Turk J Math
36 (2012) , 530 – 537.

doi:10.3906/mat-0912-19

# Lower bounds for the maximum genus of 4-regular graphs*

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**Abstract**

This paper investigates the maximum genus and upper embeddability of connected 4-regular graphs. We obtain lower bounds on the maximum genus of connected 4-regular simple graphs and connected 4-regular graphs without loops in terms of the Betti number. The definition of the Betti number is referred to [Gross and Tucker, Topological Graph Theory, New York, 1987]. Furthermore, we give examples that show that these lower bounds are tight.

**Key Words:** Maximum genus, upper embeddable, Betti number

## 1. Introduction

The *maximum genus* of a connected graph $G = (V, E)$, denoted by $\gamma_M(G)$, is the maximum integer $k$ with the property that there exists a cellular embedding of $G$ on the orientable surface with genus $k$.

The maximum genus has received considerable attention after Nordhaus et al. [18]. Xuong [22] proved that every 4-edge connected graph is upper embeddable. However, there are many examples of 3-edge connected graphs and 2-edge connected graphs that are not upper embeddable. See, for example, [2, 12, 14]. Therefore, considerable attention is given to the lower bounds on the maximum genus of many kinds of graphs in terms of some graph invariants. See, for example, [3, 5, 7, 10]. Chen et al. [2] proved that for a 2-connected simple graph with all its vertices of degree greater than 2, the maximum genus is at least $\beta(G)/3$. Chen and Huang gave some results on the maximum genus of graphs in [4]. In the paper [15], Nedela and Skoviera proved that any Eulerian graph with at most 2 vertices of degree 0 (mod 4) is necessarily upper embeddable. In particular, any connected regular graph with degree $4k+2$, for an integer $k \geq 1$, is upper embeddable. Skovieria [20, 21] has obtained the maximum genus of graphs which are 3-regular with 2-factor triangles, and its slightly weaker form of the result was earlier proved in [1]. Naturally, a question on the maximum genus for 4-regular graphs can be posed. More information on upper embeddability of graphs can be found for example in [11]-[19].

In this paper, tight lower bounds on the maximum genus of connected 4-regular simple graphs and connected 4-regular graphs without loops are obtained.

*Supported by the National Nature Science Foundation of China (Grant No.10871021).

## 2. Preliminaries

For simplification, throughout this paper, $G$ is asumed to have $n$ vertices and $m$ edges.

Since any embedding has at least one face, from the Euler formula, it can be seen that $\gamma_M(G) \leq \lfloor \beta(G)/2 \rfloor$, where $\beta(G) = m - n + 1$ is called the *Betti number* [11] of the connected graph $G$. If $\gamma_M(G) = \lfloor \beta(G)/2 \rfloor$, $G$ is said to be *upper embeddable.*

For a subset $A \subseteq E(G)$, $c(G \backslash A)$ denotes the number of connected components of $G \setminus A$, and $b(G \setminus A)$ denotes the number of connected components of $G \setminus A$ with an odd Betti number, where $G \setminus A$ means the subgraph obtained from $G$ by deleting all the edges of $A$ from $G$. Let $T$ be a spanning tree of a connected graph $G$. Define the *deficiency* $\xi(G,T)$ to be the number of components of $G \setminus E(T)$ that have an odd number of edges. The *deficiency* $\xi(G)$ of the graph $G$ is defined to be the minimum value of $\xi(G,T)$ over all spanning trees $T$ of $G$. Note that $\xi(G) = \beta(G) \pmod 2$. For $k \geq 2$, let $F_1, F_2, \cdots, F_k$ be $k$ distinct subgraphs of a graph $G$. Then denote by $E_G(F_1, F_2, \cdots, F_k)$ the edges of $G$ with ends in distinct $F_i$s and $E(F_i, G)$ the edges of $E(G)$ with just one end in $F_i$.

We follow [6] for terminologies and notations not defined here.

