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Multiplication modules with Krull dimension

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Abstract

In ring theory, it is shown that a commutative ring R with Krull dimension has classical Krull dimension and satisfies $\text{k.dim}(R) = \text{cl.k.dim}(R)$. Moreover, R has only a finite number of distinct minimal prime ideals and some finite product of the minimal primes is zero (see Gordon and Robson [9, Theorem 8.12, Corollary 8.14, and Proposition 7.3]). In this paper, we give a generalization of these facts for multiplication modules over commutative rings. Actually, among other results, we prove that if M is a multiplication R-module with Krull dimension, then: (i) M is finitely generated, (ii) R has finitely many minimal prime ideals $\mathcal{P}_1, \ldots, \mathcal{P}_n$ of $\text{Ann}(M)$ such that $\mathcal{P}_1^k \ldots \mathcal{P}_n^k M = (0)$ for some $k \geq 1$, and (iii) M has classical Krull dimension and $\text{k.dim}(M) = \text{cl.k.dim}(M) = \text{k.dim}(M/\mathcal{P}M) = \text{cl.k.dim}(M/\mathcal{P}M)$ for some prime ideal $\mathcal{P}$ of R.

Key Words: Krull dimension, classical Krull dimension, multiplication module, prime submodule

1. Introduction

All rings throughout this paper are associative, commutative with identity $1 \neq 0$, and all modules are unital. Let M be an R-module and X be either an element or a subset of R. Then the annihilator of X is the ideal $\text{Ann}(X) = \{a \in R \mid aX = 0\}$. A proper submodule P of M is called a *prime submodule*, provided $rm \in P$ for some $r \in R$ and $m \in M$, implies that $m \in P$ or $rM \subseteq P$ (i.e. $r \in \text{Ann}(M/P)$). One may easily see that a proper submodule P of M is prime if and only if $\text{Ann}(M/P) = \mathcal{P}$ is a prime ideal of R and M/P is a torsion free $R/\mathcal{P}$-module. This notion of prime submodules was first introduced by Feller [8] and studied by Karakas [10] over commutative rings, and more systematically by Dauns [6] in general (not necessarily over commutative rings). Recently this notion of primes in modules has received a good deal of attention from several authors, for example [2], [3], [4], [5], [6], [12], and many others. An R-module M is called a *multiplication module* if any submodule N of M is expressible as $N = IM$, where I is an ideal of R. Ideals generated by idempotents, and hence any ideal in a Von-Neumann regular ring, multiplication ideals, invertible ideals, and more generally projective ideals, are examples of multiplication modules. For more details and in fact some significant equivalent definitions that are crucial for understanding multiplication modules, the reader is referred to [1] and [7].

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By "Krull dimension" we will mean the module-theoretic Krull dimension (defined below), introduced by Rentschler and Gabriel [14] and Krause [11] and studied extensively by Gordon and Robson [9] amongst others. Let R be a ring and let M be an R-module. The Krull dimension of M, denoted by $\text{k.dim}(M)$, if it exists, is defined as follows: $\text{k.dim}(M) = -1$ if and only if $M = 0$. If $\alpha \geq 0$ is an ordinal such that all modules with Krull dimension strictly less than α are known, then $\text{k.dim}(M) \leq \alpha$ if for every chain $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$ of submodules of M there is a positive integer n such that $\text{k.dim}(M_i/M_{i+1}) < \alpha$ for all $i \geq n$; and $\text{k.dim}(M) = \alpha$ if α is the smallest ordinal β such that $\text{k.dim}(M) \leq \beta$. Note that $\text{k.dim}(M) = 0$ if and only if M is nonzero Artinian. In this sense, the Krull dimension of a module can be thought of as a measure of how far the module is from being Artinian. It is interesting, however, that many properties of modules with Krull dimension are similar (or identical) to those of Noetherian modules. A ring R will be said to have Krull dimension if the R-module R has Krull dimension.

For a module M, the prime spectrum $\text{Spec}(M)$ is defined to be the set of all prime submodules of M. Trivially, maximal submodules are prime submodules. Unfortunately, unlike the rings with identity, not every R-module contains a prime submodule; for example $\mathbb{Z}_{p^\infty}$ does not contain a prime submodule.

