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Best constants in second-order Sobolev inequalities on compact Riemannian manifolds in the presence of symmetries

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Abstract

Let (M,g) be a smooth compact $3 \leq n$-dimensional Riemannian manifold, and G a subgroup of the isometry group of (M,g). We establish the best constants in second-order for a Sobolev inequality when the functions are G-invariant.

Key Words: Best constants, compact Riemannian manifolds, Sobolev inequalities, isometries

1. Introduction

Let (M,g) be a compact $3 \leq n$-dimensional Riemannian manifold, and G a subgroup of the isometry group $Is(M,g)$. Assume that l is the minimum orbit dimension of G, and V is the minimum of the volume of the l-dimensional orbits. If $1 < q_1 < n$, $1 < p_1 < (n-l)$, $\tilde{q}_1 = \frac{nq_1}{n-q_1}$, and $\tilde{p}_1 = \frac{(n-l)p_1}{n-l-p_1}$, then the embeddings $W^{1,q_1}(M) \subset L^r(M)$ and $W_G^{1,p_1}(M) \subset L_G^d(M)$ are compact for any $r \in [1, \tilde{q}_1)$ and $d \in [1, \tilde{p}_1)$, respectively, where

$$W_G^{k,p}(M) = \left\{ f \in W^{k,p}(M) \mid f \circ \sigma = f \text{ for all } \sigma \in G \right\},$$

for any $1 \leq p < \infty$ and $k \in \mathbb{N}$. However, the embeddings $W^{1,q_1}(M) \subset L^{\tilde{q}_1}(M)$ and $W_G^{1,p_1}(M) \subset L_G^{\tilde{p}_1}(M)$ are only continuous. Hence, there exist real constants $A_1, B_1, \tilde{A}_1$, and $\tilde{B}_1$ so that for all $f \in W^{1,q_1}(M)$,

$$\|f\|_{L^{\tilde{q}_1}(M)} \leq A_1 \|\nabla_g f\|_{L^{q_1}(M)} + B_1 \|f\|_{L^{q_1}(M)}, \tag{$I^1_{q_1,\text{gen}}$}$$

and for all $f \in W_G^{1,p_1}(M)$,

$$\|f\|_{L_G^{\tilde{p}_1}(M)} \leq \tilde{A}_1 \|\nabla_g f\|_{L_G^{p_1}(M)} + \tilde{B}_1 \|f\|_{L_G^{p_1}(M)}. \tag{$I^1_{p_1,G,\text{gen}}$}$$

These inequalities are equivalent to the existence of constants $A_{q_1}, B_{q_1}, \tilde{A}_{p_1}$, and $\tilde{B}_{p_1}$ such that the following inequalities hold:

$$\|f\|^{q_1}_{L^{\tilde{q}_1}(M)} \leq A_{q_1} \|\nabla_g f\|^{q_1}_{L^{q_1}(M)} + B_{q_1} \|f\|^{q_1}_{L^{q_1}(M)} \tag{$I^{q_1}_{q_1,\text{gen}}$}$$

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for all $f \in W^{1,q_1}(M)$, and for all $f \in W_G^{1,p_1}(M)$,

$$\|f\|^{p_1}_{L_G^{\tilde{p}_1}(M)} \leq \tilde{A}_{p_1} \|\nabla_g f\|^{p_1}_{L_G^{p_1}(M)} + \tilde{B}_{p_1} \|f\|^{p_1}_{L_G^{p_1}(M)}. \qquad (I^{p_1}_{p_1,G,\mathrm{gen}})$$

For $s \in \{1, q_1\}$, set

$$\mathcal{A}^{s}_{q_1} = \inf \left\{ A_s \in \mathbb{R} : \text{there exists } B_s \in \mathbb{R} \text{ such that } (I^{s}_{q_1,\mathrm{gen}}) \text{ is satisfied} \right\} \qquad (1.1)$$

and

$$\mathcal{B}^{s}_{q_1} = \inf \left\{ B_s \in \mathbb{R} : \text{there exists } A_s \in \mathbb{R} \text{ such that } (I^{s}_{q_1,\mathrm{gen}}) \text{ is satisfied} \right\}. \qquad (1.2)$$

If the best constant $\mathcal{A}^{s}_{q_1}$ is attained, then there exists B_s such that for any $f \in W^{1,q_1}(M)$,

$$\|f\|^{s}_{L^{\tilde{q}_1}(M)} \leq \mathcal{A}^{s}_{q_1} \|\nabla_g f\|^{s}_{L^{q_1}(M)} + B_s \|f\|^{s}_{L^{q_1}(M)}. \qquad (I^{s}_{q_1,\mathrm{opt}})$$

Similarly, if the best constant $\mathcal{B}^{s}_{q_1}(M)$ is attained, then a real constant A_s exists such that for any $f \in W^{1,q_1}(M)$,

$$\|f\|^{s}_{L^{\tilde{q}_1}(M)} \leq B_s \|\nabla_g f\|^{s}_{L^{q_1}(M)} + \mathcal{B}^{s}_{q_1} \|f\|^{s}_{L^{q_1}(M)}. \qquad (J^{s}_{q_1,\mathrm{opt}})$$

In the same manner, we define $(I^{s}_{p_1,G,\mathrm{opt}})$ and $(J^{s}_{p_1,G,\mathrm{opt}})$ for all $f \in W_G^{1,p_1}(M)$.