The following theorems are some characterizations on the maximum genus and upper embeddability of a graph, which are due to Xuong and Nebesky, respectively.

**Theorem 1** [22, 17] *If $G$ is a connected graph, then*

*(a) $G$ is upper embeddable if and only if $\xi(G) \leq 1$;*

*(b) $\gamma_M(G) = (\beta(G) - \xi(G))/2$.*

**Theorem 2** [16] *If $G$ is a connected graph, then*

*(1) $G$ is upper embeddable if and only if $c(G \setminus A) + b(G \setminus A) - 2 \leq |A|$ for any subset $A \subseteq E(G)$;*

*(2) $\xi(G) = \max_{A \subseteq E(G)}\{c(G \setminus A) + b(G \setminus A) - |A| - 1\}$.*

The following result, which was proved by Huang, provides a structural characterization of graphs that are not upper embeddable.

**Theorem 3** [9, 8, 17] *Let $G$ be a connected graph. If $\xi(G) \geq 2$, namely $G$ is not upper embeddable, then there exists a subset $A \subseteq E(G)$ such that the following properties are satisfied:*

*(1) $c(G \setminus A) = b(G \setminus A) \geq 2$, $\beta(F) = 1 \pmod 2$ for each component $F$ of $G \setminus A$;*

*(2) $F$ is a vertex-induced subgraph of $G$ for each component $F$ of $G \setminus A$;*

*(3) for any $k$ distinct components $F_1, F_2, \ldots, F_k$ of $G \setminus A$, $|E_G(F_1, F_2, \cdots, F_k)| \leq 2k - 3$. In particular, $|E_G(F, H)| \leq 1$ for any 2 distinct components $F$ and $H$ of $G \setminus A$;*

*(4) $\xi(G) = 2c(G \setminus A) - |A| - 1$.*

## 3. Main results

**Theorem A** *If $G$ is a connected 4-regular simple graph with Betti number $\beta = \beta(G)$, then*

$$\gamma_M(G) \geq \lfloor (2\beta + 3)/5 \rfloor .$$

**Proof.** Let $G$ be a connected 4-regular simple graph with $n$ vertices and $m$ edges. Then by the definition of Betti number, $\beta = m - n + 1 = 4n/2 - n + 1 = n + 1$. The following 2 cases are considered:

**Case 1** $G$ is upper embeddable. In this case,

$\gamma_M(G) = \lfloor \beta/2 \rfloor$.

Since $G$ is a connected 4-regular simple graph, $n \geq 5$ and $\lfloor \beta/2 \rfloor \geq \lfloor (2\beta+3)/5 \rfloor$, so $\gamma_M(G) \geq \lfloor (2\beta+3)/5 \rfloor$.

**Case 2** $G$ is not upper embeddable.

By Theorem 3, there exists $A \subseteq E(G)$ such that the properties (1)–(4) of Theorem 3 are satisfied. Then $c(G \setminus A) = b(G \setminus A) \geq 2$ and $\xi(G) = 2c(G \setminus A) - |A| - 1$.

Let $a_i^j$ denote the number of connected components of $G \setminus A$ with order $i$ and Betti number $j$. Since $G$ is a simple graph and $c(G \setminus A) = b(G \setminus A)$, for each component $F$ of $G \setminus A$, we have $\beta(F) = 1 (\mathrm{mod}\ 2)$, so there exist $a_i^j \neq 0$ only for $i \geq 3$ and $j = 1(\mathrm{mod}\ 2)$, and thus the following Claim 1 is right:

**Claim 1** $a_i^j = 0$ *for any* $i \geq 3$ *and* $j = 0(\mathrm{mod}\ 2)$.

Before finishing the proof of this theorem, we first prove the following, Claim 2.

