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Extensions and s-comparability of exchange rings

Chaoling Huang

Abstract

Let S be a ring extension of R. In this note, for any positive integer s we study s-comparability related to ring extensions. We show that if S is an excellent extension of R, R and S are exchange rings, and R has the n-unperforation property. R satisfies s-comparability if and we only if so does S, and we prove that for a 2-sided ideal J of S, and an exchange subring R of the exchange ring S, which contains J as a direct summand, then R satisfies s-comparability if and only if so does R/J.

Key Words: Exchange rings, excellent extensions, s-comparability

1. Introduction

We say that S is a ring extension of R if there is a (unital) ring homomorphism $f : R \to S$. Let S be a ring and let R be a subring of S (with the same 1). S is called a finite normalizing extension of R if there exist elements $a_1, \dots, a_n \in S$ such that $a_1 = 1$, $S = Ra_1 + \dots + Ra_n$, $a_iR = Ra_i$ for all $i = 1, \dots, n$. Finite normalizing extensions have been studied in many papers such as [4, 8, 9, 10, 13]. S is called a free normalizing extension of R if $a_1 = 1$, $S = Ra_1 + \dots + Ra_n$ is finite normalizing extension and S is free with basis $\{a_1, \dots, a_n\}$ as both a right R-module and a left R-module. S is said to be an excellent extension of R in case S is a free normalizing extension of R and S is right R-projective (that is, if M_S is a right S-module and N_S is a submodule of M_S , then $N_R \mid M_R$ implies $N_S \mid M_S$, where $N|M$ means N is a direct summand of M).

For any right R-module M, Crawley and Jónsson defined M to have the exchange property if for every right R-module A and any 2 decompositions of A,

$$A = M' \oplus N = \oplus_{i \in I} A_i$$

where $M' \cong M$, there are submodules $A'_i \subseteq A_i$ such that $A = M' \oplus (\oplus_{i\in I} A'_i)$. It follows from the modular law that A'_i must be a direct summand of A_i for all i. Warfield [12] called a ring R an exchange ring if R has the exchange property as a right R-module. He proved that this definition is left-right symmetric. Many classes of rings belong to this class of rings, for instance local rings, clean rings, von Neumann regular rings, semiperfect rings, and (strongly) π-regular rings, cf. [6, 11, 12].

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For 2 R-modules M and N, we use $M \lesssim_{\oplus} N$ to denote that M is isomorphic to a direct summand of N. nM means the direct sum of n copies of M for a positive integer n and an R-module M. We denote the category of finitely generated projective R-modules by $FP(R)$. Other basic notations can be found in [3]. Throughout this note, R is an associative ring with identity and R-modules are unitary right R-modules.

R is said to be separative if the following condition holds for all $A,\ B \in FP(R)$:

$$A \oplus A \cong A \oplus B \cong B \oplus B \Rightarrow A \cong B.$$

Recall that in [3, Page 275], given a positive integer s, a von Neumann regular ring R is said to satisfy s-comparability if, for each pair of elements $x,\ y$ of R, either xR is isomorphic to a summand of $s(yR)$, or yR is isomorphic to a summand of $s(xR)$. It is clear that the notion can be generalized to exchange rings. Comparability concepts have proven to be particularly fruitful in the development of the theory of regular rings. Goodearl and Handelman showed that directly finite regular rings satisfying 1-comparability have stable rank one [3, Theorem 8.12]. Pardo [7] showed that an exchange ring satisfying s-comparability is separative and so has stable rank 1, 2, or ∞.

2. Main results

Lemma 2.1 [5, Lemma 7.2.2] *Let S be a ring extension of R. If M_S is a projective module and S_R is projective, then M_R is projective.*

Lemma 2.2 *Let S be an excellent extension of R. Given any S-module M, if M_R is projective, then M_S is projective.*

Proof. Let F_S be a free S-module with an epimorphism $g:\ F_S \to M_S$. Set $K = Kerg$. There is an exact sequence of right S-modules $0 \to K \to F \to M \to 0$ that is split as an exact sequence of right R-modules, since M_R is projective. So $K_R \mid F_R$. Since S is an excellent extension of R, $K_S \mid F_S$, therefore, $M_S \cong F/K$ is projective. □

Lemma 2.3 *Let S be an excellent extension of R, and let $A_R \cong B_R$. Given A_S, we can define an S-module multiplication on B such that $A_S \cong B_S$.*

Proof. Let $\alpha : A_R \to B_R$ and $\beta : B_R \to A_R$ be the isomorphisms. Define $bs = \alpha(\beta(b)s)$. It is easy to check that B is an S-module such that $A_S \cong B_S$. □