It is obvious that the validity of $(I^{q_1}_{q_1,\mathrm{opt}})$ implies the validity of $(I^{1}_{q_1,\mathrm{opt}})$, and the validity of $(J^{q_1}_{q_1,\mathrm{opt}})$ implies the validity of $(J^{1}_{q_1,\mathrm{opt}})$. However, the converse is generally not true (see [2] and [6]).

The best constants in Sobolev inequalities play an important role in many fields such as analysis and partial differential equations. They have received much attention from many authors (we refer the readers, in particular, to [7] and [10]). For example, when (M, g) is without boundary, it was proved that the best constant $\mathcal{A}^{1}_{q_1}$ is the same as the best constant for the Sobolev embedding for $\mathbb{R}^n$ under the Euclidean metric [2], which equals $\mathcal{J}(n,p)$, where

$$\frac{1}{\mathcal{J}(n,q_1)} = \inf_{f \in L^{\tilde{q}_1}(\mathbb{R}^n)\setminus\{0\}, \nabla f \in L^{q_1}(\mathbb{R}^n)} \frac{\|\nabla f\|_{L^{q_1}(\mathbb{R}^n)}}{\|f\|_{L^{\tilde{q}_1}(\mathbb{R}^n)}}.$$

Independently, Aubin [1] and Talenti [16] explicitly computed that

$$\begin{aligned}
\frac{1}{\mathcal{J}(n,1)} &= \frac{1}{n^{\frac{n-1}{n}} \omega_{n-1}^{1/n}}, \\
\frac{1}{\mathcal{J}(n,q_1)} &= \frac{q_1 - 1}{n - q_1} \left(\frac{n - q_1}{n(q_1 - 1)} \right)^{1/q_1} \left(\frac{\Gamma(n+1)}{\omega_{n-1} \Gamma(n/q_1) \Gamma(n + 1 - (n/q_1))} \right)^{1/n},
\end{aligned}$$

where ω_{n-1} is the volume of the unit sphere in $\mathbb{R}^n$. The same result is still true for any $f \in W_c^{1,q_1}(M)$, and (M, g) has no boundary. Furthermore, Cherrier [4] established that the same best constant equals $2^{1/n}\mathcal{J}(n, q_1)$ if $f \in W^{1,q_1}(M)$, and the compact manifold has a boundary.

For any compact $3 \leq n$-dimensional Riemannian manifold (M, g), with or without boundary, Ilias [14] proved the validity of $(J^{1}_{q_1,\mathrm{opt}})$. Moreover, he found that $\mathcal{B}^{1}_{q_1} = Vol^{-1/n}_{(M,g)}$ for any $f \in W^{1,q_1}(M)$. Further, he showed that this best constant mainly depends on an upper bound for the diameter, a lower bound for the Ricci curvature, and the lower bound for the volume.

Concerning the best constants of the inequality $(I^{p_1}_{p_1,G,\mathrm{gen}})$, Faget [9] found that the best constant in the front of the gradient term is $\frac{\mathcal{J}^{p_1}(n-l,p_1)}{V^{p_1/(n-l)}}$. Hebey and Vaugon [12] showed that if the compact manifold is without boundary, and G possesses at least one finite orbit, then this best constant is $\frac{\mathcal{J}^{p_1}(n,p_1)}{V^{p_1/n}}$. Furthermore, they proved that the inequality $(I^{2}_{2,G,\mathrm{opt}})$ is achieved under the above conditions.

Although some open problems related to the best constants in Sobolev inequalities of first order remain open, the investigation to determine the best constants in second-order Sobolev inequalities has been started (see for instance [3], [5], and [11]).

Throughout this work, we consider for $1 < p < (n-l)/2$ the Sobolev spaces

$$F^{1,p}_{G}(M) = W^{2,p}_{G}(M)$$

if M has no boundary, and

$$F^{2,p}_{G}(M) = W^{2,p}_{c,G}(M), \qquad F^{3,p}_{G}(M) = W^{2,p}_{G}(M) \cap W^{1,p}_{c,G}(M)$$

if M has boundary.