**Claim 2** $a_i^j = 0$ *for any positive integer* $j$ *with* $j > i$.

**Proof of Claim 2** By the contrary, suppose $a_i^j \neq 0$ for some positive integer $j$ with $j > i$, and then there is at least one connected component, denoted by $F$, of $G \setminus A$ with order $i$ and Betti number $j$. Since $G$ is 4-regular, thus $\sum_{v \in F} d_G(v) = 4i$. Because the number of vertices with odd degree in $F$ must be even, $E(F, G) \geq 2$. By $E(F, G) = 4i - 2(i-1) - 2j = 2i - 2j + 2$, where $i - 1$ and $j$ are the number of spanning tree edges and that of cotree edges of $F$ respectively, then $2i - 2j + 2 \geq 2$, i.e., $i \geq j$, this is a contradiction with $i < j$. □

We now continue with the proof of the theorem.

Since the number of connected components of $G \setminus A$ is sum of $a_i^j$ for all integers $i$ and $j$, by Claims 2,

$c(G \setminus A) = \sum_{j \leq i} a_i^j$.

Since $G$ is a simple graph, any connected component with order 1 of $G \setminus A$ cannot have Betti number 1; thus $a_1^1 = 0$. Similarly, it can be checked that $a_2^1 = a_3^3 = 0$ because of $G$ being a connected 4-regular simple graph. Also by Claims 1 and 2, we have

$$\begin{aligned} c(G \setminus A) = & \sum_{j \leq i} a_i^j = a_3^1 + a_4^1 + a_5^1 + \cdots + a_4^3 + a_5^3 + a_6^3 + \cdots \\ & + a_5^5 + a_6^5 + a_7^5 + \cdots + \sum_{\substack{i \geq j \geq 7 \\ j = 1(\mathrm{mod}\ 2)}} a_i^j. \end{aligned} \tag{1}$$

Since a connected component with order $i$ of $G \setminus A$ contributes to $i$ for the order $n$ of $G$, thus $a_i^j$

connected component with order $i$ contributes to $ia_i^j$ for $n$. So,

$$n = 3a_3^1 + 4a_4^1 + 5a_5^1 + \cdots + 4a_4^3 + 5a_5^3 + 6a_6^3 + \cdots + 5a_5^5 + 6a_6^5 + 7a_7^5 + \cdots + \sum_{\substack{i \geq j \geq 7 \\ j = 1 (\mathrm{mod}\ 2)}} ia_i^j. \quad (2)$$

Let $F$ be a connected component of $G \setminus A$ with order $i$ and Betti number $j$, then $\sum_{v \in F} d_G(v) = 2[i-1+j] + |E(F,G)|$, where $2(i-1+j)$ is contributed by the edges in $F$, *and* $|E(F,G)|$ is contributed by the edges not in $F$. Thus $4i = 2[i-1+j] + |E(F,G)|$, i.e., $|E(F,G)| = 4i - 2(i-1) - 2j = 2i - 2j + 2$.

Let $\varphi$ be the set of all the components of $G \setminus A$. From Theorem 3 (2),

$|A| = \frac{1}{2} \sum_{F \in \varphi} |E(F,G)|$. Also by Claims 1 and 2, then

$$\begin{aligned} |A| &= \sum_{\substack{i \geq 1 \\ j = 1 (\mathrm{mod}\ 2)}} (i-j+1)a_i^j, \\ &= 3a_3^1 + 4a_4^1 + 5a_5^1 + \cdots + 2a_4^3 + 3a_5^3 + 4a_6^3 + \cdots \\ &\quad + a_5^5 + 2a_6^5 + 3a_7^5 + \cdots + \sum_{\substack{i \geq j \geq 7 \\ j = 1 (\mathrm{mod}\ 2)}} (i-j+1)a_i^j. \end{aligned} \quad (3)$$

From $\beta = n+1$ and Theorem 1, $\xi(G) = \beta - 2\gamma_M(G) = n + 1 - 2\gamma_M(G)$.

Assume $\gamma_M(G) < \lfloor (2\beta+3)/5 \rfloor$. Then

$\xi(G) = n + 1 - 2\gamma_M(G) > n + 1 - 2\left[(2n+2)/5 + 3/5\right] = n/5 - 1$.

