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Flat surfaces in the Minkowski space $\mathbb{E}^3_1$ with pointwise 1-type Gauss map

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Abstract

In this article, we obtain all nonplanar cylindrical surfaces in the Minkowski space $\mathbb{E}^3_1$ with pointwise 1-type Gauss map of the second kind. We also prove that right circular cones and hyperbolic cones in $\mathbb{E}^3_1$ are the only cones in $\mathbb{E}^3_1$ with pointwise 1-type Gauss map of the second kind. We conclude that there is no tangent developable surface fully lying in $\mathbb{E}^3_1$ with pointwise 1-type Gauss map of the second kind.

Key Words: Gauss map, Pointwise 1-type, Ruled surface, Cone, Cylinder, Developable surface

1. Introduction

In late 1970's B.-Y. Chen introduced the notion of finite type submanifolds of Euclidean space [6]. Since then many works were done to characterize or classify submanifolds of Euclidean space or pseudo-Euclidean space in terms of finite type (cf. [7, 8, 12, 16]). Also, B.-Y. Chen and P. Piccinni extended the notion of finite type to differentiable maps, in particular, to Gauss map of submanifolds in [9]. A smooth map ϕ of a submanifold M of a Euclidean space or a pseudo-Euclidean space is said to be of *finite type* if ϕ can be expressed as a finite sum of eigenfunctions of the Laplacian Δ of M, that is, $\phi = \phi_0 + \sum_{i=1}^k \phi_i$, where ϕ_0 is a constant map, $\phi_1, \ldots, \phi_k$ nonconstant maps such that $\Delta\phi_i = \lambda_i \phi_i$, $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, k$.

If a submanifold M of a Euclidean space or a pseudo-Euclidean space has 1-type Gauss map G, then G satisfies $\Delta G = \lambda(G + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector C. In [9], B.-Y. Chen and P. Piccinni studied compact submanifolds of Euclidean spaces with finite type Gauss map. Several articles also appeared on submanifolds with finite type Gauss map (cf. [2, 3, 4, 5, 24, 25]).

However, the Laplacian of the Gauss map of several surfaces and hypersurfaces, such as helicoids of the 1st, 2nd, and 3rd kind, conjugate Enneper's surface of the second kind and B-scrolls in a 3-dimensional Minkowski space $\mathbb{E}^3_1$ [20], generalized catenoids, spherical n-cones, hyperbolical n-cones and Enneper's hypersurfaces in $\mathbb{E}^{n+1}_1$ [14], take the form

$$\Delta G = f(G + C) \tag{1}$$

for some smooth function f on M and some constant vector C. A submanifold of a pseudo-Euclidean space is said to have *pointwise 1-type Gauss map* if its Gauss map satisfies (1) for some smooth function f on M and

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some constant vector C. A submanifold with pointwise 1-type Gauss map is said to be of *the first kind* if the vector C in (1) is the zero vector. Otherwise, a submanifold with pointwise 1-type Gauss map is said to be of *the second kind* (cf. [1, 10, 11, 13, 19, 21, 22]).

Remark 1. The Gauss map G of a plane M in $\mathbb{E}_1^3$ is a constant vector and $\Delta G = 0$. For $f = 0$ if we write $\Delta G = 0 \cdot G$, then M has pointwise 1-type Gauss map of the first kind. If we choose $C = -G$ for any nonzero smooth function f, then (1) holds. In this case M has pointwise 1-type Gauss map of the second kind. Therefore we say that a plane in $\mathbb{E}_1^3$ is a trivial surface with pointwise 1-type Gauss map of the first kind or the second kind.

The complete classification of ruled surfaces in $\mathbb{E}_1^3$ with pointwise 1-type Gauss map of the first kind was obtained in [20]. Also, a complete classification of rational surfaces of revolution in $\mathbb{E}_1^3$ satisfying (1) was recently given in [19], and it was proved that a right circular cone and a hyperbolic cone in $\mathbb{E}_1^3$ are the only rational surfaces of revolution in $\mathbb{E}_1^3$ with pointwise 1-type Gauss map of the second kind. The first author described all nonplanar cylindrical surfaces in the Euclidean space $\mathbb{E}^3$ with pointwise 1-type Gauss map of the second kind [15].

In this article, we study nondegenerate flat surfaces in $\mathbb{E}_1^3$ with pointwise 1-type Gauss map of the second kind. We describe all nonplanar cylindrical surfaces with pointwise 1-type Gauss map of the second kind, and we also show that right circular cones and hyperbolic cones in $\mathbb{E}_1^3$ are the only cones in $\mathbb{E}_1^3$ with pointwise 1-type Gauss map of the second kind. We conclude that there is no tangent developable surface in $\mathbb{E}_1^3$ with pointwise 1-type Gauss map of the second kind.

Throughout this paper, we assume that all the geometric objects are smooth and all surfaces are connected unless otherwise stated.

2. Preliminaries

Let $\mathbb{E}_1^3$ be a 3-dimensional Minkowski space with the Lorentz metric $ds^2 = -dx_1^2 + dx_2^2 + dx_3^2$, where (x_1, x_2, x_3) denotes the standard coordinates of $\mathbb{E}_1^3$. A vector $X \in \mathbb{E}_1^3$ is said to be space-like if $\langle X, X\rangle > 0$ or $X = 0$, time-like if $\langle X, X\rangle < 0$, and light-like or null if $\langle X, X\rangle = 0$ and $X \neq 0$. A curve in $\mathbb{E}_1^3$ is said to be space-like, time-like or light-like (null) if its tangent vector is, respectively, space-like, time-like or light-like. A time-like or light-like vector in $\mathbb{E}_1^3$ is said to be *causal*. For the Lorentz vector space it is well known that there are no causal vectors in $\mathbb{E}_1^3$ orthogonal to a time-like vector [18].

For two vectors $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in \mathbb{E}_1^3$, the Lorentz cross-product $X \times Y$ of X and Y is defined by

$$X \times Y = (-x_2y_3 + x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

The properties that the Lorentz cross-product satisfies can be seen in [20].

Let M be a nondegenerate surface in $\mathbb{E}_1^3$. The map $G : M \to Q^2(\varepsilon_G) \subset \mathbb{E}_1^3$ which sends each point of M to the unit normal vector to M at the point is called the Gauss map of M, where $\varepsilon_G (= \pm 1)$ denotes the sign of the vector G and $Q^2(\varepsilon_G)$ is a 2-dimensional space form given by

$$Q^2(\varepsilon_G) = \begin{cases} \mathbb{S}_1^2(1) & \text{in } \mathbb{E}_1^3 \quad \text{if } \varepsilon_G = 1 \\ \mathbb{H}^2(-1) & \text{in } \mathbb{E}_1^3 \quad \text{if } \varepsilon_G = -1, \end{cases}$$

where $\mathbb{S}_1^2(1)$ and $\mathbb{H}^2(-1)$ are, respectively, the de Sitter space and hyperbolic space in $\mathbb{E}_1^3$ centered at the origin.