Let M be a multiplication R-module. Let $\text{Spec}_{-1}(M) = \emptyset$ and $\text{Spec}_0(M)$ denote the collection of maximal submodules of M and for any ordinal $\alpha \geq 1$, let $\text{Spec}_\alpha(M)$ denote the collection of prime submodules P of M such that all prime submodules of M properly containing P belong to $\bigcup_{0 \leq \beta < \alpha} \text{Spec}_\beta(\text{M})$. If there exists an ordinal $\alpha \geq 1$ such that $\text{Spec}(M) = \text{Spec}_\alpha(\text{M})$ then we shall say that M has classical Krull dimension and the classical Krull dimension of M, denoted by $\text{cl.k.dim}(M)$, shall be the least ordinal $\gamma \geq 0$ such that $\text{Spec}(M) = \text{Spec}_\gamma(\text{M})$. This notion of classical Krull dimension for modules was first introduced in [3] (see also [5]). Krause [11] shows that having classical Krull dimension for a ring R is equivalent to having ascending chain condition (*acc*) on prime ideals. This fact is also true for multiplication modules (see [3, Proposition 4.10]).

In this paper we will discuss relationships between the above different types of "Krull dimension" for multiplication modules. In ring theory, it is shown that Krull and classical Krull dimension are equal for left $FBN-$rings (fully bounded Noetherian rings); see [13, Theorem 4.8]. This generalizes to $FBK-$rings (fully bounded rings with Krull dimension); see Gordon and Robson [9, Theorem 8.12]. In particular, a commutative ring R with Krull dimension has classical Krull dimension and satisfies $\text{k.dim}(R) = \text{cl.k.dim}(R)$. Also, a ring with Krull dimension has only a finite number of distinct minimal prime ideals and some finite product of the minimal primes is zero (see [9, Proposition 7.3] or [15, Proposition 1.4.11]).

Our main aim in this paper is a further generalization of these facts for multiplication modules over commutative rings. In fact: it is shown that if M is a multiplication module with Krull dimension, then R has only a finite number of distinct minimal prime ideals $\mathcal{P}_1, \dots, \mathcal{P}_n$ of $\text{Ann}(M)$ such that $\mathcal{P}_1^k \dots \mathcal{P}_n^k M = (0)$ for some $k \geq 1$, and $\mathcal{P}_1 M, \dots, \mathcal{P}_n M$ are the only minimal prime submodules of M (see Theorem 2.12). Also, if M is a multiplication module with Krull dimension, then M is finitely generated (see Corollary 2.14). Finally, Theorem 4.6 shows that if M is a multiplication R-module with Krull dimension, then M has classical Krull dimension and $\text{k.dim}(M) = \text{cl.k.dim}(M) = \text{k.dim}(M/\mathcal{P}M) = \text{cl.k.dim}(M/\mathcal{P}M)$ for some prime ideal $\mathcal{P}$ of R.

2. On Krull dimension of multiplication modules

Let R be a ring. We recall that an element $c \in R$ is called *regular* if c is not a zero divisor of R (i.e. its $\text{Ann}(c)$ is (0))

Lemma 2.1 *Let M be a faithful multiplication R-module with Krull dimension. Then $k.dim(M/cM) < k.dim(M)$ for each regular element c of R.*

Proof. Suppose that c is a regular element of R. Clearly $M \supseteq cM \supseteq c^2M \supseteq \cdots$ is a descending chain of submodules of M. We define $\varphi : M \longrightarrow c^iM/c^{i+1}M$ with $\varphi(m) = c^im + c^{i+1}M$. Clearly, φ is an R-module epimorphism and $cM \subseteq \ker\varphi$. Also, if $m \in \ker\varphi$, then $c^im = c^{i+1}m'$ for some $m' \in M$. Since M is a multiplication R-module, the submodule $R(m - cm')$ is of the form IM for some ideal I. Therefore, $c^iIM = 0$ and since M is faithful, $c^iI = (0)$. Since c is a regular element in R, $I = (0)$, i.e. $m = cm'$. Thus $\ker\varphi = cM$ and hence for all i, $M/cM \cong c^iM/c^{i+1}M$. Now by definition of Krull dimension, $\text{k.dim}(M/cM) < \text{k.dim}(M)$. □

An R-module M is called a *prime module* provided $(0) \subsetneqq M$ is a prime submodule of M, i.e. $\text{Ann}(m) = \text{Ann}(M)$ for all $(0) \neq m \in M$. Clearly $P \subsetneqq M$ is a prime submodule if and only if M/P is a prime module.

Lemma 2.2 [7, Theorem 2.13] *Let M be a faithful multiplication R-module. A submodule E of M is essential if and only if there exists an essential ideal I of R such that $E = IM$.*

The next result is often useful in calculations.