From Theorem 3 (4), we have

$\xi(G) = 2c(G \setminus A) - |A| - 1 > n/5 - 1$. Therefore, $2c(G \setminus A) - |A| - n/5 > 0$. But from (1) – (3),

$$\begin{aligned} 2c(G \setminus A) - |A| - n/5 &= -(8/5)a_3^1 - (14/5)a_4^1 - 4a_5^1 - \cdots \\ &\quad -(4/5)a_4^3 - 2a_5^3 - (16/5)a_6^3 - \cdots - 0a_5^5 - (6/5)a_6^5 - (12/5)a_7^5 \\ &\quad + \cdots + \sum_{\substack{i \geq j \geq 7 \\ j = 1 (\mathrm{mod}\ 2)}} (1 - 6i/5 + j)a_i^j. \end{aligned} \quad (4)$$

If $i \geq j \geq 7$, then the coefficient of $a_i^j$ in (4) satisfies

$1 - 6i/5 + j \leq 1 - 6i/5 + i = 1 - i/5 \leq 0$. Thus

$2c(G \setminus A) - |A| - n/5 \leq 0$.

There is a contradiction. Consequently, $\gamma_M(G) \geq \lfloor (2\beta+3)/5 \rfloor$.

From these 2 cases, the theorem follows we complete proof of Theorem A. □

**Note 1** *In the following, we can find an infinite number of 4-regular graphs whose maximum genus equals the lower bound, that is,* $\gamma_M(G) = \lfloor (2\beta+3)/5 \rfloor$, *which means that the lower bound of Theorem A is tight.*

Let $C_5^3$ be the graph shown in Figure 1, which is obtained from $K_5$ by removing one edge.

Let $G_1, \cdots, G_k$ $(k \geq 3)$ be copies of the graph $C_5^3$. Then $G$ is obtained by adding a new edge from $G_i$ to $G_{i+1}(\bmod\ k)$, $i = 1, 2, \cdots, k$ as shown in Figure 2.

It can be seen that $G$ is a connected 4-regular simple graph. Let the set of edges from $G_i$ to $G_{i+1}$ be $A$. Then $G \backslash A$ has $k$ connected components, each with 5 vertices and its Betti number is 5. It can be checked that $n = 5k$, $\beta = n + 1 = 5k + 1$, $\gamma_M(G) = \lfloor (2\beta + 3)/5 \rfloor = 2k + 1$.

It follows from the above examples that there exist an infinite number of connected 4-regular simple graphs whose maximum genus is equal to the lower bounds, so the lower bound of Theorem A is tight.

The following Theorem B is generalized from simple 4-regular graphs to 4-regular graphs without loops.