We denote by h, A_G, $\widetilde{\nabla}$ and ∇, the second fundamental form, the Weingarten map, the Levi-Civita connection of $\mathbb{E}_1^3$ and the induced Riemannian connection on M, respectively. We choose a local oriented orthonormal moving frame $\{e_1, e_2, e_3\}$ on M in $\mathbb{E}_1^3$ with $\varepsilon_i = \langle e_i, e_i\rangle\,(=\mp 1)$, $i = 1, 2, 3$, such that e_1, e_2 are tangent to M and $e_3 = G$ is normal to M.

We denote by $\{\omega_1, \omega_2, \omega_3\}$ the dual 1-forms to $\{e_1, e_2, e_3\}$ defined by $\omega_A(e_B) = \langle e_A, e_B\rangle = \varepsilon_A\delta_{AB}$ and by $\{\omega_{AB}\}, A, B = 1, 2, 3$, the connection 1-forms associated with $\{\omega_1, \omega_2, \omega_3\}$ satisfying $\omega_{AB} + \omega_{BA} = 0$. Thus we have $\widetilde{\nabla}_{e_k} e_i = \sum_{j=1}^{2} \varepsilon_j \omega_{ij}(e_k) e_j + \varepsilon_3 h_{ik} e_3$, $\widetilde{\nabla}_{e_k} e_3 = \sum_{j=1}^{2} \varepsilon_j \omega_{3j}(e_k) e_j$, where h_{ik}'s are the coefficients of the second fundamental form h. By Cartan's Lemma, we also have $\omega_{j3} = \sum_{k=1}^{2} \varepsilon_k h_{jk} \omega_k$, $h_{jk} = h_{kj}$.

The mean curvature H and the Gauss curvature K of M in $\mathbb{E}_1^3$ are, respectively, defined by

$$H = \frac{1}{2}\mathrm{tr}A_G = \frac{1}{2}\sum_{i=1}^{2} \varepsilon_i \langle A_G(e_i), e_i\rangle \quad \text{and} \quad K = \varepsilon_G \det A_G.$$

A nondegenerate surface in $\mathbb{E}_1^3$ with zero Gauss curvature is called a developable surface. The developable surfaces in Minkowski space $\mathbb{E}_1^3$ are the same as in Euclidean space. In particular, they are plane, cone, cylinder and tangent developable surfaces.

Let I and J be open intervals containing the origin in the real line. Let $\alpha = \alpha(s)$ be a curve from J into $\mathbb{E}_1^3$ and $\beta(s)$ a vector field along $\alpha(s)$ orthogonal to $\alpha(s)$. A ruled surface M in $\mathbb{E}_1^3$ is defined as a semi-Riemannian surface swept out by the vector $\beta(s)$ along the curve $\alpha(s)$. Then M has always a parametrization

$$x(s,t) = \alpha(s) + t\beta(s), \quad s \in J,\ t \in I.$$

The curve $\alpha = \alpha(s)$ is called a base curve and $\beta = \beta(s)$ is a director curve. If β is constant, then the ruled surface is said to be cylindrical, and noncylindrical otherwise.

We consider the curve α is space-like or time-like. As it is explained in [20] we have five different ruled surfaces according to the character of the base curve α and the director β as follows: If the curve α is space-like or time-like, the ruled surface M is said to be of type M_+ or type M_-, respectively. Also the ruled surface of type M_+ is divided into three types. When β is space-like, it is said to be of type M_+^1 or M_+^2 if β' is non-null or light-like, respectively. When β is time-like, then β' must be space-like because there is no causal vector in $\mathbb{E}_1^3$ orthogonal to a time-like vector. In this case, M is said to be of type M_+^3. On the other hand, for the ruled surface M_- it is said to be of type M_-^1 or M_-^2 if β' is non-null or light-like, respectively. The ruled surface type M_+^1 or M_+^2 (resp. M_+^3, M_-^1 or M_-^2) is space-like (resp. time-like).

However if the base curve α is a light-like curve and the vector field β along α is a light-like vector field, then the ruled surface is called a null scroll. A null scroll with zero Gauss curvature is a plane in $\mathbb{E}_1^3$. In particular, a null scroll with Cartan frame is said to be a B-scroll [17] which is a time-like surface. It is known that a B-scroll has 1-type Gauss map of the first kind [20].

For the Frenet equations of a space-like or time-like curve in $\mathbb{E}_1^3$ we have the following theorem.

Theorem 2.1 [*23*] *Let α be a space-like or time-like curve which we assume to be parametrized by arc length and satisfies $\langle\alpha'', \alpha''\rangle \neq 0$. Then this curve induces a Frenet 3-frame $T = \alpha'(s)$, $N = \frac{\alpha''(s)}{\sqrt{|\langle\alpha'', \alpha''\rangle|}}$, $B = T \times N$*

for which the following Frenet equations hold:

$$\begin{aligned}T'(s) =&\varepsilon_N k(s)N(s),\\ N'(s) =&-\varepsilon_T k(s)T-\varepsilon_T\varepsilon_N\tau(s)B(s),\\ B'(s) =&-\varepsilon_N\tau(s)N(s),\end{aligned}\tag{2}$$

where $k=\langle T',N\rangle$ *and* $\tau=\langle N',B\rangle$ *are called the curvature and torsion of* α*, and* $\varepsilon_T=\langle T,T\rangle=\mp1$*,* $\varepsilon_N=\langle N,N\rangle=\mp1$*.*

Let M be an oriented nondegenerate surface (time-like or space-like) in $\mathbb{E}_1^3$ with corresponding unit normal field G, and let α be an arc length parametrized curve in M. Let V be a unit tangent vector field along $\alpha(s)$ such that $V(s)=G(\alpha(s))\times\alpha'(s)$ with $\varepsilon_V=\langle V,V\rangle=-\varepsilon_G\varepsilon_T$. Then the functions

$$k_n(s)=\langle\alpha''(s),G(\alpha(s))\rangle\quad\text{and}\quad k_g(s)=\langle\alpha''(s),V(s)\rangle$$

are, respectively, called the *normal curvature* and the *geodesic curvature* of $\alpha(s)$ at s if α'' is non-null. If α'' is non-null, then we can write $\alpha''(s)$ as follows

$$\alpha''(s)=\varepsilon_N k(s)N(s)=\varepsilon_V k_g(s)V(s)+\varepsilon_G k_n(s)G(\alpha(s)),$$

where $k(s)$ is the curvature of $\alpha''(s)$, and thus we have

$$\varepsilon_N k^2(s)=\varepsilon_G(-\varepsilon_T k_g^2(s)+k_n^2(s)).\tag{3}$$

Note that there is no a definition of curvature when $\alpha''(s)$ is null.

3. Cylindrical ruled surfaces with pointwise 1-type Gauss map

Considering Remark 1, a plane in the Minkowski space $\mathbb{E}_1^3$ which is a cylinder has pointwise 1-type Gauss map of the second kind. Here we determine nonplanar cylindrical ruled surfaces in $\mathbb{E}_1^3$ with pointwise 1-type Gauss map of the second kind. A cylindrical ruled surface M is only of type M_+^1, M_-^1 or M_+^3.

The following lemma is obtained in [14].