The Sobolev embedding theorem ensures that the inclusion $F^{i,p}_{G}(M) \subset L^{p^*}(M)$ is continuous for $p^* = p(n-l)/(n-l-2p)$. Thus, two real constants A_m, B_m exist such that for any $f \in F^{i,p}_{G}(M)$ and $m \in \{1, p\}$,

$$\|f\|^{m}_{L^{p^*}_{G}(M)} \leq A_m \|\triangle_g f\|^{m}_{L^{p}_{G}(M)} + B_m \|f\|^{m}_{L^{p}_{G}(M)}. \qquad (I^{m}_{p,G,\mathrm{gen}})$$

Similar to what we did before, we define

$$\alpha^{m}_{p} = \inf\left\{A_m \in \mathbb{R} : \text{there exists } B_m \in \mathbb{R} \text{ such that inequality } (I^{m}_{p,G,\mathrm{gen}}) \text{ holds}\right\}$$

and

$$\beta^{m}_{p} = \inf\left\{B_m \in \mathbb{R} : \text{there exists } A_m \in \mathbb{R} \text{ such that inequality } (I^{m}_{p,G,\mathrm{gen}}) \text{ holds}\right\}.$$

We denoted the inequality $(I^{m}_{p,G,\mathrm{gen}})$ by $(I^{m}_{p,G,\mathrm{opt}})$ and $(J^{m}_{p,G,\mathrm{opt}})$ if the best constants α^{m}_{p} and β^{m}_{p} are achieved, respectively.

For $1 < p < n/2$, and (M, g) is a Riemannian manifold, with or without boundary, and the functions are not G- invariant Biezuner and Montenegro [3] determined for any $f \in F^{i,p}(M)$ that α^{p}_{p} equals $\mathcal{K}^{p}(n,p)$, where

$$\frac{1}{\mathcal{K}(n,p)} = \inf_{f \in E^{2,p}(\mathbb{R}^n)\setminus\{0\}} \frac{\|\triangle f\|_{L^{p}(\mathbb{R}^n)}}{\|f\|_{L^{p^*}(\mathbb{R}^n)}},$$

and $E^{2,p}(\mathbb{R}^n)$ is the completion of $C_c^{\infty}(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_{E^{2,p}(\mathbb{R}^n)} = \left(\int\limits_{\mathbb{R}^n} |\triangle f|^p \, dx \right)^{\frac{1}{p}}.$$

In addition, it was computed in [8] and [15] that

$$\mathcal{K}(n,2) = \frac{16}{n(n-4)(n^2-4)\omega_{n-1}^{4/n}}.$$

In view of the results in the first and second orders in Sobolev inequalities, a question naturally arises: For any $f \in F_G^{i,p}(M)$, $1 < p < (n-l)/2$, and $p^* = p(n-l)/(n-l-2p)$, can we obtain the best constants in second-order for a Sobolev inequality when the functions are G- invariant?

The next sections provide an affirmative answer to this question.

2. Establishing the best constant $\overset{p}{\alpha}_p(M)$

In this section, we find the best constant $\overset{p}{\alpha}_p(M)$, and prove some lemmas used in the sequel. The following theorem is the main result of this section.

Theorem 2.1 *Let (M,g) be a compact $3 \leq n$-dimensional Riemannian manifold, with or without boundary, and G be a subgroup of the isometry group $Is(M,g)$. Assume that l is the minimum orbit dimension of G, and V is the minimum of the volume of the l-dimensional orbits (if G has finite orbits, then $l=0$ and $V = \min\limits_{x\in M} Card(O_G^x)$). Let $1 < p < (n-l)/2$, and $q = \frac{(n-l)p}{n-l-2p}$. Then*

$$\overset{p}{\alpha}_p(M) = \overset{p}{\mathcal{K}}_G = \frac{\mathcal{K}^p(n-l,p)}{V^{2p/(n-l)}}$$

for any $f \in F_G^{i,p}(M)$.

Following the arguments used in [9] and [13], we achieve the following lemma.

Lemma 2.2 *Let (M,g) be a compact $3 \leq n$-dimensional Riemannian manifold without boundary, and G a compact subgroup of the isometry group $Is(M,g)$. Let x be in M with orbit of dimension $N < n$. Then there exists a chart (Ω, Ψ) around x such that the following properties are valid:*

(1) $\Psi(\Omega) = U_1 \times U_2$, where $U_1 \in \mathbb{R}^N$ and $U_2 \in \mathbb{R}^{n-N}$.

(2) $\Psi = \Psi_1 \times \Psi_2$ and Ψ_1, Ψ_2 can be chosen in the following way:

$\Psi_1 = \Phi_1 \circ \gamma \circ \Gamma_1$, γ defined from a neighborhood of Id in G to O_G^x, and $\gamma \circ \Gamma_1(\Omega) = \mathcal{V}_x$, where $\mathcal{V}_x$ is an open neighborhood of x in O_G^x.

$\Psi_2 = \Phi_2 \circ \Gamma_2$ with $\Gamma_2(\Omega) = \mathcal{W}_x$, where $\mathcal{W}_x$ is a submanifold of dimension $n-N$

orthogonal to O_G^x *at* x.