Lemma 2.3 [13, Lemma 2.2.8] *Let M be an R-module with Krull dimension. Then*

$$k.dim(M) \leq sup\{k.dim(M/E) + 1 \mid E \text{ is an essential submodule of } M\}.$$

Lemma 2.4 [13, Lemma 6.2.4] *Let M be an R-module. If N is a submodule of M then*

$$k.dim(M) = sup\{k.dim(N), k.dim(M/N)\}.$$

Lemma 2.5 [7, Corollary 2.11] *The following statements are equivalent for a proper submodule P of multiplication R-module M.*

(i) *P is a prime submodule of M.*

(i) *$\mathcal{P} = \text{Ann}(M/P)$ is a prime ideal of R.*

(i) *$P = \mathcal{P}M$ for some prime ideal $\mathcal{P}$ of R with $\mathcal{P} \supseteq \text{Ann}(M)$.*

Theorem 2.6 *Let M be a prime multiplication R-module with Krull dimension. Then*

$$k.dim(M) = sup\ \{k.dim(M/E) + 1 \mid E \text{ is an essential submodule of } M\}.$$

Proof. Since M is a prime module by Lemma 2.5, $\mathcal{P} = Ann_R(M)$ is a prime ideal. Also, it is easy to see that the Krull dimension of M as an R-module is equal to the Krull dimension of M as an $R/\mathcal{P}$-module. Thus without loss of generality we can assume that R is a domain and M is a faithful R-module. Since every nonzero ideal of a domain is essential, by Lemma 2.2, every nonzero submodule of M is also essential. Let the supremum be α. By Lemma 2.3, $\text{k.dim}(M) \leq \alpha$. For each nonzero submodule $E = IM$ of M, the ideal I contains a regular element; say c, so $cM \subseteq IM = E$. Thus by using Lemma 2.1 and Lemma 2.4

$$\text{k.dim}(M/E) + 1 \leq \text{k.dim}(M/cRM) + 1 \leq \text{k.dim}(M),$$

and this completes the proof. □

Corollary 2.7 *Let M be a prime multiplication R-module with Krull dimension. Then for any nonzero submodule N of M, $k.dim(M/N) < k.dim(M)$.*

Proof. Clear by the proof of Theorem 2.6. □

Recall that an element of a ring is called *nilpotent* if some power of it is zero and that a set is called nil if each of its elements is nilpotent. Also, we recall the definition of the nilpotent element in a module. An element m of an R-module M is called *nilpotent* if $m = \sum_{i=1}^{r} a_i m_i$ for some $a_i \in R$, $m_i \in M$, and $r \in N$, such that $a_i^k m_i = 0$ $(1 \leq i \leq r)$ for some $k \in N$. A submodule N of M is called a *nil submodule* if each element of N is nilpotent. We denote the set of all nilpotent elements of M by $\text{Nil}_*({}_RM)$. Clearly $\text{Nil}_*({}_RM)$ is a submodule of M (see [4]). M is called a *nil module* if every element of M is nilpotent. Also, a submodule $N = IM$ of a multiplication R-module M is called *nilpotent* (resp., *idempotent*) if $I^n M = 0$ for some $n \in N$ (resp., $I^2 M = IM$).

Minimal prime submodules are defined in a natural way. It is clear that whenever $\{P_i\}_{i \in I}$ is a chain of prime submodules of an R-module M, then $\cap_{i \in I} P_i$ is always a prime submodule. Therefore, by Zorn's lemma each prime submodule of M contains a minimal one. Let $\text{rad}_R(M)$ be the intersection of all (minimal) prime submodules of M. Then $\text{Nil}_*(R)M \subseteq \text{Nil}_*({}_RM) \subseteq \text{rad}_R(M)$ (see [4, Lemma 3.2]). Note that for an ideal I of R, $\sqrt{I} := \{r \in R \mid r^k \in I \text{ for some } k \geq 1\}$ and $\text{Nil}_*(R) = \sqrt{(0)}$.

Lemma 2.8 *Let M be a multiplication R-module. Then*

$$\sqrt{Ann(M)}M = Nil_*({}_RM) = rad_R(M).$$

In particular, if M is faithful, then $\text{Nil}_*(R)M = \text{Nil}_*({}_RM) = \text{rad}_R(M)$.