Lemma 3.1 *Let* M_q *be a hypersurface with index* q *in a Lorentz-Minkowski space* L^{n+1}*. Then the Laplacian of the Gauss map* G *is given by*

$$\Delta G=\varepsilon_G\|A_G\|^2G+n\nabla H,\tag{4}$$

where $\|A_G\|^2=tr(A_GA_G)$*,* $\varepsilon_G=\langle G,G\rangle$ *and* H *is the mean curvature of* M_q*.*

We prove the following lemma for later use.

Lemma 3.2 *Let* M *be an oriented nondegenerate surface in the Minkowski space* $\mathbb{E}_1^3$*. Let* $\{e_1,e_2\}$ *be a local orthonormal tangent frame on* M *with* $\varepsilon_i=<e_i,e_i>$*,* $i=1,2$*. If* C *is a constant vector in* $\mathbb{E}_1^3$*, then the components of* $C=\varepsilon_1C_1\,e_1+\varepsilon_2C_2e_2+\varepsilon_GC_3G$ *in the basis* $\{e_1,e_2,G\}$ *of* $\mathbb{E}_1^3$ *satisfy the following equations:*

$$e_1(C_1)+\varepsilon_2\omega_{21}(e_1)\,C_2-\varepsilon_Gh_{11}\,C_3=0,\tag{5}$$

$$e_1(C_2) + \varepsilon_1\omega_{12}(e_1)\, C_1 - \varepsilon_G h_{12}\, C_3 = 0, \tag{6}$$

$$e_1(C_3) + \varepsilon_1 h_{11}\, C_1 + \varepsilon_2 h_{21}\, C_2 = 0, \tag{7}$$

$$e_2(C_1) + \varepsilon_2\omega_{21}(e_2)\, C_2 - \varepsilon_G h_{21}\, C_3 = 0, \tag{8}$$

$$e_2(C_2) + \varepsilon_1\omega_{12}(e_2)\, C_1 - \varepsilon_G h_{22}\, C_3 = 0, \tag{9}$$

$$e_2(C_3) + \varepsilon_1 h_{12}\, C_1 + \varepsilon_2 h_{22}\, C_2 = 0, \tag{10}$$

where $C_i = \langle C, e_i\rangle$, $i = 1, 2$ *and* $C_3 = \langle C, G\rangle$.

Proof. Taking derivative of the vector C in direction e_k and using the formulas of Gauss and Weingarten, we obtain

$$\begin{aligned}\widetilde{\nabla}_{e_k} C =& \varepsilon_1[e_k(C_1) + \varepsilon_2\omega_{21}(e_k)\, C_2 - \varepsilon_G h_{k1}\, C_3]e_1\\ &+ \varepsilon_2[e_k(C_2) + \varepsilon_1\omega_{12}(e_k)\, C_1 - \varepsilon_G h_{k2}\, C_3]e_2\\ &+ \varepsilon_G[e_k(C_3) + \varepsilon_1 h_{1k}\, C_1 + \varepsilon_2 h_{2k}\, C_2]G = 0\end{aligned}$$

which produces equations (5)–(10) for $k = 1, 2$. □

Theorem 3.3 *A nonplanar cylindrical ruled surface M in the Minkowski space $\mathbb{E}^3_1$ has pointwise 1-type Gauss map of the second kind if and only if M is congruent to an open part of the following surfaces:*

1. *the time-like cylinder M^3_+ parametrized by*

$$x(k,t) = \Big(t,\ \pm\Big(\frac{(k+k_0)\sqrt{R(k)}}{2c_0k_0k^2} + \frac{c_0}{2k_0}\arctan\Big(\frac{k-k_0}{\sqrt{R(k)}}\Big)\Big),\ -\frac{k_0}{2c_0k^2}\Big), \tag{11}$$

where $R(k) = c_0^2k^2 - (k-k_0)^2 > 0$;

2. *the space-like cylinder M^1_+ parametrized by*

$$x(k,t) = \Big(\pm\varphi(k), \frac{k_0}{2c_0k^2},\ t\Big); \tag{12}$$

3. *the space-like cylinder M^1_+ parametrized by*

$$x(k,t) = \Big(\frac{k_0}{2c_0k^2}, \pm\psi(k),\ t\Big); \tag{13}$$

4. *the space-like cylinder M^1_+ parametrized by*

$$x(k,t) = \Big(\pm\big(\frac{k_0}{4k^2} - \theta(k)\big),\ \frac{k_0}{4k^2} + \theta(k),\ t\Big); \tag{14}$$

5. the time-like cylinder M^1_- parametrized by

$$x(k,t)=\Big(\pm\,\psi(k),\ \frac{k_0}{2c_0k^2},\ t\Big); \tag{15}$$

6. the time-like cylinder M^1_- parametrized by

$$x(k,t)=\Big(\frac{k_0}{2c_0k^2},\ \pm\varphi(k),\ t\Big); \tag{16}$$

7. the time-like cylinder M^1_- parametrized by

$$x(k,t)=\Big(\frac{k_0}{4k^2}+\theta(k),\ \pm\Big(\frac{k_0}{4k^2}-\theta(k)\Big),\ t\Big); \tag{17}$$

where

$$\varphi(k)=\frac{(k+k_0)}{2c_0k_0k^2}\sqrt{c_0^2k^2+(k-k_0)^2}-\frac{c_0}{2k_0}\ln\Big|\frac{k_0-k+\sqrt{c_0^2k^2+(k-k_0)^2}}{k}\Big|,$$

$$\psi(k)=\frac{(k+k_0)}{2c_0k_0k^2}\sqrt{(k-k_0)^2-c_0^2k^2}+\frac{c_0}{2k_0}\ln\Big|\frac{k_0-k+\sqrt{(k-k_0)^2-c_0^2k^2}}{k}\Big|,$$

with $(k-k_0)^2-c_0^2k^2>0$*,*

$$\theta(k)=\frac{1}{2(k-k_0)}-\frac{1}{2k_0}\ln\Big|\frac{k}{k-k_0}\Big|,$$

and, p_0*,* c_0 *and* k_0 *are nonzero constants.*

Proof. Suppose that M has pointwise 1-type Gauss map of the second kind. Then the gradient vector ∇H of the mean curvature H is nonzero on M because of (3.1). If ∇H were zero, then the Gauss map would be of pointwise 1-type of the first kind. So the mean curvature H is a nonconstant function on M.

Let $\{e_1,e_2\}$ be a local orthonormal tangent frame on M with $\varepsilon_i=<e_i,e_i>$, $i=1,2$. By equations (1) and (4), we have

$$\varepsilon_G\|A_G\|^2G+2\nabla H=f(G+C) \tag{18}$$

for some nonzero smooth function f on M and some nonzero constant vector $C\in\mathbb{E}^3_1$. In the basis $\{e_1,e_2,G\}$ we can write

$$C=\varepsilon_1C_1\,e_1+\varepsilon_2C_2e_2+\varepsilon_GC_3G,$$

where $C_i=\langle C,e_i\rangle$, $i=1,2$ and $C_3=\langle C,G\rangle$ which satisfy equations (5)–(10) in Lemma 3.2. Considering $\nabla H=\varepsilon_1e_1(H)\,e_1+\varepsilon_2e_2(H)e_2$ equation (18) implies

$$\varepsilon_G||A_G||^2=f(1+\varepsilon_GC_3), \tag{19}$$

$$e_1(\varepsilon_1h_{11}+\varepsilon_2h_{22})=f\,C_1, \tag{20}$$

$$e_2(\varepsilon_1h_{11}+\varepsilon_2h_{22})=f\,C_2. \tag{21}$$

Since M is a cylindrical surface, then it is parametrized by

$$x(s,t) = \alpha(s) + t\beta, \tag{22}$$

where the base curve $\alpha(s)$ which is a smooth time-like or space-like curve with the arc length parameter s lies in a plane with a time-like or space-like unit normal vector β which is the director of the cylinder.