(3) (Ω,Ψ) *is a normal chart of* M *around* x, $(\mathcal{V}_x,\Phi_1)$ *is a normal chart around* x *of the submanifold* O_G^x, $(\mathcal{W}_x,\Phi_2)$ *is a geodesic normal chart around* x *of the submanifold* $\mathcal{W}_x$. *In particular, for any* $\varepsilon>0$, (Ω,Ψ) *can be chosen such that:*

$$(i) \qquad \left|(g^{ij})-(\delta^{ij})\right|\le\varepsilon, \qquad \left|\Gamma_{ij}^l\right|\le\varepsilon\,,$$

and

$$1-\varepsilon\le\sqrt{\det(g_{ij})}\le 1+\varepsilon \quad \text{on } \Omega, \text{ for } 1\le i,j\le n,$$

$$(ii) \qquad \left|(\tilde g^{ij})-(\delta^{ij})\right|\le\varepsilon, \qquad \left|\Gamma_{ij}^l\right|\le\varepsilon\,,$$

and

$$1-\varepsilon\le\sqrt{\det(\tilde g_{ij})}\le 1+\varepsilon \quad \text{on } \mathcal{V}_x, \text{ for } 1\le i,j\le N,$$

where $\tilde g$ *is the metric induced by* g *on* O_G^x.

Furthermore, $(1-\varepsilon)(\delta_{ij})\le(g_{ij})\le(1+\varepsilon)(\delta_{ij})$ *as bilinear forms.*

(4) For any $f\in F_G^{i,p}$, $f\circ\Psi^{-1}$ *depends only on* U_2 *variables.*

In order to prove Theorem 2.1, it suffices to prove the following lemmas.

Lemma 2.3 *Let* (M,g) *be a compact* $3\le n$*-dimensional Riemannian manifold, and* G *a subgroup of the isometry group* $Is(M,g)$. *Suppose that* l *is the minimum orbit dimension of* G, *and* V *is the minimum of the volume of the* l*-dimensional orbits. Assume that for any* $1<p<(n-l)/2$ *there exist real numbers* A *and* B *such that*

$$\|f\|^p_{L_G^{p^*}(M)}\le A\,\|\triangle_g f\|^p_{L_G^p(M)}+B\,\|f\|^p_{L_G^p(M)} \tag{2.1}$$

for all $f\in F_G^{i,p}(M)$, *then* $A\ge\mathcal{K}_G^p$.

Proof. Suppose by contradiction that real numbers A, B exist with $A<\mathcal{K}_G^p$ such that (2.1) holds for any $f\in F_G^{i,p}(M)$. Fix $x_0\in M\setminus\partial M$. Given $\varepsilon>0$, let $B(x_0,\delta)$ be a geodesic ball of radius δ and center x_0 such that in normal coordinators of $B(x_0,\delta)$, the properties of Lemma 2.2 are verified.

For any $f\in C_c^\infty(B_\delta)$ and ε small enough, there exist two real numbers A, B with $A<\mathcal{K}^p(n-l,p)$ such that

$$\|f\|^p_{L_G^{p^*}(B_\delta)}\le(1+\varepsilon_1)A\,\|\triangle_g f\|^p_{L_G^p(B_\delta)}+B\,\|f\|^p_{L_G^p(B_\delta)}\,, \tag{2.2}$$

where $\varepsilon_1=\mathrm{O}(\varepsilon)$, and B_δ is the Euclidean ball with radius δ and center 0.

Using the inequality

$$\int_{B_\delta}|\partial f|^p\,dx\le\alpha\int_{B_\delta}\left|\partial^2 f\right|^p dx+C_{\alpha,\delta}\int_{B_\delta}|f|^p\,dx \tag{2.3}$$

with $\alpha>0$ and $C_{\alpha,\delta}=O(\delta^{-p})$, plus the inequalities

$$\int_{B_\delta}\left|\partial^2 f\right|^p dx\le C_{n,p}\int_{B_\delta}|\triangle f|^p\,dx, \tag{2.4}$$

$$(x+y)^p \leq (1+\varepsilon_2)\, x^p + C_{\varepsilon_2} y^p \tag{2.5}$$

with $\varepsilon_2 = \mathrm{O}(\varepsilon)$, implies that there exists $A' < \mathcal{K}^p(n-l,p)$ such that

$$\|f\|^p_{L^{p^*}_G(B_\delta)} \leq A' \|\triangle f\|^p_{L^p_G(B_\delta)} + C_{1,\delta} \|f\|^p_{L^p_G(B_\delta)}.$$

Applying Hölder's inequality, and choosing δ small enough so that $C_{1,\delta}\left(|B_\delta|^{2p/(n-l)}\right)$ is small enough, we obtain that

$$\begin{aligned}\|f\|^p_{L^{p^*}_G(B_\delta)} &\leq A' \|\triangle f\|^p_{L^p_G(B_\delta)} + C_{1,\delta}\left(|B_\delta|^{2p/(n-l)} \|f\|^p_{L^{p^*}_G(B_\delta)}\right) \\ &\leq A_1' \|\triangle f\|^p_{L^p_G(B_\delta)}\end{aligned}$$

for some real number $A_1' < \mathcal{K}^p(n-l,p)$.