Proof. Clearly $\sqrt{\text{Ann}(M)}M \subseteq \text{Nil}_*({}_RM)$ and by [4, Lemma 3.2], $\text{Nil}_*({}_RM) \subseteq \text{rad}_R(M)$. Also, by [7, Theorem 2.12], $\sqrt{\text{Ann}(M)}M = \text{rad}_R(M)$, and hence the proof is complete. □

Lemma 2.9 ([7, Corollary 4.7] *Let M be a faithful multiplication R-module. Then the ring R has Krull dimension if and only if M has Krull dimension. In this case $k.dim(R) = k.dim(M)$.*

It is well known that if R is a ring with Krull dimension, then every nil ideal of R is nilpotent (see for example, [13, Theorem 6.3.7]). Next, we give a generalization of this fact for multiplication modules over commutative rings.

Proposition 2.10 *Let M be a multiplication R-module with Krull dimension. Then every nil submodule of M is nilpotent.*

Proof. Clearly $\mathrm{Nil}_*(M)$ is nilpotent as an R-submodule of M if and only if $\mathrm{Nil}_*(M)$ is nilpotent as an $R/\mathrm{Ann}(M)$-submodule. Also, M has Krull dimension as an R-module if and only if M has Krull dimension as an $R/\mathrm{Ann}(M)$-module. Thus we may assume that M is faithful. Since each nil submodule of M contained in $\mathrm{Nil}_*(M)$, it suffices to show that $\mathrm{Nil}_*(M)$ is nilpotent. By Lemma 2.8, $\mathrm{Nil}_*(M) = \mathrm{Nil}_*(R)M$. By Lemma 2.9, R has Krull dimension and since $\mathrm{Nil}_*(R)$ is a nil ideal of R, by [13, Theorem 6.3.7], $\mathrm{Nil}_*(R)$ is nilpotent. It follows that $\mathrm{Nil}_*(M)$ is a nilpotent submodule of M. $\square$

In [15, Proposition 1.4.11] it was shown that, if R is a ring with right Krull dimension, then there are only finitely many minimal prime ideals of R and some finite product of these is zero. The following theorem is a generalization of this fact for multiplication modules over commutative rings.

Lemma 2.11 *Let M be a multiplication R-module and P be a proper submodule of M. Then P is a prime submodule of M if and only if for all ideals I, J of R, $IJM \subseteq P$ implies that $IM \subseteq P$ or $JM \subseteq P$.*

Proof. Evident. $\square$

Theorem 2.12 *Let R be a ring and M be a multiplication R-module with Krull dimension. Then R has finitely many minimal prime ideals $\mathcal{P}_1, \ldots, \mathcal{P}_n$ of Ann(M) such that $\mathcal{P}_1^k \ldots \mathcal{P}_n^k M = (0)$ for some $k \geq 1$ and $\mathcal{P}_1 M, \ldots, \mathcal{P}_n M$ are the only minimal prime submodules of M.*

Proof. By Lemma 2.8, $\sqrt{\mathrm{Ann}(M)}M = \mathrm{rad}_R(M)$ and by Lemma 2.9, $R/\mathrm{Ann}(M)$ has Krull dimension. Also, by [13, Corollary 6.3.8], $R/\mathrm{Ann}(M)$ contain only finitely many minimal primes, $\mathcal{P}_1/\mathrm{Ann}(M), \cdots, \mathcal{P}_n/\mathrm{Ann}(M)$ say; and the prime radical $\mathrm{Nil}_*(R/\mathrm{Ann}(M)) = (\bigcap_{i=1}^n \mathcal{P}_i)/\mathrm{Ann}(M)$ is nilpotent, i.e. $(\bigcap_{i=1}^n \mathcal{P}_i)^k M = (0)$ for some $k \geq 1$. It follows that $\mathcal{P}_1^k \cdots \mathcal{P}_n^k M = (0)$. Since each $\mathcal{P}_i$ contains $\mathrm{Ann}(M)$, by Lemma 2.5, for each i, $\mathcal{P}_i$ is a minimal prime of $\mathrm{Ann}(M)$ and $\mathcal{P}_i M$ is a prime submodule of M. On the other hand, if $\mathcal{P}M$ is a minimal prime submodule of M, then $\mathcal{P}_1^k \cdots \mathcal{P}_n^k M = (0)$ implies that $\mathcal{P}_j M \subseteq \mathcal{P}M$ (see Lemma 2.11), whence by minimality, $\mathcal{P}_j M = \mathcal{P}M$. Therefore, $\mathcal{P}_1 M, \ldots, \mathcal{P}_n M$ are the only minimal prime submodules of M. $\square$

