Turk J Math
36 (2012) , 538 – 543.

doi:10.3906/mat-1007-31

Results of generalized local cohomology modules of $\mathfrak{a}$-minimax modules

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Abstract

Let R be a commutative Noetherian ring, M a finitely generated R-module, and N a minimax R-module. It is shown that if $\mathfrak{a}$ is an ideal of R, such that $\operatorname{cd}(\mathfrak{a}) = 1$, where cd is the cohomological dimension of $\mathfrak{a}$ in R, then $H^j_{\mathfrak{a}}(M,N)$ and $\operatorname{Ext}^i_R(M, H^j_{\mathfrak{a}}(N))$ are $\mathfrak{a}$-cominimax for all i, j. Furthermore, if t is a non-negative integer such that $H^j_{\mathfrak{a}}(M,N)$ is $\mathfrak{a}$-minimax for all $j < t$, then for any $\mathfrak{a}$-minimax R-submodule L of $H^t_{\mathfrak{a}}(M,N)$, the R-module $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M,N)/L)$ is $\mathfrak{a}$-minimax. As a consequence, it follows that the Goldie dimension of $H^t_{\mathfrak{a}}(M,N)/L$ is finite, and so the associated primes of $H^t_{\mathfrak{a}}(M,N)/L$ is finite.

Key Words: Generalized local cohomology module, minimax module

1. Introduction

Throughout this paper, we assume that R is a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ an ideal of R, and M, N 2 R-modules.

The generalized local cohomology module

$$H^j_{\mathfrak{a}}(M,N) = \varinjlim_{n} \operatorname{Ext}^j_R(M/\mathfrak{a}^n M, N)$$

was introduced by Herzog in [9] and studied further by Suzuki [18]. Clearly, this notion is a natural generalization of the ordinary local cohomology module (cf. [5]). There are some basic problems concerning ordinary local cohomology modules (cf. [10]) and several articles are devoted to studying these problems (see for example [6], [11], [4], [1], [14], [8], [12], and [13]). In general, it is hard to extend these results to generalized local cohomology modules. Therefore it is worthwhile to find methods to transfer these results from the ordinary local cohomology modules to the above generalized local cohomology modules.

Recall that an R-module N is said to have finite Goldie dimension (written $\operatorname{Gdim} N < \infty$) if N does not contain an infinite direct sum of non-zero submodules, or equivalently the injective hull $E(N)$ of N decomposes

2000 *AMS Mathematics Subject Classification:* 13D45, 13E99, 13E05.

as a finite direct sum of indecomposable (injective) submodules. Also, an R-module N is said to have finite $\mathfrak{a}$-relative Goldie dimension if the Goldie dimension of the $\mathfrak{a}$-torsion submodule $\Gamma_{\mathfrak{a}}(N)$ of N is finite. Azami, Naghipour, and Vakili [2] defined that an R-module N is $\mathfrak{a}$-minimax if the $\mathfrak{a}$-relative Goldie dimension of any quotient module of N is finite. Moreover, an R-module N is $\mathfrak{a}$-cominimax if the support of N is contained in $V(\mathfrak{a})$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, N)$ is $\mathfrak{a}$-minimax for all i.

The main aim of this paper is to prove Theorem 1.2. Firstly, we recall the following definition.

Definition 1.1 *The cohomological dimension of* $\mathfrak{a}$ *in* R*, denoted by* $\mathrm{cd}(\mathfrak{a})$*, is the smallest integer* n *such that the local cohomology modules* $H^i_{\mathfrak{a}}(M) = 0$ *for all* R*-modules* M*, and for all* $i > n$*.*

Theorem 1.2 *Let* M *be a finitely generated* R*-module and* N *be a minimax* R*-module. Then the following statements hold:*

(i) If $\mathrm{cd}(\mathfrak{a}) = 1$*, then* $H^j_{\mathfrak{a}}(M, N)$ *and* $\mathrm{Ext}^i_R(M, H^j_{\mathfrak{a}}(N))$ *are* $\mathfrak{a}$*-cominimax for all* i, j*.*

(ii) If $\mathfrak{b}$ *is an ideal of* R *with* $\mathfrak{b} \subseteq \mathfrak{a}$ *and* $\mathrm{cd}(\mathfrak{a}) = 1$*, then* $H^i_{\mathfrak{a}}(H^j_{\mathfrak{b}}(N))$ *is* $\mathfrak{a}$*-cominimax for all* i, j*.*