Lemma 2.4 *Let $A,\ B$ be finitely generated projective right modules over an exchange ring R. If $A \lesssim_{\oplus} nB$ for some positive integer n, then there is a decomposition $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ such that $A_1 \lesssim_{\oplus} A_2 \lesssim_{\oplus} \cdots \lesssim_{\oplus} A_n \lesssim_{\oplus} B$.*

Proof. We prove it by induction. By [1, Proposition 1.2], there is a decomposition $A = U \oplus W$, where $U \lesssim_{\oplus} (n-1)B$, and $W \lesssim_{\oplus} B$. By the inductive hypothesis, there is a chain $C_1 \subseteq C_2 \subseteq \cdots \subseteq C_{n-1}$ of B such that $U \cong C_1 \oplus C_2 \oplus \cdots \oplus C_{n-1}$. Let $D_1, ..., D_n$ be submodules of B such that $D_1 = C_1$, $C_i = C_{i-1} \oplus D_i$ for $i = 2, ..., n-1$ and $B = C_{n-1} \oplus D_n$. Then $B = D_1 \oplus D_2 \oplus \cdots \oplus D_n$. Using [1, Proposition 1.2] again, it follows from $W \lesssim_{\oplus} B$ that there are $X_i \subseteq D_i,\ i = 1, ..., n$ such that $W \cong X_1 \oplus X_2 \oplus \cdots \oplus X_n$. Hence

$A = U \oplus W \cong C_1 \oplus C_2 \oplus \cdots \oplus C_{n-1} \oplus X_1 \oplus X_2 \oplus \cdots \oplus X_n = X_1 \oplus (C_1 \oplus X_2) \oplus \cdots \oplus (C_{n-1} \oplus X_n)$, where $X_1 \subseteq (C_1 \oplus X_2) \subseteq \cdots \subseteq (C_{n-1} \oplus X_n) \subseteq B$. In fact, since $X_1 \subseteq D_1$ and $D_1 = C_1$, $X_1 \subseteq C_1 \subseteq C_1 \oplus X_2$. Since $X_2 \subseteq D_2$ and $C_2 = C_1 \oplus D_2$, $C_1 \oplus X_2 \subseteq C_1 \oplus D_2 = C_2 \subseteq C_2 \oplus X_2$. Continue this procedure. Set $A_1 = X_1$, $A_i = (C_{i-1} \oplus X_i)$ for $i = 2, ..., n$ as desired. □

For a positive integer s, we say that a ring R satisfies s-comparability, if for any idempotents $x,\ y \in R$, $xR \lesssim_{\oplus} s(yR)$ or $yR \lesssim_{\oplus} s(xR)$. The finitely generated projective right R-modules are said to satisfy s-comparability if $A \lesssim_{\oplus} sB$ or $B \lesssim_{\oplus} sA$ for any finitely generated projective right R-modules A and B.

Proposition 2.5 *Let R be an exchange ring satisfying s-comparability. Then the finitely generated projective right R-modules also satisfy s-comparability.*

Proof. Let A, B be finitely generated projective right R-modules. There is a positive integer n such that $A,\ B \lesssim_{\oplus} nR$. We prove the assertion by induction on n. If $n = 1$, it is true since R satisfies s-comparability. Assume that the assertion holds for $n-1$ and suppose $A,\ B \lesssim_{\oplus} nR$. Since R is an exchange ring, by [1, Proposition 1.2], we can write $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ where $A_i,\ B_i \lesssim_{\oplus} (n-1)R$ for $i = 1,\ 2$. By the induction, we have $A_1 \lesssim_{\oplus} sB_1$ or $B_1 \lesssim_{\oplus} sA_1$, and $A_2 \lesssim_{\oplus} sB_2$ or $B_2 \lesssim_{\oplus} sA_2$. If $A_1 \lesssim_{\oplus} sB_1$ and $A_2 \lesssim_{\oplus} sB_2$, or $B_1 \lesssim_{\oplus} sA_1$ and $B_2 \lesssim_{\oplus} sA_2$, the assertion is obviously true. Now assume that $A_1 \lesssim_{\oplus} sB_1$ and $B_2 \lesssim_{\oplus} sA_2$. By Lemma 2.4, there is a direct summand V of A_1 such that $V \lesssim_{\oplus} B_1$ and $A_1 \lesssim_{\oplus} sV$ and a direct summand U of B_2 such that $U \lesssim_{\oplus} A_2$ and $B_2 \lesssim_{\oplus} sU$. So $B_1 \cong V \oplus C$ and $A_2 \cong U \oplus D$, where C and D are finitely generated projective right R-modules. Since $C,\ D \lesssim_{\oplus} (n-1)R$, by the inductive hypothesis, we get $C \lesssim_{\oplus} sD$, or $D \lesssim_{\oplus} sC$. If the former is true, then

$$\begin{aligned} B = B_1 \oplus B_2 &\cong V \oplus C \oplus B_2 \\ &\lesssim_{\oplus} V \oplus sD \oplus sU \\ &\cong V \oplus sA_2 \\ &\lesssim_{\oplus} sA_1 \oplus sA_2 \\ &= sA. \end{aligned}$$

Thus $B \lesssim_{\oplus} sA$. Similarly, if the latter is true, then $A \lesssim_{\oplus} sB$, as desired. □

Lemma 2.4 and Proposition 2.5 for regular rings is well-known [2, Lemma 1.1 and Proposition 2.1].