For any $f \in C_c^\infty(\mathbb{R}^{n-l})$, we define $f^\nu = \nu^{(l-n)/p^*} f(x/\nu)$. Then, choosing ν small enough such that $f^\nu \in C_c^\infty(B_\delta)$, we get

$$\|f\|_{L^{p^*}_G(\mathbb{R}^{n-l})} = \left\|f^\nu\right\|_{L^{p^*}_G(\mathbb{R}^{n-l})} < a_1' \left\|\triangle f^\nu\right\|_{L^p_G(\mathbb{R}^{n-l})} = A_1' \|\triangle f\|_{L^p_G(\mathbb{R}^{n-l})},$$

which contradicts the definition of $\mathcal{K}(n-l,p)$. □

Lemma 2.4 *Let (M,g) be a compact $3 \leq n$-dimensional Riemannian, and (O_j,η_j) a partition of unity of M. Assume that for any $f \in F^{i,p}_G(M)$, $|\eta_j|^{1/p} f$ is in $F^{i,p}_{c,G}(O_j)$. Suppose that $\mathcal{K}^p_G$ exists such that for any j there exists C_j such that*

$$\left\|\eta_j^{1/p} f\right\|^p_{L^{p^*}_G(M)} \leq \left(\mathcal{K}^p_G + \varepsilon\right)\left\|\triangle_g(\eta_j^{1/p} f)\right\|^p_{L^p_G(M)} + C_j \left\|\eta_j^{1/p} f\right\|^p_{L^p_G(M)}, \tag{2.6}$$

where p and q are as in the above lemma, then for any $\varepsilon^* > 0$ there exists real constant C_{ε^*} such that for all $f \in F^{i,p}_G(M)$,

$$\|f\|^p_{L^{p^*}_G(M)} \leq (\mathcal{K}^p_G + \varepsilon^*) \|\triangle_g f\|^p_{L^p_G(M)} + C_{\varepsilon^*} \|f\|^p_{L^p_G(M)}. \tag{2.7}$$

Proof. For any $f \in F^{i,p}_G(M)$, Minkowski's inequality with (2.6) yields that

$$\begin{aligned}\|f\|^p_{L^{p^*}_G(M)} &= \left(\int_M \left(\sum_j \left(\eta_j^{1/p}|f|\right)^p\right)^{p^*/p} dv(g)\right)^{p/p^*} \leq \sum_j \left\|\eta_j^{1/p} f\right\|^p_{L^{p^*}_G(M)} \\ &\leq (\mathcal{K}^p_G + \varepsilon_1)\sum_j \left\|\triangle_g(\eta_j^{1/p} f)\right\|^p_{L^p_G(M)} + C\|f\|^p_{L^p_G(M)} \\ &\leq (\mathcal{K}^p_G + \varepsilon_2)\|\triangle_g f\|^p_{L^p_G(M)} + C_{\varepsilon,\delta}\|\nabla_g f\|^p_{L^p_G(M)} + C_{\varepsilon,\delta}\|f\|^p_{L^p_G(M)},\end{aligned}$$

where $\varepsilon_1 = \mathrm{O}(\varepsilon)$ and $\varepsilon_2 = \mathrm{O}(\varepsilon)$.

Using the L^p-theory of linear elliptic operations, the interpolation inequality of lower-order derivatives, and Lemma 2.2, we find that

$$\begin{aligned}
\|\nabla_g f\|^p_{L^p_G(M)} &= \sum_j \left\|\eta_j^{1/p}\nabla_g f\right\|^p_{L^p_G(M)} \leq (1+\varepsilon)^{(p+1)} \sum_j \int_{O_j} \left|\eta_j^{1/p}\partial f\right|^p dx \\
&\leq (1+\varepsilon_3)\sum_j \left(\int_{O_j} \left|\partial\left(\eta_j^{1/p} f\right)\right|^p dx + C_\varepsilon \int_{O_j} \left|f\partial\eta_j^{1/p}\right|^p dx \right) \\
&\leq \alpha(1+\varepsilon_3)\sum_j \left(\left\|\partial^2\left(\eta_j^{1/p} f\right)\right\|^p_{L^p_G(O_j)} + C_{\varepsilon,\alpha,\delta}\|f\|^p_{L^p_G(O_j)} \right) \\
&\leq \frac{\alpha(1+\varepsilon_3)C_\delta}{1-\varepsilon}\|\triangle_g f\|^p_{L^p_G(M)} + C_{\varepsilon,\alpha,\delta}\|\nabla_g f\|^p_{L^p_G(M)} + C_{\varepsilon,\alpha,\delta}\|f\|^p_{L^p_G(M)},
\end{aligned}$$

with $\varepsilon_3 = \mathrm{O}(\varepsilon)$. Choosing ε and α sufficiently small implies

$$\|\nabla_g f\|^p_{L^p_G(M)} \leq \alpha\tilde{C}_{\varepsilon,\delta}\|\triangle_g f\|^p_{L^p_G(M)} + C\|f\|^p_{L^p_G(M)}. \tag{2.8}$$

We finish the proof by choosing $\alpha C_{\varepsilon,\delta}\tilde{C}_{\varepsilon,\delta} < \varepsilon_2 < \varepsilon^*/2$. □

Lemma 2.5 *Let (M,g) and G be as in Theorem 2.1. Suppose that l is the minimum orbit dimension of G, and V is the minimum of the volume of the l-dimensional orbits. Then for any $\varepsilon > 0$ there exists real constant $B = B_{\varepsilon,M,g}$ such that for any $1 < p < (n-l)/2$,*

$$\|f\|^p_{L^{p^*}_G(M)} \leq \left(\mathcal{K}^p_G + \varepsilon\right)\|\triangle_g f\|^p_{L^p_G(M)} + B\|f\|^p_{L^p_G(M)}$$

for all $f \in F^{i,p}_G(M)$.