Now we take a local orthonormal tangent frame $\{e_1, e_2\}$ on M as $e_1 = \frac{\partial}{\partial t}$ and $e_2 = \frac{\partial}{\partial s}$ with $\varepsilon_1 = \langle e_1, e_1\rangle = \langle \beta, \beta\rangle = \pm 1$, $\varepsilon_2 = \langle e_2, e_2\rangle = \langle \alpha'(s), \alpha'(s)\rangle = \pm 1$ and $\langle e_1, e_2\rangle = \langle \beta, \alpha'(s)\rangle = 0$. By taking the Gauss map G as $G = e_1 \times e_2$ with $\varepsilon_G = \langle G, G\rangle = -\varepsilon_1\varepsilon_2$, then the Frenet 3-frame for the curve α and the frame $\{e_1, e_2, G\}$ on M in $\mathbb{E}^3_1$ have the same orientation.

By a direct calculation we obtain $\widetilde{\nabla}_{e_2} e_2 = \alpha''(s) = \varepsilon_G k(s) G$ because α is a plane curve and the principal normal vector of the curve α is the normal of the cylinder, and $\widetilde{\nabla}_{e_1} e_1 = \widetilde{\nabla}_{e_1} e_2 = \widetilde{\nabla}_{e_2} e_1 = 0$, where $k(s)$ is the curvature of $\alpha(s)$. All these relations imply that $\omega_{21}(e_1) = \omega_{21}(e_2) = 0$, $h_{11} = h_{12} = h_{21} = 0$, and $h_{22} = k(s)$. Therefore the mean curvature is $H = \varepsilon_2 k(s)/2$ which is the function of s, and $\|A_G\|^2 = k^2(s)$, where $k(s) \neq 0$, i.e. $k(s)$ is strictly positive or strictly negative. Without losing generality we suppose that $k(s) > 0$. Thus equations (20) and (21) give, respectively, $C_1 = 0$ and $C_2 \neq 0$.

On the other hand it follows from equations (5)–(7) that $C_1, C_2,$ and C_3 are functions of s, and equations (9) and (10) give, respectively

$$C_2'(s) - \varepsilon_G k(s)\, C_3(s) = 0 \tag{23}$$

and

$$C_3'(s) + \varepsilon_2 k(s)\, C_2(s) = 0. \tag{24}$$

It is seen from (21) that f is also a function of s. As the vector C is constant, we have

$$\varepsilon_2 C_2^2(s) - \varepsilon_1\varepsilon_2 C_3^2(s) = \langle C, C\rangle = \varepsilon_C c_0^2, \tag{25}$$

where c_0 is a constant and $\varepsilon_C = \operatorname{sgn}(\langle C, C\rangle)$.

Equations (19) and (21) yield

$$\frac{k'(s)}{k^2(s)} = \frac{\varepsilon_2\varepsilon_G C_2}{1 + \varepsilon_G C_3} \tag{26}$$

from which and equation (24) we obtain

$$\frac{k'(s)}{k(s)} = -\frac{\varepsilon_G C_3'}{1 + \varepsilon_G C_3} \tag{27}$$

and from its solution we get

$$C_3(s) = \varepsilon_1\varepsilon_2\Big(1 - \frac{k_0}{k(s)}\Big), \tag{28}$$

where k_0 is a nonzero constant. Also, by using (24) and (28)

$$C_2(s) = -\varepsilon_1 \frac{k_0 k'(s)}{k^3(s)}. \tag{29}$$

Moreover, from (21) and (29) we obtain

$$f(s) = -\varepsilon_1\varepsilon_2\frac{k^3(s)}{k_0}. \tag{30}$$

If C is a non-null vector, then using (28) and (29) equation (25) yields the differential equation

$$k_0^2{k'}^2 = k^4[\varepsilon_2\varepsilon_C c_0^2k^2 + \varepsilon_1(k-k_0)^2]. \tag{31}$$

For later use we need

$$\int C_2(s)ds = -\varepsilon_1\int\frac{k_0k'}{k^3}ds + d_2 = \varepsilon_1\frac{k_0}{2k^2} + d_2 \tag{32}$$

and by considering (31)

$$\int C_3(s)ds = \varepsilon_1\varepsilon_2\int\frac{k-k_0}{k}ds = \pm\varepsilon_1\varepsilon_2k_0\int\frac{(k-k_0)dk}{k^3\sqrt{\varepsilon_2\varepsilon_C c_0^2k^2+\varepsilon_1(k-k_0)^2}} + d_3,$$

where d_1 and d_2 are integration constants. From the evaluation of the last integral for $\varepsilon_1 = 1$ we have

$$\int C_3(s)ds = \pm\Big(\varepsilon_2\frac{(k+k_0)}{2k_0k^2}\sqrt{R(k)} - \frac{\varepsilon_C c_0^2}{2k_0}\ln\Big(\frac{k_0-k+\sqrt{R(k)}}{k}\Big)\Big) + d_3, \tag{33}$$

where $R(k) = \varepsilon_2\varepsilon_C c_0^2k^2 + (k-k_0)^2$, and for $\varepsilon_1 = -1$ (in this case $\varepsilon_2 = \varepsilon_C = 1$) we have

$$\int C_3(s)ds = \mp\Big(\frac{(k+k_0)}{2k_0k^2}\sqrt{R(k)} + \frac{c_0^2}{2k_0}\arctan\Big(\frac{k-k_0}{\sqrt{R(k)}}\Big)\Big) + d_3, \tag{34}$$

where $R(k) = c_0^2k^2 - (k-k_0)^2 > 0$.

A cylindrical ruled surface M is only of type M_+^1, M_-^1 or M_+^3.

Case 1. M is of type M_+^3, i.e., the vector β is time-like. Hence $\varepsilon_1 = -1$ and $\varepsilon_2 = \varepsilon_G = \varepsilon_C = 1$. Considering equation (25), we may put

$$C_2(s) = c_0\sin\lambda(s), \quad C_3(s) = c_0\cos\lambda(s), \tag{35}$$

which satisfy equations (23) and (24) if $\lambda'(s) = k(s)$, that is, $\lambda(s) = \lambda_0 + \int k(s)ds$, where λ_0 is an integration constant. Thus we have

$$\sin\lambda(s) = \frac{C_2}{c_0} = \frac{k_0k'}{c_0k^3} \quad \text{and} \quad \cos\lambda(s) = \frac{C_3}{c_0} = \frac{k_0-k}{c_0k} \tag{36}$$

for later use.