Theorem 2.6 *Let S be an excellent extension of R. If R and S are exchange rings, and R has the n-unperforation property (i.e., $nA \lesssim_{\oplus} nB$ implies that $A \lesssim_{\oplus} B$ for any finitely generated projective R-modules A and B), then R satisfies s-comparability if and only if so does S.*

Proof. $\Rightarrow$: Let $x = x^2$, $y = y^2 \in S$. xS and yS are finitely generated projective S-modules. By Lemma 2.1, $(xS)_R$ and $(yS)_R$ are finitely generated projective. Since R satisfies s-comparability, by Proposition 2.5, finitely generated projective R-modules satisfy s-comparability. Thus $(xS)_R \lesssim_{\oplus} s(yS)_R$ or $(yS)_R \lesssim_{\oplus} s(xS)_R$. If $(xS)_R \lesssim_{\oplus} s(yS)_R$, let T be the direct summand of $s(yS)_R$ such that $(xS)_R \cong T_R$. Since xS is an S-module, we can consider T as an S-module such that $(xS)_S \cong T_S$ as S-modules by Lemma 2.3. Since $T_R \mid s(yS)_R$, by the R-projectivity of S, $T_S \mid s(yS)_S$. Thus $(xS)_S \lesssim_{\oplus} s(yS)_S$. Similarly, we have $(yS)_S \lesssim_{\oplus} s(xS)_S$, if $(yS)_R \lesssim_{\oplus} s(xS)_R$.

$\Leftarrow$: For any $x = x^2,\ y = y^2 \in R,\ (xR)_R \lesssim_\oplus R_R \lesssim_\oplus nR_R \cong S_R$. So $(xR) \otimes_R S$ and $(yR) \otimes_R S$ are finitely generated projective S-modules. Since S satisfies s-comparability, $((xR) \otimes_R S)_S \lesssim_\oplus s((yR) \otimes_R S)_S$ or $((yR) \otimes_R S)_S \lesssim_\oplus s((xR) \otimes_R S)_S$. $((xR) \otimes_R S)_R \lesssim_\oplus s((yR) \otimes_R S)_R$ or $((yR) \otimes_R S)_R \lesssim_\oplus s((xR) \otimes_R S)_R$. Since S is a free R-module with basis $\{a_1, \ldots, a_n\}$, we have $((xR) \otimes_R S)_R \cong ((xR) \otimes_R nR)_R \cong n(xR)_R$. Similarly, $((yR) \otimes_R S)_R \cong n(yR)_R$. Thus, $n(xR)_R \lesssim_\oplus s(n(yR))_R$ or $n(yR)_R \lesssim_\oplus s(n(xR))_R$. By the hypothesis, we have $(xR)_R \lesssim_\oplus s(yR)_R$ or $(yR)_R \lesssim_\oplus s(xR)_R$. $\square$

Lemma 2.7 [1, Proposition 1.4] *Any exchange ring R has the following properties:*

(1) for any $M' \in FP(R/I)$, there is $M \in FP(R)$ such that $R/I \otimes_R M \cong M'$;

(2) whenever $A,\ B \in FP(R)$ with $A/AI \cong B/BI$, there exist decompositions $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ such that $A_1 \cong B_1,\ A_2 = A_2 I,\ B_2 = B_2 I$.

We thank Ken Goodearl for giving the proof of the following lemma (in private communication).

Lemma 2.8 *Let J be a 2-sided ideal in an exchange ring R, and let A and B be finitely generated projective right R-modules such that $A/AJ \lesssim_\oplus B/BJ$. Then there exist decompositions $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ such that $A_1 \cong B_1$ and $A_2 = A_2 J$.*

Proof. Suppose that A and B are finitely generated projective right R-modules, and let J be an ideal of R. Assume that A/AJ is isomorphic to a direct summand of B/BJ. Set $(A/AJ) \oplus C' \cong B/BJ$. Then C' is a finitely generated projective R/J-module. By Lemma 2.7, it lifts to a finitely generated projective R-module C, that is, $C/CJ \cong C'$. At this point, we have $(A \oplus C)/(A \oplus C)J$ isomorphic to B/BJ.