Proof. The proof depends on the proof of [9, Theorem 1] and the approaches used in [13]. Given $\varepsilon > 0$. Let $\delta > 0$ be taken as small as we need. Fix $x \in M$; let O^x_G be its G-orbit, and (Ω, Ψ) be a chart around x such that the above properties are satisfied. Let $y \in O^x_G$, $\sigma \in G$ be such that $\sigma(x) = y$, and $(\sigma(\Omega), \Psi\circ\sigma^{-1})$ be a chart around y isometric to (Ω, Ψ). Then, O^x_G is covered by such charts. From that and due to the compactness of O^x_G, we say that $\{\Omega_m\}_1^L$ is a finite extract covering. Choose $\delta > 0$, depending on ε and x, small enough such that

$$(O^x_G)_\delta = \{y \in M \mid d_g(y, O^x_G) < \delta\}$$

is covered by $\{\Omega_m\}_1^L$, $d^2(., O^x_G)$ is a C^∞ function on $(O^x_G)_\delta$, and $\overline{(O^x_G)_\delta}$ is a submanifold of M with boundary. Obviously, the manifold M is covered by $\bigcup_{x\in M}(O^x_G)_\delta$; therefore, there exists a finite extract cover; say $\left\{(O^x_G)_{i,\delta}\right\}_1^J$. Assume that (η_i) is a partition of unity relative to $(O^x_G)_{i,\delta}$ such that $\eta_i \in C^\infty_G\left((O^x_G)_{i,\delta}\right)$ for any i. Hence, $\eta_i f$ has a compact support in $(O^x_G)_{i,\delta}$ for any $f \in W^{k,p}_G(M)$.

Furthermore, for each m we let $\alpha_m = \frac{\beta_m \circ \Psi_m}{\sum\limits_{m=1}^{L} (\beta_m \circ \Psi_m)}$, where $\beta_m \in C_c^\infty(U_{1m})$ and $\beta_m \geq 0$. Thus, (α_m) is a partition of unity relative to Ω_m's, which covers $(O_G^x)_{i,\delta}$. As β_m is a function, defined on $U_{1m} \times U_{2m}$, depending only on U_{1m} variables, $\alpha_m \circ \Psi_m^{-1}$ is depending only on U_{1m} variables.

For any integer $1 \leq i \leq J$, and for any $f \in F_{c,G}^{i,p}((O_G^x)_{i,\delta})$, we have that

$$\begin{aligned} \int_M f dv(g) &= \sum_m \int_{\Omega_m} \alpha_m f dv(g) = \sum_m \int_{U_{1m} \times U_{2m}} \sqrt{det(g_{ij})} \alpha_m f \circ \Psi_m^{-1} dx dy \\ &\leq (1+\varepsilon) \sum_m \int_{U_{1m}} \alpha_m \circ \Psi_m^{-1} dx \int_{U_{2m}} f \circ \Psi_m^{-1} dy. \end{aligned} \tag{2.9}$$

For each m, let $\alpha_{1m} = \alpha_m \circ \Psi_m^{-1}$, which is independent of U_{2m} variables, and $f_{2m} = f \circ \Psi_m^{-1}$, which is independent of U_{1m} variables.

As f is G-invariant, and as (Ω_m, Ψ_m) are isometric to each other, we conclude that $\int\limits_{U_{2m}} f_2$ does not depend on m. From this and the inequality (2.9), we obtain that

$$\begin{aligned} \int_M f dv(g) &\leq (1+\varepsilon) \int_{U_2} f_2 dy \sum_m \int_{U_{1m}} \alpha_{1m} dx \\ &\leq (1+\varepsilon) \int_{U_2} f_2 dy \sum_m \frac{1}{1-\varepsilon} \int_{U_{1m}} \alpha_{1m} \sqrt{\det(\tilde{g}_{ij})} dx \\ &= \frac{1+\varepsilon}{1-\varepsilon} \int_{U_2} f_2 dy \int_{O_G^x} \sum_m \alpha_{1m} \circ \Phi_{1m} dv(\tilde{g}). \end{aligned}$$

Thus,

$$\int_M f dv(g) \leq (1+\varepsilon_1) Vol(O_G^x) \int_{U_2} f_2 dy \tag{2.10}$$

with $\varepsilon_1 = \mathrm{O}(\varepsilon)$. In the same manner, we derive

$$\int_M |f|^p dv(g) \geq C \int_{U_2} |f_2|^p dy \tag{2.11}$$

and

$$\int_M |\triangle f|^p dv(g) \geq C \int_{U_2} |\triangle f_2|^p dy. \tag{2.12}$$