Since α is a plane curve, acting a Lorentz transformation we can write

$$\alpha(s) = (0, \alpha_2(s), \alpha_3(s)) \text{ and } \beta = (1,0,0)$$

without loss of generality. Then the Gauss map of the cylinder M_+^3 is

$$G = e_1\times e_2 = (0, -\alpha_3'(s), \alpha_2'(s)),$$

as $e_2 = \alpha'(s) = (0, \alpha_2'(s), \alpha_3'(s))$. Now we may put $\alpha_2'(s) = \cos\mu(s)$ and $\alpha_3'(s) = \sin\mu(s)$ because of ${\alpha_2'}^2(s) + {\alpha_3'}^2(s) = 1$, where $\mu(s)$ is a differentiable function.

The equation $\alpha''(s) = \varepsilon_G k(s)G$ implies that $\mu'(s) = k(s)$. For simplicity we take $\mu(s) = \lambda(s) = \lambda_0 + \int k(s)ds$. In view of (32), (34) and (36) the base curve $\alpha(s)$ of the cylinder M_+^3 is determined uniquely, up to a rigid motion, by

$$\begin{aligned} \alpha(s) &= (0,\ d_3 + \frac{1}{c_0}\int C_3(s)ds,\ d_2 + \frac{1}{c_0}\int C_2(s)ds), \\ &= \Big(0,\ d_3 \pm \Big(\frac{(k+k_0)\sqrt{R(k)}}{2c_0k_0k^2} + \frac{c_0}{2k_0}\arctan\Big(\frac{k-k_0}{\sqrt{R(k)}}\Big)\Big),\ d_2 - \frac{k_0}{2c_0k^2}\Big), \end{aligned} \tag{37}$$

where $R(k) = c_0^2k^2 - (k-k_0)^2 > 0$. It is seen that the base curve of the cylinder M_+^3 can be parametrized in terms of the curvature function k, that is, $\alpha = \alpha(k)$. Therefore we obtain the parametrization (11) for the cylinder M_+^3 which has pointwise 1-type Gauss map of the second kind for $f(k) = \frac{k^3}{k_0}$ and $C = (0,\ 0,\ c_0)$.

Case 2. M is of type M_+^1, i.e., $\varepsilon_1 = \varepsilon_2 = 1, (\varepsilon_G = -1)$. From equation (25), the vector C is space-like, time-like or null.

Considering equation (25) we may put

$$C_2(s) = c_0\cosh\lambda(s), \quad C_3(s) = c_0\sinh\lambda(s) \quad \text{for} \quad \varepsilon_C = 1$$

or

$$C_2(s) = c_0\sinh\lambda(s), \quad C_3(s) = c_0\cosh\lambda(s) \quad \text{for} \quad \varepsilon_C = -1$$

which hold for equations (23) and (24) if $\lambda'(s) = -k(s)$, that is, $\lambda(s) = \lambda_0 - \int k(s)ds$, where λ_0 is an integration constant. Thus we have

$$\cosh\lambda(s) = \frac{C_2}{c_0} = -\frac{k_0k'}{c_0k^3} \quad \text{and} \quad \sinh\lambda(s) = \frac{C_3}{c_0} = \frac{k_0-k}{c_0k} \quad \text{for } \varepsilon_C = 1 \tag{38}$$

or

$$\cosh\lambda(s) = \frac{C_3}{c_0} = \frac{k_0-k}{c_0k} \quad \text{and} \quad \sinh\lambda(s) = \frac{C_2}{c_0} = -\frac{k_0k'}{c_0k^3} \quad \text{for } \varepsilon_C = -1. \tag{39}$$

For the plane curve α, acting a Lorentz transformation we can write

$$\alpha(s) = (\alpha_1(s), \alpha_2(s), 0) \ \text{ and } \beta = (0,0,1)$$

without loss of generality. The Gauss map of the cylinder M_+^1 is

$$G = e_1 \times e_2 = (\alpha_2'(s),\ \alpha_1'(s),\ 0)$$

as $e_2 = \alpha'(s) = (\alpha_1'(s), \alpha_2'(s), 0)$. Considering $-{\alpha_1'}^2(s) + {\alpha_2'}^2(s) = 1$, we may put $\alpha_1'(s) = \sinh\mu(s)$ and $\alpha_2'(s) = \cosh\mu(s)$ to determine $\alpha(s)$, where μ is a differentiable function. From the equation $\alpha''(s) = \varepsilon_G k(s)G$ we obtain $\mu'(s) = -k(s)$. For simplicity we take $\mu(s) = \lambda(s) = \lambda_0 - \int k(s)ds$.

Now we suppose that C is space-like, i.e., $\varepsilon_C = 1$. By using (32), (33) and (38) the base curve $\alpha(s)$ of the cylinder M^1_+ is determined uniquely, up to a rigid motion, by

$$\alpha(s) = (d_3 + \frac{1}{c_0}\int C_3(s)ds,\ d_2 + \frac{1}{c_0}\int C_2(s)ds,\ 0),$$
$$= \Big(d_3 \pm \Big(\frac{(k+k_0)}{2c_0k_0k^2}\sqrt{R(k)} - \frac{c_0}{2k_0}\ln\Big|\frac{k_0 - k + \sqrt{R(k)}}{k}\Big|\Big),\ d_2 + \frac{k_0}{2c_0k^2},\ 0\Big),$$

where $R(k) = c_0^2k^2 + (k-k_0)^2$. It is seen the base curve of the cylinder M^1_+ can be parametrized in terms of the curvature function k, that is, $\alpha = \alpha(k)$.

Therefore we obtain the parametrization (12) for the cylinder M^1_+ which has pointwise 1-type Gauss map of the second kind for $f(k) = -\frac{k^3}{k_0}$ and $C = (0,\ c_0, 0\ )$.

If C is time-like, i.e., $\varepsilon_C = -1$, then by a similar argument we obtain the base curve of the cylinder M^1_+ as

$$\alpha(k) = \Big(d_2 + \frac{k_0}{2c_0k^2}, d_3 \pm \Big(\frac{(k+k_0)}{2c_0k_0k^2}\sqrt{R(k)} + \frac{c_0}{2k_0}\ln\Big|\frac{k_0 - k + \sqrt{R(k)}}{k}\Big|\Big),\ 0\Big),$$

where $R(k) = (k-k_0)^2 - c_0^2k^2 > 0$. So we get the parametrization (13) for the cylinder M^1_+ which has pointwise 1-type Gauss map of the second kind for $f(k) = -\frac{k^3}{k_0}$ and $C = (-c_0, 0, 0)$.