Lemma 2.13 [7, Theorem 3.7] *Let M be a multiplication R-module such that M contains only finitely many minimal prime submodules. Then M is finitely generated.*

Corollary 2.14 *Let M be a multiplication R-module with Krull dimension. Then M is finitely generated.*

Proof. Let M be a multiplication R-module with Krull dimension. By Theorem 2.12, M contains only finitely many minimal prime submodules and by Lemma 2.13, M is finitely generated. $\square$

By Woodward [15], if R is a ring with Krull dimension, then $\mathrm{k.dim}(R) = \mathrm{k.dim}(R/\mathcal{P})$ for some prime ideal $\mathcal{P}$ of R. Next we prove the following extension of this result for multiplication modules.

Theorem 2.15 *Let M be a multiplication R-module with Krull dimension. Then $k.dim(M) = k.dim(M/\mathcal{P}M)$ for some prime submodule $\mathcal{P}M$ of M.*

Proof. It is easy to see that the Krull dimension of M as an R-module is equal to the Krull dimension of M as an $R/\mathrm{Ann}(M)$-module. Thus we can assume that M is a faithful R-module. Since M has Krull dimension, by Corollary 2.14, M is finitely generated and by Theorem 2.12, there exist minimal prime ideals $\mathcal{P}_1,\ldots,\mathcal{P}_t$ (not necessarily distinct) of $\mathrm{Ann}(M)$ such that $\mathcal{P}_1\ldots\mathcal{P}_tM=(0)$. Consider the descending chain $M\supseteq\mathcal{P}_1M\supseteq\mathcal{P}_1\mathcal{P}_2M\supseteq\ldots\supseteq\mathcal{P}_1\ldots\mathcal{P}_tM=(0)$. If $t=1$, then $\mathcal{P}_1=(0)$ (since M is faithful) and hence (0) is a prime ideal of R, and $\text{k.dim}(M)=\text{k.dim}(M/(0)M)$. Thus we can assume that $t\geq 2$. Then for each $2\leq i\leq t$ the factor $(\mathcal{P}_1...\mathcal{P}_{i-1})M/(\mathcal{P}_1...\mathcal{P}_{i-1}\mathcal{P}_i)M$ is an $(R/\mathcal{P}_i)-$module and so by [15, Lemma 1.4.6], $\text{k.dim}((\mathcal{P}_1...\mathcal{P}_{i-1})M/(\mathcal{P}_1...\mathcal{P}_{i-1}\mathcal{P}_i)M)\leq\text{k.dim}(R/\mathcal{P}_i)$ as an $R/\mathcal{P}_i$-module. Since M is finitely generated and each $\mathcal{P}_i$ is a minimal prime of $\mathrm{Ann}(M)$, we conclude that $M/\mathcal{P}_iM$ is a finitely generated faithful $R/\mathcal{P}_i$-module. Thus by Lemma 2.9, $\text{k.dim}(M/\mathcal{P}_iM)=\text{k.dim}(R/\mathcal{P}_i)$ for each i. Now by Lemma 2.4,

$$\text{k.dim}(M)=\sup\ \{\text{k.dim}(M/\mathcal{P}_1M),...,\text{k.dim}(M/\mathcal{P}_tM)\}$$

and hence, $\text{k.dim}(M)=\text{k.dim}(M/\mathcal{P}_iM)$ for some $1\leq i\leq t$. □

3. On classical Krull dimension of multiplication modules

Let R be a ring, let I be an ideal of R, and let $\alpha\geq 0$ be an ordinal. If $\mathcal{P}$ is a prime ideal of R containing I, then $\mathcal{P}\in\mathrm{Spec}_\alpha(R)$ if and only if $\mathcal{P}/I\in\mathrm{Spec}_\alpha(R/I)$ (see [15, Lemma 5.2.3(i)]). By using the same method, we still extend this result for modules as follows:

Lemma 3.1 *Let M be an R-module, let N be a submodule of M, and let $\alpha\geq 0$ be an ordinal. If P is a prime submodule of M containing N, then $P\in Spec_\alpha(M)$ if and only if $P/N\in Spec_\alpha(M/N)$.*

Let M be an R-module and N_1, N_2 be submodules of M. As [3], we say that N_1 is strongly properly contained in N_2, and write $N_1\subset_s N_2$, if $N_1\subset N_2$ and also $\mathrm{Ann}(M/N_1)\subset\mathrm{Ann}(M/N_2)$.