Using Lemma 2.7 again, there are decompositions $A \oplus C = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ such that A_1 is isomorphic to B_1 while $A_2 = A_2 J$ and $B_2 = B_2 J$. Since R is an exchange ring, by [1, Proposition 1.2], there are decompositions $A = A' \oplus A''$ and $C = C' \oplus C''$ such that $A_1 \cong A' \oplus C'$ and $A_2 \cong A'' \oplus C''$. Since $A_2 = A_2 J$, we have that $A'' = A'' J$. On the other hand, $B_1 = B' \oplus B''$ with $A' \cong B'$ and $C' \cong B''$. So $A = A' \oplus A''$ $B = B_1 \oplus B_2 = B' \oplus (B'' \oplus B_2)$, $A' \cong B'$ and $A'' = A'' J$. $\square$

Lemma 2.9 [2, Lemma 1.2] *Let $A,\ B,\ C$ be finitely generated projective modules over any exchange ring R. If $A \oplus B \cong kC$ for some positive integer k, then there is a decomposition $C = C_0 \oplus C_1 \oplus \cdots \oplus C_k$ such that*

$$A \cong C_1 \oplus 2C_2 \oplus \cdots \oplus kC_k$$

$$B \cong kC_0 \oplus (k-1)C_1 \oplus \cdots \oplus C_{k-1}.$$

Theorem 2.10 *Let S be an exchange ring satisfying s-comparability, let J be a 2-sided ideal of S, and let R be an exchange subring of S which contains J as a direct summand. Then R satisfies s-comparability if and only if so does R/J.*

Proof. Assume that $x = x^2 \in R,\ y = y^2 \in R$. We distinguish the following cases:

(1) If $x,\ y \in J$, then $xR = xJ = xS$ and $yR = yJ = yS$. Since S satisfies s-comparability, $xR \lesssim_\oplus s(yR)$ or $yR \lesssim_\oplus s(xR)$.

(2) If $x \in J$, and $y \notin J$, then $y \notin SxS$. If $yS \lesssim_{\oplus} s(xS)$, since $xRxR = xR$ and $yR \cong yRxS$, $yS \lesssim_{\oplus} SxS$. We have $y \in SxS$, which is a contradiction. Thus we have $xS \lesssim_{\oplus} syS$. $xR = xJ \cong xS \otimes_S J \lesssim_{\oplus} s(yS) \otimes_S J \cong s(yJ) \leq_{\oplus} s(yR)$.

(3) If $x \notin J$, since R/J satisfies s-comparability, we have that $xR/(xR)J \lesssim_{\oplus} s(yR/(yR)J)$ or $yR/(yR)J \lesssim_{\oplus} s(xR/(xR)J)$. Without loss of the generality, we assume the former is true. By Lemma 2.8, there are decompositions $xR = x_1R \oplus x_2R$ and $yR = y_1R \oplus y_2R$ with $x_1R \cong y_1R$ and $x_2R = (x_2R)J$. Clearly, $y_2 = y_2^2 \in R$. Thus by the above discussion, $x_2R \lesssim_{\oplus} s(y_2R)$ or $y_2R \lesssim_{\oplus} s(x_2R)$. If the former is true,

$$xR = x_1R \oplus x_2R \lesssim_{\oplus} s(y_1R) \oplus s(y_2R) \cong s(yR).$$

If the latter is true, by Lemma 2.4 and 2.9, there exist finitely generated projective R-modules U, V such that

$$x_1R \lesssim_{\oplus} sU,\ U \lesssim_{\oplus} y_1R,\ U \lesssim_{\oplus} x_1R,$$

$$y_2R \lesssim_{\oplus} sV,\ V \lesssim_{\oplus} x_2R,\ V \lesssim_{\oplus} y_2R.$$

Set $U \oplus W \cong y_1R$, $V \oplus T \cong x_2R$. So there are $f = f^2$, $g = g^2 \in R$ such that $W = fR$, $T = gR$. It is clear that $g \in x_2R \subseteq J$, thus $W \lesssim_{\oplus} sT$ or $T \lesssim_{\oplus} sW$. Without loss of generality, we assume that the former is true,

$$\begin{aligned} yR = y_1R \oplus y_2R &\cong U \oplus W \oplus y_2R \\ &\lesssim_{\oplus} U \oplus sT \oplus y_2R \\ &\lesssim_{\oplus} x_1R \oplus sT \oplus sV \\ &\lesssim_{\oplus} s(x_1R) \oplus s(T \oplus V) \\ &\lesssim_{\oplus} s(x_1R) \oplus s(x_2R) \\ &\lesssim_{\oplus} s(xR). \end{aligned}$$

Similarly, if the latter is true, then $xR \lesssim_{\oplus} s(yR)$, as desired. □

We end this note by raising the following question: let S be an excellent extension of R. Is it true that R is an exchange ring satisfying s-comparability if and only if S is an exchange ring satisfying s-comparability?

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Chaoling HUANG
Department of Mathematics,
Jiangxi Agricultural University,
Nanchang 330045, CHINA
e-mail: chaolinghuang43@163.com

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