Now let the vector C be null. From (25) we get $C_2(s) = \pm C_3(s)$. We will consider the case $C_2(s) = C_3(s)$. Hence, from (23) we get $C_2(s) = e^{\mu(s)}$, where $\mu(s) = \lambda_0 - \int k(s)ds$, and then

$$\alpha_1'(s) = \sinh\mu(s) = \frac{1}{2}(C_2(s) - \frac{1}{C_2(s)})$$

and

$$\alpha_2'(s) = \cosh\mu(s) = \frac{1}{2}(C_2(s) + \frac{1}{C_2(s)}).$$

Using (28) and (29) we obtain the first two components of $\alpha(s)$ as

$$\alpha_i(s) = \frac{k_0}{4k^2} + \frac{(-1)^i}{2}\Big(\frac{1}{k-k_0} - \frac{1}{k_0}\ln\Big|\frac{k}{k-k_0}\Big|\Big) + d_i,\quad i = 1, 2. \tag{40}$$

Similarly if we take $C_2(s) = -C_3(s)$, then the first two components of $\alpha(s)$ are

$$\alpha_i(s) = (-1)^i\frac{k_0}{4k^2} + \frac{1}{2(k-k_0)} - \frac{1}{2k_0}\ln\Big|\frac{k}{k-k_0}\Big| + d_i,\quad i = 1, 2. \tag{41}$$

Therefore, considering (40) and (41) we obtain the parametrization (14) for cylinder M^1_+ which has pointwise 1-type Gauss map of the second kind for $f(k) = -\frac{k^3}{k_0}$ and $C = (-1, 1, 0)$ if $C_2(s) = C_3(s)$ or $C = (1, 1, 0)$ if $C_2(s) = -C_3(s)$.

Case 3. M is of type M^1_-, i.e., $\varepsilon_1 = 1, \varepsilon_2 = -1, (\varepsilon_G = 1)$. From (25) the vector C is space-like, time-like or null.

Considering equation (25) we may put

$$C_2(s) = c_0 \sinh \lambda(s), \quad C_3(s) = c_0 \cosh \lambda(s) \quad \text{for} \quad \varepsilon_C = 1$$

or

$$C_2(s) = c_0 \cosh \lambda(s), \quad C_3(s) = c_0 \sinh \lambda(s) \quad \text{for} \quad \varepsilon_C = -1$$

which hold equations (23) and (24) if $\lambda'(s) = k(s)$, that is, $\lambda(s) = \lambda_0 + \int k(s)ds$, where λ_0 is an integration constant. Thus we have

$$\sinh \lambda(s) = \frac{C_2}{c_0} = -\frac{k_0 k'}{c_0 k^3} \quad \text{and} \quad \cosh \lambda(s) = \frac{C_3}{c_0} = \frac{k_0 - k}{c_0 k} \quad \text{for } \varepsilon_C = 1 \tag{42}$$

or

$$\sinh \lambda(s) = \frac{C_3}{c_0} = \frac{k_0 - k}{c_0 k} \quad \text{and} \quad \cosh \lambda(s) = \frac{C_2}{c_0} = -\frac{k_0 k'}{c_0 k^3} \quad \text{for } \varepsilon_C = -1. \tag{43}$$

For the plane curve α, acting a Lorentz transformation we can write

$$\alpha(s) = (\alpha_1(s), \alpha_2(s), 0) \ \text{ and } \beta = (0, 0, 1)$$

without loss of generality. The Gauss map of the cylinder M^1_+ is

$$G = e_1 \times e_2 = (\alpha_2'(s),\ \alpha_1'(s),\ 0)$$

as $e_2 = \alpha'(s) = (\alpha_1'(s), \alpha_2'(s), 0)$. Considering $-{\alpha_1'}^2(s) + {\alpha_2'}^2(s) = -1$, we may put $\alpha_1'(s) = \cosh \mu(s)$ and $\alpha_2'(s) = \sinh \mu(s)$ to determine $\alpha(s)$, where μ is a differentiable function of s. From the equation $\alpha''(s) = \varepsilon_G k(s) G$ we obtain $\mu'(s) = k(s)$. For simplicity we take $\mu(s) = \lambda(s) = \lambda_0 + \int k(s)ds$.

Now we suppose that C is space-like, i.e., $\varepsilon_C = 1$. By using (32), (33) and (42) the base curve $\alpha(s)$ of the cylinder M^1_- is determined uniquely, up to a rigid motion, by

$$\begin{aligned}\alpha(s) =& \Big(d_3 + \frac{1}{c_0}\int C_3(s)ds,\ d_2 + \frac{1}{c_0}\int C_2(s)ds,\ 0\Big),\\ =& \Big(d_3 \pm \Big(-\frac{(k+k_0)}{2c_0k_0k^2}\sqrt{R(k)} - \frac{c_0}{2k_0}\ln\Big(\frac{k_0 - k + \sqrt{R(k)}}{k}\Big)\Big),\\ & d_2 + \frac{k_0}{2c_0k^2},\ 0\Big),\end{aligned}$$

where $R(k) = (k - k_0)^2 - c_0^2 k^2 > 0$. It is seen that the base curve of the cylinder M^1_- can be parametrized in terms of the curvature function k, that is, $\alpha = \alpha(k)$. Therefore we obtain the parametrization (15) for the cylinder M^1_- which has pointwise 1-type Gauss map of the second kind for $f(k) = \frac{k^3}{k_0}$ and $C = (0,\ c_0, 0\ )$.

If C is time-like, i.e., $\varepsilon_C = -1$, then by a similar argument we obtain the base curve of the cylinder M^1_- as

$$\begin{aligned}\alpha(k) =& \Big(d_2 + \frac{k_0}{2c_0k^2},\\ & d_3 \pm \Big(-\frac{(k+k_0)}{2c_0k_0k^2}\sqrt{R(k)} + \frac{c_0}{2k_0}\ln\Big(\frac{k_0 - k + \sqrt{R(k)}}{k}\Big)\Big),\ 0\Big)\end{aligned} \tag{44}$$

where $R(k)=c_0^2k^2+(k-k_0)^2$. So we have the parametrization (16) for the cylinder M_+^1 which has pointwise 1-type Gauss map of the second kind for $f(k)=\frac{k^3}{k_0}$ and $C=(0,\,-c_0,\,0)$.

Now let the vector C be null. From equation (25) we get $C_2(s)=\pm C_3(s)$. We will consider the case $C_2(s)=C_3(s)$. Hence, from (23) we get $C_2(s)=e^{\mu(s)}$ and then

$$\alpha_1'(s)=\cosh\mu(s)=\frac{1}{2}(C_2(s)+\frac{1}{C_2(s)})$$

and

$$\alpha_2'(s)=\sinh\mu(s)=\frac{1}{2}(C_2(s)-\frac{1}{C_2(s)}).$$

Using (28) and (29) we obtain the first two components of $\alpha(s)$ as

$$\alpha_i(s)=\frac{k_0}{4k^2}+\frac{(-1)^{i-1}}{2}\Big(\frac{1}{k-k_0}-\frac{1}{k_0}\ln\Big|\frac{k}{k-k_0}\Big|\Big)+d_i,\quad i=1,2. \tag{45}$$

Similarly if we take $C_2(s)=-C_3(s)$, then the first two components of $\alpha(s)$ are

$$\alpha_i(s)=(-1)^{i-1}\frac{k_0}{4k^2}+\frac{1}{2(k-k_0)}-\frac{1}{2k_0}\ln\Big|\frac{k}{k-k_0}\Big|+d_i,\quad i=1,2. \tag{46}$$

Therefore, considering (45) and (46) we obtain the parametrization (17) for cylinder M_-^1 which has pointwise 1-type Gauss map of the second kind for $f(k)=\frac{k^3}{k_0}$ and $C=(-1,1,0)$ if $C_2(s)=C_3(s)$ or $C=(-1,-1,0)$ if $C_2(s)=-C_3(s)$.