(iii) If t *is a non-negative integer such that* $H^j_{\mathfrak{a}}(M, N)$ *is* $\mathfrak{a}$*-minimax for all* $j < t$*, then for any* $\mathfrak{a}$*-minimax* R*-submodule* L *of* $H^t_{\mathfrak{a}}(M, N)$ *the* R*-module* $\mathrm{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M, N)/L)$ *is* $\mathfrak{a}$*-minimax. As a consequence it follows that the Goldie dimension of* $H^t_{\mathfrak{a}}(M, N)/L$ *is finite, and so the associated primes of* $H^t_{\mathfrak{a}}(M, N)/L$ *is finite.*

Clearly, (i) is an improvement of [11, Theorem 1] and [14, Theorems 2.2, 2.10], (ii) is a generalization of [15, Proposition 3.15], and (iii) extends [4, Theorem 2.2] and [1, Theorem 1.3].

2. The results

Following [2], for an R-module N the *Goldie dimension* of N is defined as the cardinal of the set of indecomposable submodules of $E(N)$, which appear in a decomposition of $E(N)$ into a direct sum of indecomposable submodules. We shall use $\mathrm{Gdim}\, N$ to denote the Goldie dimension of N. For a prime ideal $\mathfrak{p}$, let $\mu^0(\mathfrak{p}, N)$ denote the 0-th Bass number of N with respect to the prime ideal $\mathfrak{p}$. It is known that $\mu^0(\mathfrak{p}, N) > 0$ if and only if $\mathfrak{p} \in \mathrm{Ass}(N)$. It is clear by the definition of the Goldie dimension that

$$\mathrm{Gdim}\, N = \sum_{\mathfrak{p} \in \mathrm{Ass}(N)} \mu^0(\mathfrak{p}, N).$$

Also, for any ideal $\mathfrak{a}$ of R and any R-module N, the $\mathfrak{a}$-*relative Goldie dimension* of N is defined as

$$\mathrm{Gdim}_{\mathfrak{a}}\, N := \sum_{\mathfrak{p} \in V(\mathfrak{a})} \mu^0(\mathfrak{p}, N).$$

The $\mathfrak{a}$-relative Goldie dimension of an R-module N has been studied in [7]. In [19], H. Zöschinger introduced the interesting class of minimax modules, and in [19] and [20] he has given many equivalent conditions for a module to be minimax. The R-module N is said to be a *minimax module* if there is a finitely generated submodule L of N such that N/L is Artinian. It is known that a module is minimax if and only if each of its quotients has finite Goldie dimension [20].

Example 2.1 *Let* $\mathfrak{a}$ *be an ideal of* R *and* N *be an* R*-module.*

(i) If $\mathfrak{a}=0$*, then* N *is* $\mathfrak{a}$*-minimax if and only if* N *is minimax.*

(ii) If N *is* $\mathfrak{a}$*-torsion, then* N *is* $\mathfrak{a}$*-minimax if and only if* N *is minimax [7, Lemma 2.6].*

(iii) If N *is Noetherian or Artinian, then* N *is* $\mathfrak{a}$*-minimax.*

(iv) If $\mathfrak{b}\subseteq\mathfrak{a}$ *are 2 ideals of* R *and* N *is* $\mathfrak{b}$*-minimax, then* N *is* $\mathfrak{a}$*-minimax. In particular, every minimax module is* $\mathfrak{a}$*-minimax.*

Theorem 2.2 *Let* $\mathfrak{a}$ *be an ideal of* R *such that* $\operatorname{cd}(\mathfrak{a})=1$ *and let* N *be an* $\mathfrak{a}$*-minimax* R*-module. Then, for all finitely generated* R*-module* M *and all* j*,* $H^j_{\mathfrak{a}}(M,N)$ *is* $\mathfrak{a}$*-cominimax.*

Proof. Consider the convergent spectral sequence

$$E_2^{p,q}:=H^p_{\mathfrak{a}}(\operatorname{Ext}^q_R(M,N)) \underset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{a}}(M,N).$$