Proposition 3.2 *Let M be a prime multiplication R-module with classical Krull dimension. Then for any nonzero prime submodule $\mathcal{P}M$ of M, cl.k.dim$(M/\mathcal{P}M)$ exists and*

$$cl.k.dim(M/\mathcal{P}M)<cl.k.dim(M).$$

Proof. Assume that $\mathcal{P}M$ is a nonzero prime submodule of M. Since M is a multiplication R-module, by [3, Lemma 4.4], for any 2 prime submodules P, Q of M, $P\subset_s Q$ if and only if $P\subset Q$. Since $(0)\subset\mathcal{P}M$, so $(0)\subset_s\mathcal{P}M$ and hence by [3, Lemma 3.10], cl.k.dim$(M/\mathcal{P}M)$ exists and cl.k.dim$(M/\mathcal{P}M)<$ cl.k.dim(M). □

Lemma 3.3 (see [3, Lemma 3.6]) *Let M be multiplication R-module for which cl.k.dim(M) exists. Then for any submodule N of M, cl.k.dim(M/N) exists and is no larger than cl.k.dim(M).*

Proposition 3.4 *Let M be a multiplication R-module with classical Krull dimension cl.k.dim$(M)\geq\alpha$ for some ordinal $\alpha\geq 0$. If cl.k.dim$(M/IM)<\alpha$ for every nonzero submodule IM of M, then M is a prime module with cl.k.dim$(M)=\alpha$.*

Proof. Since cl.k.dim(M) $\geq \alpha \geq 0$, M has a prime submodule. Let $\mathcal{P}_1M \in \operatorname{Spec}(M)$. It suffices to show that $\mathcal{P}_1M \in \operatorname{Spec}_\alpha(M)$. If every prime submodule of M is maximal, then cl.k.dim$(M) = 0 = \alpha$. Take $\mathcal{P}_2M$ strictly containing $\mathcal{P}_1M$. Since $\mathcal{P}_2M \neq (0)$, cl.k.dim$(M/\mathcal{P}_2M) = \beta$ for some ordinal $\beta < \alpha$ and so $\mathcal{P}_2M \in \operatorname{Spec}_\beta(M)$ by Lemma 3.1. It follows that $\mathcal{P}_1M \in \operatorname{Spec}_\alpha(M)$. Therefore, cl.k.dim$(M) \leq \alpha$ and so cl.k.dim$(M) = \alpha$. Now suppose, contrary to our claim, that M is not a prime module. By Lemma 2.11, there exist nonzero submodules IM and JM of M where I, J are ideals of R and $IJM = (0)$. Let $\mathcal{P}M$ be a prime submodule of M and put $\beta = \max\{\text{cl.k.dim}(M/IM), \text{cl.k.dim}(M/JM)\}$. Then $\beta < \alpha$ and we may assume that $IM \subseteq \mathcal{P}M$ (since $IJM = (0)$ and $\mathcal{P}M \subsetneqq M$ is prime). Thus by Lemma 3.3,

$$\text{cl.k.dim}(M/\mathcal{P}M) = \text{cl.k.dim}(\frac{M/IM}{\mathcal{P}M/IM}) \leq \text{cl.k.dim}(M/IM) \leq \beta < \alpha,$$

so $\mathcal{P}M \in \operatorname{Spec}_\beta(M)$. Thus $\operatorname{Spec}(M) = \operatorname{Spec}_\beta(M)$ with $\beta < \alpha$, a contradiction (since cl.k.dim$(M) = \alpha$). Thus M is a prime module. □

Theorem 3.5 *Let M be a Noetherian multiplication R-module. Then the following statements are equivalent:*
(i) *M is a prime module.*
(ii) *cl.k.dim$(M/\mathcal{P}M) <$ cl.k.dim(M) for every nonzero prime submodule $\mathcal{P}M$ of M.*
(iii) *cl.k.dim$(M/IM) <$ cl.k.dim(M) for every nonzero submodule IM of M.*

Proof. Since M is Noetherian, by [3, Proposition 4.10], M has classical Krull dimension. Thus (i) $\Rightarrow$ (ii) is by Lemma 3.3 and (iii) $\Rightarrow$ (i) is by Proposition 3.4.
(ii) $\Rightarrow$ (iii). Let IM be a submodule of M which is maximal with respect to the property that cl.k.dim$(M/IM) =$ cl.k.dim$(M) = \alpha$. If KM/IM is a nonzero submodule of M/IM then cl.k.dim$((M/IM)/(KM/IM)) =$ cl.k.dim$(M/KM) < \alpha =$ cl.k.dim(M/IM), by the maximality of IM. It follows, by Proposition 3.4, that M/IM is a prime module, that is IM is a prime submodule of M. Thus $IM = (0)$ by (ii) and, hence, cl.k.dim$(M/IM) <$ cl.k.dim(M) for every nonzero submodule IM of M. □

4. The relationship between Krull and classical Krull dimension of multiplication modules

We begin this section with the following interesting result that shows that as rings with identity, every nonzero multiplication module M over a Noetherian ring R has a maximal submodule.