4. Noncylindrical flat surfaces with pointwise 1-type Gauss map of the second kind

In this section we study noncylindrical flat surfaces, i.e., cones and tangent developable surfaces with pointwise 1-type Gauss map of the second kind in $\mathbb{E}_1^3$.

Theorem 4.1 *Let M be a noncylindrical flat surface in the Minkowski space $\mathbb{E}_1^3$. Then, M has pointwise 1-type Gauss map of the second kind if and only if it is an open part of a right circular cone or a hyperbolic cone in $\mathbb{E}_1^3$.*

Proof. Suppose that M has pointwise 1-type Gauss map of the second kind. Since M is a regular noncylindrical flat surface in the Minkowski space $\mathbb{E}_1^3$, then M is an open part of a cone or an open part of a tangent developable surface in $\mathbb{E}_1^3$. We consider two cases.

Case 1. M is an open part of a cone. Then, by an appropriate rigid motion, M can be parametrized locally by

$$x(s,t)=\alpha_0+t\beta(s),\quad t\neq 0,$$

where $\langle\beta(s),\beta(s)\rangle=\pm1$, $\langle\beta'(s),\beta'(s)\rangle=\pm1$, and α_0 is a constant vector. The coordinate vector fields $x_s=t\beta'(s)$ and $x_t=\beta(s)$ are orthogonal because of $\langle\beta(s),\beta(s)\rangle=\pm1$, and the surface M is regular if $t\beta'(s)\times\beta(s)\neq 0$. So we take the orthonormal tangent frame $\{e_1,e_2\}$ on M as $e_1=\frac{1}{t}\frac{\partial}{\partial s}$ and $e_2=\frac{\partial}{\partial t}$ with

$\varepsilon_1 = \langle e_1, e_1 \rangle = \pm 1$ and $\varepsilon_2 = \langle e_2, e_2 \rangle = \pm 1$. The Gauss map of M is given by $G = e_1 \times e_2 = \beta'(s) \times \beta(s)$ with $\varepsilon_G = \langle G, G \rangle = -\varepsilon_1 \varepsilon_2$.

By a straightforward calculation we obtain

$$\widetilde{\nabla}_{e_1} e_1 = -\frac{\varepsilon_1 \varepsilon_2}{t} e_2 - \frac{\varepsilon_G k_g(s)}{t} G, \quad \widetilde{\nabla}_{e_1} e_2 = \frac{1}{t} e_1, \quad \widetilde{\nabla}_{e_2} e_1 = \widetilde{\nabla}_{e_2} e_2 = 0,$$

where $k_g(s) = \langle \beta''(s), \beta(s) \times \beta'(s) \rangle \neq 0$ which is the geodesic curvature of β in the hyperbolic space $\mathbb{H}^2(-1)$ or in the de Sitter space $\mathbb{S}_1^2(1)$. All these relations imply that

$$\omega_{12}(e_1) = -\frac{\varepsilon_1}{t}, \ \omega_{12}(e_2) = 0, \ h_{11} = -\frac{k_g(s)}{t}, \ h_{12} = h_{21} = h_{22} = 0,$$

and thus we have the mean curvature $H = -\frac{\varepsilon_1 k_g(s)}{2t}$ and $\|A_G\|^2 = \frac{k_g^2(s)}{t^2}$.

Now (8)–(10) imply that C_1, C_2, and C_3 are functions of s, and equations (5)–(7) become

$$C_1'(s) + \varepsilon_1 \varepsilon_2 C_2(s) + \varepsilon_G k_g(s) C_3(s) = 0, \tag{47}$$

$$C_2'(s) - C_1(s) = 0, \tag{48}$$

$$C_3'(s) - \varepsilon_1 k_g(s) C_1(s) = 0. \tag{49}$$

On the other hand, we have from (19), (20), and (21)

$$\varepsilon_G \frac{k_g^2(s)}{t^2} = f(1 + \varepsilon_G C_3), \tag{50}$$

$$-\frac{\varepsilon_1}{t^2} \frac{dk_g(s)}{ds} = f\, C_1, \tag{51}$$

$$\frac{\varepsilon_1 k_g(s)}{t^2} = f\, C_2. \tag{52}$$

It follows from (52) that $C_2 \neq 0$. Also equations (50) and (52) give

$$\varepsilon_1 k_g(s) C_2(s) - C_3(s) = \varepsilon_G \tag{53}$$

from which by taking derivative with respect to s, we get

$$\varepsilon_1 k_g'(s) C_2(s) + \varepsilon_1 k_g(s) C_2'(s) = C_3'(s) \tag{54}$$

that gives $\varepsilon_1 k_g'(s) C_2(s) = 0$ in view of (48) and (49). Hence we obtain $k_g'(s) = 0$ as $C_2 \neq 0$, that is, $k_g(s)$ is a nonzero constant.

Now we assume that β'' is non-null. By considering (3) for the curve β in the hyperbolic space $\mathbb{H}^2(-1)$ (resp., in the de Sitter space $\mathbb{S}_1^2(1)$) we have $\varepsilon_N k^2(s) = k_g^2(s) - 1$ (resp., $\varepsilon_N k^2(s) = -\varepsilon_1 k_g^2(s) + 1$), where ε_N is the sign of the principal normal vector N of the curve β. Note that we take $\varepsilon_G = -1$ for $\mathbb{H}^2(-1)$ and

$\varepsilon_G = 1$ for $\mathbb{S}_1^2(1)$ while we use formula (3). Thus, the curvature k of β is also constant, and $k \neq 0$ because if the curvature k were zero, then β would be a line, and M would be a part of plane which is a cylindrical surface.

Therefore, taking the derivative of $k_g(s) = \langle \beta''(s), \beta(s) \times \beta'(s) \rangle = \text{const.} \neq 0$, and using the Frenet equations (2) it can be shown that the torsion of β is zero, that is, β is a plane curve with nonzero constant curvature. A plane curve in $\mathbb{E}_1^3$ with nonzero constant curvature is a part of a circle or a hyperbola. Thus the curve β is a part of a circle or a hyperbola in $\mathbb{H}^2(-1)$ or in the de Sitter space $\mathbb{S}_1^2(1)$ such that the plane containing the curve β does not pass through the origin. Therefore the ruled surface M is an open part of a right circular cone or a hyperbolic cone in $\mathbb{E}_1^3$.

Moreover equation (51) implies $C_1 = 0$, and equations (48) and (49) imply $C_2' = 0$ and $C_3' = 0$, respectively, i.e., C_2 and C_3 are constants. Then we obtain from equations (47) and (53)

$$C_2 = \frac{\varepsilon_1 \varepsilon_2 k_g}{1 - \varepsilon_1 k_g^2} \quad \text{and} \quad C_3 = \frac{\varepsilon_1 \varepsilon_2}{1 - \varepsilon_1 k_g^2}.$$

Also, we get from (52) $f = \frac{\varepsilon_2(1-\varepsilon_1 k_g^2)}{t^2}$. Therefore M has pointwise 1-type Gauss map of the second kind, that is, equation (1) holds for $f = \frac{\varepsilon_2(1-\varepsilon_1 k_g^2)}{t^2}$ and for the constant vector $C = \frac{1}{1-\varepsilon_1 k_g^2}(\varepsilon_2 k_g e_2 - G)$.