Hence, for all $j\geq 0$, there is a finite filtration of the module $H^j=H^j_{\mathfrak{a}}(M,N)$

$$0=\phi^{j+1}H^j\subseteq\phi^jH^j\subseteq\ldots\subseteq\phi^1H^j\subseteq\phi^0H^j=H^j$$

such that $E_\infty^{p,j-p}\cong\phi^pH^j/\phi^{p+1}H^j$ for all $0\leq p\leq j$. By hypothesis, $E_2^{p,q}=0$ for all $p\geq 2$ and all $q\geq 0$. Moreover, since $E_\infty^{p,q}$ is a subquotient of $E_2^{p,q}$ for all $p,q\geq 0$, it implies that $E_\infty^{p,q}=0$ for all $p\geq 2$ and all $q\geq 0$. It therefore follows $0=\phi^{j+1}H^j=\phi^jH^j=\ldots=\phi^2H^j$. On the other hand, it is easily seen that $E_2^{1,j-1}\cong E_\infty^{1,j-1}$ and $E_2^{0,j}\cong E_\infty^{0,j}$. From the exact sequence

$$0\longrightarrow E_2^{1,j-1}\longrightarrow H^j_{\mathfrak{a}}(M,N)\longrightarrow E_2^{0,j}\longrightarrow 0,$$

and using [2, Corollaries 2.4, 2.5, and 3.9], the result follows. □

The following result immediately follows by Theorem 2.2.

Corollary 2.3 (Compare [2, Corollary 3.10]) *Let* $\mathfrak{a}$ *be an ideal of* R *such that* $\operatorname{cd}(\mathfrak{a})=1$ *and let* N *be an* $\mathfrak{a}$*-minimax* R*-module. Then, for all* j*,* $H^j_{\mathfrak{a}}(N)$ *is* $\mathfrak{a}$*-cominimax.*

Theorem 2.4 *Let* $\mathfrak{a}$ *be an ideal of* R *such that* $\operatorname{cd}(\mathfrak{a})=1$ *and let* N *be an* $\mathfrak{a}$*-minimax* R*-module. Then, for all finitely generated* R*-module* M *and all* i,j*,* $\operatorname{Ext}^i(M,H^j_{\mathfrak{a}}(N))$ *is* $\mathfrak{a}$*-cominimax.*

Proof. By [16, Theorem 11.38], we consider the Grothendieck spectral sequence

$$E_2^{p,q}:=\operatorname{Ext}^p(M,H^q_{\mathfrak{a}}(N)) \underset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{a}}(M,N). \qquad (\star)$$

By hypothesis $H^j_{\mathfrak{a}}(N)=0$ for all $j>1$ and $E_2^{i,0}$ is $\mathfrak{a}$-minimax for all i; thus it suffices to show that $E_2^{i,1}$ is $\mathfrak{a}$-cominimax for all i. For all $p\geq 2$, we consider the exact sequence

$$0\longrightarrow ker d_p^{i,1}\longrightarrow E_p^{i,1}\overset{d_p^{i,1}}{\longrightarrow} E_p^{i+p,2-p}. \qquad (\dagger)$$

Since $E_{p+1}^{i,1} = kerd_p^{i,1}/imd_p^{i-p,p}$, we obtain $kerd_2^{i,1} \cong E_3^{i,1} \cong \ldots \cong E_\infty^{i,1}$ for all i. By using $(\star)$, there is a finite filtration of the module $H^{i+1} = H_{\mathfrak{a}}^{i+1}(M, N)$

$$0 = \phi^{i+2}H^{i+1} \subseteq \phi^{i+1}H^{i+1} \subseteq \ldots \subseteq \phi^1 H^{i+1} \subseteq \phi^0 H^{i+1} = H^{i+1}$$

such that $E_\infty^{p,i+1-p} \cong \phi^p H^{i+1}/\phi^{p+1}H^{i+1}$ and $E_\infty^{p,i+1-p}$ is a subquotient of $E_2^{p,i+1-p}$ for all $0 \leq p \leq i+1$. It therefore follows that $\phi^i H^{i+1} = \phi^{i-1}H^{i+1} = \ldots \phi^1 H^{i+1} = \phi^0 H^{i+1} = H_{\mathfrak{a}}^{i+1}(M, N)$. Now, the exact sequence

$$0 \longrightarrow E_\infty^{i+1,0} \longrightarrow H_{\mathfrak{a}}^{i+1}(M, N) \longrightarrow E_\infty^{i,1} \longrightarrow 0$$

in conjunction with $E_\infty^{i+1,0}$ is $\mathfrak{a}$-minimax and $H_{\mathfrak{a}}^{i+1}(M, N)$ is $\mathfrak{a}$-cominimax by Theorem 2.2, yielding that $E_\infty^{i,1}$ is $\mathfrak{a}$-cominimax and so is $kerd_2^{i,1}$. By using $p = 2$ in $(\dagger)$, the result is easily seen, as required. □