Lemma 2.2 and the techniques used in the proof of Lemma 2.4 imply that for any $f \in F^{i,p}_{c,G}((O^x_G)_{i,\delta})$,

$$\begin{aligned} Vol(O^x_G)\left\|\triangle f_2\right\|^p_{L^p(U_2)} &\leq (1+\varepsilon_2)\sum_m \int_{O^x_G} \alpha_{1m}\circ\Phi_{1m}dv(\tilde{g}) \int_{U_{2m}} \left|\triangle f_{2m}\right|^p dy \\ &\leq (1+\varepsilon_3)\sum_m \int_{U_{1m}} \alpha_{1m}dx \int_{U_{2m}} \left|\triangle f\right|^p \circ \Psi_m^{-1}dy \\ &\leq (1+\varepsilon_4)\sum_m \int_{U_{1m}} \alpha_{1m}dx \int_{U_{2m}} \left|\triangle_g f\right|^p \circ \Psi_m^{-1}dy \\ &+ C_\varepsilon \sum_m \int_{U_{1m}} \alpha_{1m}dx \int_{U_{2m}} \left|f\right|^p \circ \Psi_m^{-1}dy \\ &\leq \frac{1+\varepsilon_4}{1-\varepsilon}\sum_m \int_{U_{1m}\times U_{2m}} \sqrt{\det(g^m_{ij})}\alpha_m \left|\triangle_g f\right|^p \circ \Psi_m^{-1}dxdy \\ &+ \frac{C_\varepsilon}{1-\varepsilon}\sum_m \int_{U_{1m}\times U_{2m}} \sqrt{\det(g^m_{ij})}\alpha_m \left|f\right|^p \circ \Psi_m^{-1}dxdy \\ &\leq (1+\varepsilon_5)\left\|\triangle_g f\right\|^p_{L^p_G(M)} + C_\varepsilon \left\|f\right\|^p_{L^p_G(M)} \end{aligned} \tag{2.13}$$

with $\varepsilon_i = \mathrm{O}(\varepsilon)$ for $i = 2,3,4,5$. Thus, by using [3, Lemma 1] plus the inequalities (2.10) and (2.13), we conclude that

$$\begin{aligned} \left\|f\right\|^p_{L^{p^*}_G(M)} &\leq (1+\varepsilon_6)\left(Vol(O^x_G)\right)^{p/p^*}\left\|f_2\right\|^p_{L^{p^*}(U_2)} \\ &\leq \left[(Vol(O^x_G))^{p/p^*}\mathcal{K}^p(n-l,p)+\varepsilon_6\right]\left\|\triangle f_2\right\|^p_{L^p(U_2)} \\ &\leq \left[(Vol(O^x_G))^{(p-p^*)/p^*}\mathcal{K}^p(n-l,p)+\varepsilon_7\right]\left\|\triangle_g f\right\|^p_{L^p_G(M)} + C_\varepsilon\left\|f\right\|^p_{L^p_G(M)}, \end{aligned}$$

where $\varepsilon_6 = \mathrm{O}(\varepsilon)$, and $\varepsilon_7 = \mathrm{O}(\varepsilon)$.

On one hand, if O^x_G is of minimum dimension V, then

$$\left\|f\right\|^p_{L^{p^*}_G(M)} \leq \left(\mathcal{K}^p_G+\varepsilon\right)\left\|\triangle_g f\right\|^p_{L^p_G(M)} + C_\varepsilon\left\|f\right\|^p_{L^p_G(M)}. \tag{2.14}$$

On the other hand, if O^x_G is of minimum dimension $\tilde{V} > V$, let U_2 be an open set of dimension $n-\tilde{V}$; hence, the compactness of the embedding $F^{i,p}_G(U_2) \subset L^{p^*}(U_2)$ plus the inequalities (2.7)-(2.9), gives, by applying the approach used in the proof of Lemma 2.4, that for any $\varepsilon_0 > 0$ there exist $C, C_1, C_2 \in \mathbb{R}$ such that

$$\begin{aligned} \left(\int_M |f|^{p^*}dv(g)\right)^{p/p^*} &\leq \varepsilon_0 C\int_M |\triangle f|^p dv(g) + C\int_M |f|^p dv(g) \\ &\leq \varepsilon_0 C_1\int_M |\triangle_g f|^p dv(g) + C_2\int_M |f|^p dv(g). \end{aligned}$$

We finish the proof by choosing ε_0 small enough such that

$$\|f\|^p_{L^{p^*}_G(M)} \leq \left(\mathcal{K}^p_G + \varepsilon\right) \|\triangle_g f\|^p_{L^p_G(M)} + C\,\|f\|^p_{L^p_G(M)} \tag{2.15}$$

for any $f \in F^{i,p}_{c,G}\left((O^x_G)_{i,\delta}\right)$. □

As an immediate consequence of Lemmas 2.3-2.5, we obtain that the best constant $\alpha^p_p(M) = \frac{\mathcal{K}^p(n-l,p)}{V^{2p/(n-l)}}$.