Proposition 4.1 *Let R be a Noetherian ring. Then every nonzero multiplication R-module M has a maximal submodule.*

Proof. Without loss of generality we can assume that M is a faithful R-module. First we show that M has a prime submodule, for if not, then by Lemma 2.5, $\mathcal{P}M = M$ for all prime ideals of R. It follows that $IM = M$ for all nonzero ideals I of R and so R must be a domain and (0) is the only proper submodule of M. Thus M is a simple R-module and so (0) is prime submodule of M, a contradiction. Therefore, M has a prime submodule, say $\mathcal{P}M$. Now by [7, Corollary 3.5], $M/\mathcal{P}M$ is a finitely generated $R/\mathcal{P}$-module. It follows that $M/\mathcal{P}M$ has a maximal $R/\mathcal{P}$-submodule, say $K/\mathcal{P}M$. It is clear that K is a maximal R-submodule of M. □

Corollary 4.2 *Let M be a multiplication module over a Noetherian ring R. Then*

(i) *$k.dim(M)=-1$ if and only if $cl.k.dim(M)=-1$ if and only if $M=(0)$.*

(ii) *$cl.k.dim(M)=0$ if and only if $M\neq(0)$ and every prime submodule of M is maximal.*

(iii) *$k.dim(M)=0$ if and only if M is an Artinian module with $cl.k.dim(M)=0$, if and only if M is an Artinian cyclic module.*

Proof. (i) and (ii) are clear by Proposition 4.1 and the coincidence of Krull and classical Krull dimensions. For (iii), we note that by [7, Corollary 2.9], Artinian multiplication modules are cyclic. $\mathrm{k.dim}(M)=0$ if and only if M is an Artinian module with $\mathrm{cl.k.dim}(M)=0$, if and only if, M is an Artinian cyclic module. □

The following example shows that a multiplication module M (even if $M=R$) with $\mathrm{cl.k.dim}(M)=0$ need not have Krull dimension.

Example 4.3 Put

$$R:=\frac{\mathbb{Z}_2[\{X_i\mid i\in\mathbb{N}\}]}{<X_iX_j\mid i,j\in\mathbb{N}>}$$

where $<X_iX_j\mid i,j\in\mathbb{N}>$ is the ideal of $\mathbb{Z}_2[\{X_i\mid i\in\mathbb{N}\}]$ generated by the set $\{X_iX_j\mid i,j\in\mathbb{N}\}$. Then the ring R is a non-Noetherian local ring with only prime (maximal) ideal $\mathcal{M}=<x_i\mid i\in\mathbb{N}>$, where $x_i=X_i+<X_iX_j\mid i,j\in\mathbb{N}>$. Thus $\mathrm{cl.k.dim}(R)=0$, but R has no krull dimension (since $\mathcal{M}\cong\bigoplus_{i\in\mathbb{N}}Rx_i$ where each $Rx_i=\{0,x_i\}$ is a simple R-module, and so $\mathcal{M}$ has no Krull dimension).

Proposition 4.4 *Let R be a commutative ring with Krull dimension. Then R has classical Krull dimension and $cl.k.dim(R)=k.dim(R)$.*

Proof. By [9, Theorem 8.12]. □

Proposition 4.5 *Let M be a be nonzero finitely generated multiplication R-module with $Ann_R(M)=I$. Then $cl.k.dim(R/I)$ exists if and only if $cl.k.dim(M)$ exists. Moreover, if one of them exists, then*

$$cl.k.dim(R/I)=cl.k.dim(M).$$

Proof. By [5, Corollary 2.5] is clear. □

Now, we are in position to prove the main theorem of this paper.

Theorem 4.6 *Let M be a multiplication R-module with Krull dimension. Then M has classical Krull dimension and*

$$k.dim(M)=cl.k.dim(M)=k.dim(M/\mathcal{P}M)=cl.k.dim(M/\mathcal{P}M)$$

for some prime submodule $\mathcal{P}M$ of M.