Proposition 2.5 *Let $\mathfrak{b} \subseteq \mathfrak{a}$ be 2 ideals of R such that $\operatorname{cd}(\mathfrak{a}) = 1$ and let N be a $\mathfrak{b}$-minimax R-module. Then $H_{\mathfrak{a}}^i(H_{\mathfrak{b}}^j(N))$ is $\mathfrak{a}$-cominimax for all i, j.*

Proof. The only non-trivial case is $i = 0, 1$, and so it is an immediate consequence of the Grothendieck spectral sequence

$$E_2^{p,q} := H_{\mathfrak{a}}^p(H_{\mathfrak{b}}^q(N)) \underset{p}{\Longrightarrow} H_{\mathfrak{a}}^{p+q}(N).$$

□

Lemma 2.6 *Suppose that N and $H_{\mathfrak{a}}^j(N)$ are $\mathfrak{a}$-minimax for all $j < t$. Then $H_{\mathfrak{a}}^j(M, N)$ is $\mathfrak{a}$-minimax for all finitely generated module M and all $j < t$.*

Proof. Consider the Grothendieck spectral sequence

$$E_2^{p,q} := \operatorname{Ext}_R^p(M, H_{\mathfrak{a}}^q(N)) \underset{p}{\Longrightarrow} H_{\mathfrak{a}}^{p+q}(M, N).$$

By using the same argument as in the proof of Theorem 2.4 the result follows. □

Corollary 2.7 *Let N be an $\mathfrak{a}$-minimax R-module. Then, for all finitely generated R-module M,*
$\inf\{j : H_{\mathfrak{a}}^j(N)$ *is not* $\mathfrak{a}$*-minimax*$\} \leq \inf\{j : H_{\mathfrak{a}}^j(M, N)$ *is not* $\mathfrak{a}$*-minimax*$\}$.

The following theorem is a generalization of [3, Theorem 2.2].

Theorem 2.8 *Let N be a minimax R-module and M a finitely generated R-module. Let t be a non-negative integer such that $H_{\mathfrak{a}}^i(M, N)$ is $\mathfrak{a}$-minimax for all $i < t$. Then $\operatorname{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M, N))$ is $\mathfrak{a}$-minimax. In particular the set $\operatorname{Ass}(H_{\mathfrak{a}}^t(M, N))$ is finite.*

Proof. By using the same method that appeared in the proof of Theorem 4.3 in [17], the result follows. □

The following theorem is a generalization of [3, Corollary 2.3].

Theorem 2.9 *Let M be finitely generated and N be minimax. Let t be a non-negative integer such that $H^t_{\mathfrak{a}}(M,N)$ is $\mathfrak{a}$-minimax for all $i < t$. Then for any $\mathfrak{a}$-minimax submodule L of $H^t_{\mathfrak{a}}(M,N)$, the R-module $\mathrm{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M,N)/L)$ is $\mathfrak{a}$-minimax. In particular, the Goldie dimension of $H^t_{\mathfrak{a}}(M,N)/L$ is finite and so the set $\mathrm{Ass}(H^t_{\mathfrak{a}}(M,N)/L)$ is finite.*

Proof. From the exact sequence

$$0 \longrightarrow L \longrightarrow H^t_{\mathfrak{a}}(M,N) \longrightarrow H^t_{\mathfrak{a}}(M,N)/L \longrightarrow 0$$

we get the following exact sequence

$$\mathrm{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M,N)) \longrightarrow \mathrm{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M,N)/L) \longrightarrow \mathrm{Ext}^1_R(R/\mathfrak{a}, L).$$

Hence by [2, Corollary 2.5] and Theorem 2.8, we conclude that $\mathrm{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M,N)/L)$ is $\mathfrak{a}$-minimax and so the result follows. □

Acknowledgment

The author is deeply grateful to the referee for careful reading of the original manuscript and the valuable suggestions. The author also would like to thank Dr. Amir Mafi for many helpful discussions during the preparation of this paper.

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Received: 09.07.2010