3. Establishing the best constant $\beta^p_p(M)$

The goal of this section is to determine the best constant $\beta^p_p(M)$. More precisely, we prove the following theorem.

Theorem 3.1 *Let (M,g) be a compact $3 \leq n$-dimensional Riemannian manifold, and G a subgroup of the isometry group $Is(M,g)$. Let l be the minimum orbit dimension of G, V be the minimum of the volume of the l-dimensional orbits, and $1 < p < (n-l)/2$. Then for any $f \in F^{i,p}_G(M)$,*

$$\beta^p_p(M) = \left(Vol_{(M,g)}\right)^{-2p/(n-l)}.$$

To achieve this result, we need the following lemma.

Lemma 3.2 *Let (M,g) be a compact $3 \leq n$-dimensional Riemannian manifold, and G a subgroup of the isometry group $Is(M,g)$. Then there exists $C \in \mathbb{R}$ such that for any $f \in F^{i,p}_G(M)$ with $\|\triangle_g f\|_{L^p_G(M)} \neq 0$ one has*

$$\|f - (f)_M\|_{L^p_G(M)} \leq C\,\|\triangle_g f\|_{L^p_G(M)}, \tag{3.1}$$

where $(f)_M = \frac{1}{Vol_{(M,g)}} \int\limits_M f dv(g)$.

Proof. To prove (3.1), it is enough to show

$$\inf_{f \in \mathcal{H}^{i,p}_G(M)} \|\triangle_g f\|_{L^p_G(M)} > 0, \tag{3.2}$$

where

$$\mathcal{H}^{i,p}_G(M) = \left\{ f \in F^{i,p}_G(M) : \|f\|_{L^p_G(M)} = 1 \text{ and } (f)_M = 0 \right\}.$$

Let $\{f_j\} \subset \mathcal{H}^{i,p}_G(M)$ be such that

$$\lim_{j\to\infty} \|\triangle_g f_j\|^p_{L^p_G(M)} = \inf_{f\in\mathcal{H}^{i,p}(M)} \|\triangle_g f\|^p_{L^p_G(M)}.$$

The Rellich-Kondrakov theorem (see, for instance, [10]) with the reflexivity of $F^{i,p}_G(M)$, implies that there exists a subsequence $\{f_{j_m}\}$ of $\{f_j\}$ such that $f_{j_m} \rightharpoonup h$ in $F^{i,p}_G(M)$, and $f_{j_m} \to h$ in $L^p_G(M) \cap L^1_G(M)$. Hence, $h \in \mathcal{H}^{i,p}_G(M)$ and then

$$\inf_{f\in\mathcal{H}^{i,p}_G(M)} \|\triangle_g f\|_{L^p_G(M)} \geq \|\triangle_g h\|_{L^p_G(M)} > 0.$$

This completes the proof of the lemma. □

Proof of Theorem 3.1. We prove Theorem 3.1 by applying the same approaches that Hebey [12] used. On one hand, suppose $\|\triangle_g f\|_{L^p(M)} > 0$, then by using Minkowski's inequality, Hölder's inequality, Lemma 2.5, and Lemma 3.1, we find $A, B, C \in \mathbb{R}$ so that

$$\begin{aligned}
\|f\|_{L_G^{p^*}(M)} &\leq \|f-(f)_M\|_{L_G^{p^*}(M)} + \|(f)_M\|_{L_G^{p^*}(M)} \\
&\leq A\|f-(f)_M\|_{L_G^{p}(M)} + B\|\triangle_g f\|_{L_G^{p}(M)} + \|(f)_M\|_{L_G^{p^*}(M)} \\
&\leq C\|\triangle_g f\|_{L_G^{p}(M)} + \left(Vol_{(M,g)}\right)^{1/p^*-1/p}\|f\|_{L_G^{p}(M)} \\
&= C\|\triangle_g f\|_{L_G^{p}(M)} + \left(Vol_{(M,g)}\right)^{-2/(n-l)}\|f\|_{L_G^{p}(M)}\,. \qquad (3.3)
\end{aligned}$$

Consequently, for any $\varepsilon > 0$, there exists a real number B_1 such that

$$\|f\|^p_{L_G^{p^*}(M)} \leq \left(\left(Vol_{(M,g)}\right)^{-2p/(n-l)} + \varepsilon\right)\|f\|^p_{L_G^{p}(M)} + B_1\|\triangle_g f\|^p_{L_G^{p}(M)}\,. \qquad (3.4)$$

On the other hand, setting $f = 1$ in $(I^p_{p,G,\text{gen}})$ gives that $A \geq \left(Vol_{(M,g)}\right)^{-2p/(n-l)}$. Therefore,

$$\beta^p_p(M) \geq \left(Vol_{(M,g)}\right)^{-2p/(n-l)}\,. \qquad (3.5)$$

In particular, this proof showed more than what we desire; it proved that $(I^1_{p,G,\text{opt}})$ is attained for all $1 < p < (n-l)/2$.

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