Proof. First we show that cl.k.dim(M) exists and also cl.k.dim$(M) \leq$ k.dim(M). So suppose that k.dim$(M) = \alpha$ for some ordinal $\alpha \geq -1$. We prove the result by induction on α. By Corollary 4.2, the result is clear if $\alpha = -1$ or 0, so suppose that $\alpha \geq 1$. Let $\mathcal{P}M$ be any prime submodule of M. Then $M/\mathcal{P}M$ is a prime module with Krull dimension. Let $\mathcal{Q}M$ be any prime submodule of M such that $\mathcal{P}M \subsetneqq \mathcal{Q}M$. Since $M/\mathcal{Q}M \cong \frac{M/\mathcal{P}M}{\mathcal{Q}M/\mathcal{P}M}$, by Corollary 2.7,

$$\text{k.dim}(M/\mathcal{Q}M) = \text{k.dim}(\frac{M/\mathcal{P}M}{\mathcal{Q}M/\mathcal{P}M}) < \text{k.dim}(M/\mathcal{P}M) \leq \text{k.dim}(M) = \alpha.$$

Now by hypotheses cl.k.dim$(M/\mathcal{Q}M) \leq$ k.dim$(M/\mathcal{Q}M)$, and so $\mathcal{Q}M \in \text{Spec}_\beta(M)$ for some ordinal $0 \leq \beta < \alpha$, by Lemma 3.1. Thus $\mathcal{P}M \in \text{Spec}_\gamma(M)$ for some ordinal $\beta < \gamma \leq \alpha$. It follows that $\text{Spec}_\gamma(M) = \text{Spec}(M)$ and hence M has classical Krull dimension and cl.k.dim$(M) \leq \gamma \leq \alpha =$ k.dim(M). Now by Theorem 2.15, there exists a prime submodule $\mathcal{P}_0M$ of M with k.dim$(M) =$ k.dim$(M/\mathcal{P}_0M)$. If cl.k.dim$(M/\mathcal{P}_0M) =$ k.dim$(M/\mathcal{P}_0M)$, then, by the first part of the proof

$$\text{k.dim}(M/\mathcal{P}_0M)=\text{cl.k.dim}(M/\mathcal{P}_0M)\leq \text{cl.k.dim}(M)\leq \text{k.dim}(M)=\text{k.dim}(M/\mathcal{P}_0M)$$

and so k.dim$(M) =$ cl.k.dim$(M) =$ k.dim$(M/\mathcal{P}_0M) =$ cl.k.dim$(M/\mathcal{P}_0M)$. Therefore, without loss of generality, we may assume that M is a prime module. We claim that for any ordinal $\beta < \alpha$ there is a nonzero prime submodule $\mathcal{Q}M$ of M such that $\beta \leq$ k.dim$(M/\mathcal{Q}M) < \alpha$. By Theorem 2.6, there is certainly an essential submodule E of M such that $\beta \leq$ k.dim$(M/E) < \alpha$. Therefore, by choosing a nonzero prime submodule $\mathcal{Q}M$ of M satisfying k.dim$(M/E) =$ k.dim$(M/\mathcal{Q}M)$, we are done by Theorem 2.15. By induction on α, assume that the result is true for all ordinals strictly less than α. If cl.k.dim$(M) \neq \alpha$ then cl.k.dim$(M) < \alpha$, so there is a nonzero prime submodule $\mathcal{Q}M$ of M such that cl.k.dim$(M) \leq$ k.dim$(M/\mathcal{Q}M) < \alpha$. By induction hypothesis we have k.dim$(M/\mathcal{Q}M) =$ cl.k.dim$(M/\mathcal{Q}M)$ and thus

$$\text{cl.k.dim}(M) \leq \text{k.dim}(M/\mathcal{Q}M) = \text{cl.k.dim}(M/\mathcal{Q}M).$$

Since M is a prime module, by Proposition 3.2, cl.k.dim$(M/\mathcal{Q}M) <$ cl.k.dim(M) and so cl.k.dim$(M) <$ cl.k.dim(M), a contradiction. Therefore, cl.k.dim$(M) = \alpha =$ k.dim(M). □

Finally, we conclude this article with the following corollary.

Corollary 4.7 *Let M be a multiplication R-module with Krull dimension. Then M is finitely generated and has acc on prime submodules.*

Proof. By Corollary 2.14, M is finitely generated and by Theorem 4.6 and [3, Proposition 4.10], M has *acc* on prime submodules. □

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