(k)} < \infty$. In fact,

$$\|B\| = \sup_k \frac{\beta(k-1)}{\beta(k)}.$$

Theorem 2 *Let n be an arbitrary positive integer. Suppose that the weight sequence of $H^2(\beta)$, denoted by $\{\beta(k)\}_{k=0}^\infty$, satisfies the following two conditions:*

- (i) $\{\beta(k)\}_k$ is a non-decreasing sequence;
- (ii) there exists $c > 0$ so that for every non-negative integer k ,

$$\beta(n(k+1) - 1) \leq c\beta(k).$$

Then B^n is similar to $\bigoplus_1^n B$ on $H^2(\beta)$.

Proof. Put $B_n = \bigoplus_1^n B$, and for every $m \geq 0$ define

$$H_m^2(\beta) = \bigvee \{z^{nm}, z^{nm+1}, \dots, z^{n(m+1)-1}\}.$$

Thus, $H^2(\beta) = \bigoplus_{m=0}^\infty H_m^2(\beta)$ and so every $f \in H^2(\beta)$ can be represented as $\bigoplus_0^\infty f_m$ in which for every f_m in $H_m^2(\beta)$ there exist scalars $\alpha_{m1}, \dots, \alpha_{mn}$ so that

$$f_m = \alpha_{m1}z^{nm} + \alpha_{m2}z^{nm+1} + \dots + \alpha_{mn}z^{n(m+1)-1}.$$

Define the mapping S_2 on $H^2(\beta)$ by

$$S_2(\bigoplus_0^\infty f_m) = \bigoplus_{i=1}^n \left(\sum_{m=0}^\infty \alpha_{mi}z^m \right).$$

It is clear that S_2 is linear and $\ker S_2 = \{0\}$. Since $\{\beta(k)\}_k$ is a non-decreasing sequence, for $i = 1, \dots, n$, we obtain

$$\begin{aligned} \sum_{m=0}^\infty |\alpha_{mi}|^2 \beta(m)^2 &\leq \sum_{i=1}^n \sum_{m=0}^\infty |\alpha_{mi}|^2 \beta(m)^2 \\ &\leq \sum_{m=0}^\infty \sum_{i=1}^n |\alpha_{mi}|^2 \beta(mn + i - 1)^2 \\ &= \|\bigoplus_0^\infty f_m\|^2 < \infty. \end{aligned}$$

Thus, $\sum_{m=0}^{\infty} \alpha_{mi} z^m \in H^2(\beta)$, and so S_2 maps $H^2(\beta)$ into $\bigoplus_{i=1}^n H^2(\beta)$. Besides, the continuity of S_2 can be deduced from the above computation. Also, S_2 is surjective. Indeed, if

$$g = \bigoplus_{i=1}^n \left(\sum_{m=0}^{\infty} \alpha_{mi} z^m \right) \in \bigoplus_{i=1}^n H^2(\beta),$$

then taking $f_m = \sum_{i=1}^n \alpha_{mi} z^{nm+i-1}$ we see that $S_2(\bigoplus_0^{\infty} f_m) = g$; moreover, the hypotheses (i) and (ii) along with a straightforward computation show that $\|\bigoplus_0^{\infty} f_m\| \leq \frac{1}{c} \|g\| < \infty$. Furthermore, $S_2 B^n = B_n S_2$. Hence, B^n and B_n are similar operators. \square

For $\alpha \geq 0$, by \mathcal{D}_α we denote the space of all analytic functions f on the open unit disc \mathbb{D} for which

$$\sum_{k=0}^{\infty} (k+1)^\alpha |\hat{f}(k)|^2 < \infty.$$

Taking $\beta(k) = (k+1)^{\frac{\alpha}{2}}$ we observe that \mathcal{D}_α is the weighted Hardy space $H^2(\beta)$; thus, the following result will be obtained as a consequence of Theorem 2.

Corollary 2 *The backward shift operator B^n is similar to $\bigoplus_1^n B$ on \mathcal{D}_α , $\alpha \geq 0$.*

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