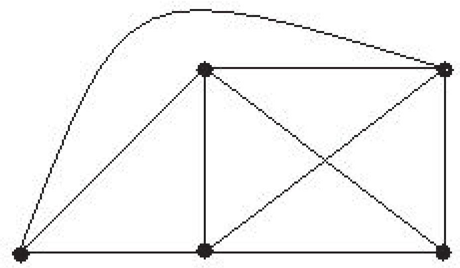

**Figure 1**. $C_5^3$.

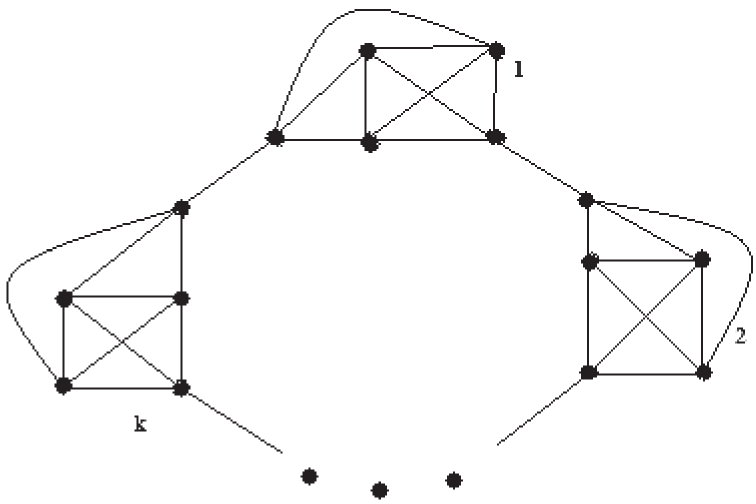


**Figure 2**. A connected 4-regular simple graph

**Theorem B** *If $G$ is a connected 4-regular graph without loops, then*

$$\gamma_M(G) \geq \lfloor (\beta + 2)/3 \rfloor .$$

The proof of Theorem B is similar to that of Theorem A, so it is omitted.

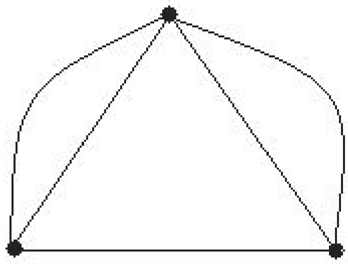

**Figure 3**. $C_3^3$.

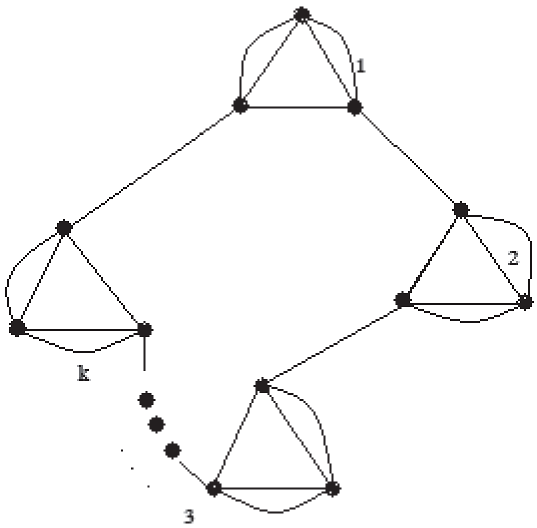


**Figure 4**. A connected 4-regular graph without a loop.

**Note 2** *We now give an infinite number of graphs whose maximum genus equal the lower bound, that is,* $\gamma_M(G) = \lfloor(\beta+2)/3\rfloor$. *It means that the lower bound of Theorem B is tight.*

Let $C_3^3$ be the graph shown in Figure 3, which is obtained from $K_3$ by doubling 2 of its edges. Let $G_1, \cdots, G_k$ $(k \geq 3)$ be copies of $C_3^3$; $G$ is obtained by adding a new edge from $G_i$ to $G_{i+1}(\text{mod } k)$ $(i = 1, 2, \cdots, k)$ as shown in Figure 4.

Then it can be seen that $G$ is a connected 4-regular graph and has no loops, Let $A$ be the set of edges from $G_i$ to $G_{i+1}$, then $G\backslash A$ has $k$ connected components, each with 3 vertices and the Betti number is 3. It can be checked that $\beta = n + 1 = 3k + 1$, $\gamma_M(G) = \lfloor(\beta+2)/3\rfloor = k + 1$.

This example shows that the lower bound of Theorem B is tight.

**Note 3** *If $G$ is a graph with loops, Theorems A and B do not hold.*

For example, let $C_n$ be the circuit with $n$ vertices; the graph denoted by $C_n^1$ is obtained from $C_n$ by adding one loop at each vertex. If $n \geq 4$, so $\beta \geq 5$, then $\gamma_M(C_n^1) = 1$, but $\lfloor(2\beta+3)/5\rfloor \geq 2$ and $\lfloor(\beta+2)/3\rfloor \geq 2$.

**Acknowledgment**

The authors would like to thank the anonymous reviewers for valuable comments that helped to revise the error in the first version and improve the quality of this paper. Corresponding author: Rongxia Hao, this work was supported by the National Natural Science Foundation of China (10871021).

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Received: 12.12.2009