Now let β'' be null. If β lies in $\mathbb{H}^2(-1)$, we have $k_g^2 = 1$, and also $\varepsilon_2 = -1, \varepsilon_1 = \varepsilon_G = 1$. Then equation (51) implies $C_1 = 0$, and equations (50) and (52) imply $k_g C_2 - C_3 = 1$. Also, from (47) we have $C_2 = k_g C_3$. In view of the last two equations we obtain $(k_g^2 - 1)C_3 = 1$, which is not valid as $k_g^2 = 1$. If β lies in $\mathbb{S}_1^2(1)$ we have $\varepsilon_2 = 1$ and $k_g^2 = \varepsilon_1$, which holds if $\varepsilon_1 = 1$. By a similar argument given above we have $(1 - k_g^2)C_3 = 1$, which is not valid as $k_g^2 = 1$. As a result, if β'' is null, then the Gauss map of the cone M is not of pointwise 1-type of the second kind.

Case 2. M is an open part of a tangent developable surface fully lying in $\mathbb{E}_1^3$. We will show that there is no tangent developable surface in $\mathbb{E}_1^3$ with pointwise 1-type Gauss map of the second kind. The surface M is locally parametrized by

$$x(s,t) = \alpha(s) + t\alpha'(s),$$

where $\alpha(s)$ is a unit speed curve with nonzero curvature $k(s)$. Note that if α is a null curve or α'' is null, then the tangent surface is degenerate. We assume that the torsion $\tau(s)$ of $\alpha(s)$ is nonzero. If $\tau = 0$, then the tangent surface is a part of a plane which is a cylindrical.

Let $T(s), N(s)$, and $B(s)$ denote the unit tangent vector, principal normal vector and binormal vector of the curve α with signatures $\varepsilon_T, \varepsilon_N$ and $\varepsilon_B = -\varepsilon_T \varepsilon_N$, respectively. The coordinate vector fields of M are $x_s = \alpha'(s) + t\alpha''(s) = T(s) + \varepsilon_N t k(s) N(s)$ and $x_t = \alpha'(s) = T(s)$ which are not orthogonal. The parametrization x is regular if $tk(s) \neq 0$. We take the orthonormal tangent frame $\{e_1, e_2\}$ on M as $e_1 = \frac{\partial}{\partial t}$ and $e_2 = \frac{\varepsilon_N}{tk(s)}\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t}\right)$ with $\varepsilon_1 = \langle e_1, e_1 \rangle = \pm 1$ and $\varepsilon_2 = \langle e_2, e_2 \rangle = \pm 1$. It is seen that $e_1 = T$, $e_2 = N$, $\varepsilon_1 = \varepsilon_T$ and $\varepsilon_2 = \varepsilon_N$. Then the Gauss map of M is given by $G = e_1 \times e_2 = T \times N = B$ with $\varepsilon_G = -\varepsilon_1 \varepsilon_2$.

By a direct calculation we obtain

$$\widetilde{\nabla}_{e_1} e_1 = \widetilde{\nabla}_{e_1} e_2 = 0, \;\; \widetilde{\nabla}_{e_2} e_1 = \frac{1}{t} e_2, \;\; \widetilde{\nabla}_{e_2} e_2 = -\frac{\varepsilon_1 \varepsilon_2}{t} e_1 - \frac{\varepsilon_1 \tau(s)}{tk(s)} G.$$

So we have $\omega_{21}(e_1)=0,\ \omega_{21}(e_2)=-\frac{\varepsilon_2}{t},\ h_{11}=h_{12}=h_{21}=0$ and $h_{22}=\frac{\varepsilon_2\tau(s)}{tk(s)}$. Therefore the mean curvature is $H(s,t)=\frac{\tau(s)}{2tk(s)}$, and $\|A_G\|^2=(\frac{\tau(s)}{tk(s)})^2$.

Now, it follows from equations (5)–(7) that C_1, C_2, and C_3 are functions of s, and thus equations (8)–(10) become

$$C_1'(s)-\varepsilon_2 k(s)C_2(s)=0, \tag{55}$$

$$C_2'(s)+\varepsilon_1 k(s)C_1(s)+\varepsilon_1\varepsilon_2\tau(s)C_3(s)=0, \tag{56}$$

$$C_3'(s)+\varepsilon_2\tau(s)C_2(s)=0. \tag{57}$$

On the other hand, we have from (19), (20), and (21) that

$$\varepsilon_G\frac{\tau^2}{t^2k^2}=f(1+\varepsilon_G C_3), \tag{58}$$

$$-\frac{\tau}{t^2k}=f\,C_1, \tag{59}$$

$$\frac{\varepsilon_2}{t^2k}\Big(\frac{d}{ds}\Big(\frac{\tau}{k}\Big)+\frac{\tau}{tk}\Big)=f\,C_2. \tag{60}$$

Equation (59) implies that $C_1\neq 0$ as $\tau\neq 0$. Also, from (58) and (59) we obtain

$$\tau(s)C_1(s)+k(s)C_3(s)=\varepsilon_1\varepsilon_2 k(s) \tag{61}$$

from which, by taking the derivative we get

$$\tau'(s)C_1(s)+\tau(s)C_1'(s)+k'(s)C_3(s)+k(s)C_3'(s)=\varepsilon_1\varepsilon_2 k'(s). \tag{62}$$

Using equations (55) and (57), equation (62) turns into

$$\tau'(s)C_1(s)+k'(s)C_3(s)=\varepsilon_1\varepsilon_2 k'(s). \tag{63}$$

If $\tau'(s)k(s)-k'(s)\tau(s)\neq 0$, then equations (61) and (63) give $C_1=0$ and $C_3=-\varepsilon_G$. Hence, we have $\tau=0$ from (58) or (59), which is a contradiction.

Now suppose that $\tau'(s)k(s)-k'(s)\tau(s)=0$, which means that $\tau(s)/k(s)=r_0$ is a nonzero constant. In this case, by (59) and (60) we get

$$tk(s)C_2(s)+\varepsilon_2 C_1(s)=0$$

which implies that $C_1=C_2=0$, that is, $\tau=0$ by (59). This is a contradiction. Therefore the torsion τ is zero, and there is no tangent developable surface fully lying in $\mathbb{E}^3_1$ with pointwise 1-type Gauss map of the second kind.

The converse of the proof follows from a straightforward calculation. □

We then have the following.

Corollary 4.2 *Right circular cones and hyperbolical cones in Minkowski space* $\mathbb{E}_1^3$ *are the only cones in* $\mathbb{E}_1^3$ *with pointwise* 1*-type Gauss map of the second kind.*

Corollary 4.3 *There is no tangent developable surface fully lying in Minkowski space* $\mathbb{E}_1^3$ *with pointwise* 1*-type Gauss map of the second kind.*

Combining Theorem 3.3 and Theorem 4.1 we have

Theorem 4.4 *Let* M *be a flat ruled surface in the Minkowski space* $\mathbb{E}_1^3$*. Then,* M *has pointwise 1-type Gauss map of the second kind if and only if it is a part of a plane, cylinders given by* (11)–(17)*, a right circular cone or a hyperbolic cone.*

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