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p-Rank and p-groups in algebraic groups

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Abstract

A few remarks on the measures of the p-rank of a group equipped with a dimension, including the refutation of a result of Burdges and Cherlin.

Key Words: Algebraic groups, groups of finite Morley rank, torsion, p-groups

Groups of finite Morley rank are abstract analogues of algebraic groups; like them they bear a dimension enabling various genericity arguments. They do not come from geometry but from logic; yet the Cherlin-Zilber conjecture and related work suggest tight relationships between both aspects. My reader may thus view what follows as naive properties of algebraic groups obtained by elementary means; the word "definable" stands for "constructible". Should he desire more on groups of finite Morley rank, [1] would provide references.

I wish to thank Éric Jaligot for his many suggestions.

A group of finite Morley rank is *connected* if it has no proper definable subgroup of finite index. One lets H° be the *connected component* of a definable subgroup H, i.e. its smallest definable connected subgroup of finite index. This extends to arbitrary H: H is included in a smallest definable subgroup $d(H)$, one takes the connected component $d(H)^\circ$ and sets $H^\circ = H \cap d(H)^\circ$.

Throughout, p will be a prime (possibly 2). A *p-torus* T is a finite power of the Prüfer quasi-cyclic p-group $\mathbb{Z}_{p^\infty}$; $T \simeq \mathbb{Z}_{p^\infty}^d$ is injective among abelian groups. For $H \leq T$, H° consistently denotes the maximal subtorus of H.

A group of finite Morley rank is $U_p^\perp$ if it has no infinite elementary abelian p-subgroup. $U_p^\perp$ groups conjugate their *Sylow p-subgroups* [3, Theorem 4], i.e. their maximal (non-necessarily definable) p-subgroups; these are finite extensions of p-tori. Hence, for S a Sylow p-subgroup of a $U_p^\perp$ group, S° is a p-torus.

Given a $U_p^\perp$ group, 3 measures of its Sylow p-subgroups are available. One can consider the *Prüfer p-rank* $\Pr_p(G)$, which is the number of $\mathbb{Z}_{p^\infty}$ factors in a Sylow p-subgroup. One can also estimate the *normal p-rank* $n_p(G)$, which is the maximal p-rank of an elementary abelian p-group normal in a Sylow p-subgroup. Or one can simply compute the *p-rank* $m_p(G)$, which is the maximal p-rank of an elementary abelian p-subgroup. All 3 numbers are well defined by conjugacy of the Sylow p-subgroups, and $m_p(G) \geq n_p(G) \geq \Pr_p(G)$.

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1. The n-rank

Lemma 1 *If G is a connected, $U_p^\perp$ group, then $n_p(G) = \mathrm{Pr}_p(G)$.*

Proof. Let S be a Sylow p-subgroup of G, $V \lhd S$ an elementary abelian normal subgroup, and $v \in V$. As $V \lhd S$, $v^{S^\circ} \subseteq V$ which is finite; by connectedness, S° centralizes v. So $v \in C_S(S^\circ) = S^\circ$ by [3, Corollary 3.1], and $V \leq S^\circ$. □

2. Not quite a digression

For a p-torus $T \simeq \mathbb{Z}_{p^\infty}^d$, $\Omega_{p^n}(T)$ denotes the set of elements of order at most p^n.

Fact 1 *Let φ be an automorphism* of finite order *of a p-torus $T \simeq \mathbb{Z}_{p^\infty}^d$.*

1. *Suppose $\Omega_{p^2}(T) \leq C_T(\varphi)$. Then $\varphi = \mathrm{Id}$.*
2. *Suppose $\Omega_p(T) \leq C_T(\varphi)$. If $p = 2$, then $\varphi^2 = \mathrm{Id}$. If $p \neq 2$, then $\varphi = \mathrm{Id}$.*

Proof. This must be classical but I know no reference.

1. Up to taking a power of φ, we may assume that φ has prime order q. Let $x \notin C_T(\varphi)$ have minimal order. Then $\varphi(x^p) = x^p$ so there is $y \in \Omega_p(T) \setminus \{1\}$ with $\varphi(x) = xy$. By assumption $y \in C_T(\varphi)$, so $x = \varphi^q(x) = xy^q$ and $q = p$. Let $\hat{x}$ and $\hat{y}$ be such that $\hat{x}^p = x$ and $\varphi(\hat{x}) = \hat{x}\hat{y}$. Then $\hat{y}^p = y$ so $\hat{y} \in \Omega_{p^2}(T) \leq C_T(\varphi)$ and $\hat{x} = \varphi^p(\hat{x}) = \hat{x}\hat{y}^p = \hat{x}y$, a contradiction.

2. Since φ centralizes $\Omega_p(T)$, for $x \in \Omega_{p^2}(T)$ there is $y \in \Omega_p(T)$ with $\varphi(x) = xy$; hence $\varphi^p(x) = xy^p = x$ and $\Omega_{p^2}(T) \leq C_T(\varphi^p)$. So $\varphi^p = \mathrm{Id}$; we may assume $p \neq 2$. Represent $\varphi_{|\Omega_{p^3}(T)}$ by a matrix $M \in \mathrm{GL}_d(\mathbb{Z}/p^3\mathbb{Z})$. As $\Omega_p(T) \leq C_T(\varphi)$, the reduction of M modulo p is the identity: there is a matrix N with $M = \mathrm{Id} + pN$. Since $\varphi^p = \mathrm{Id}$,

$$0 \equiv \sum_{\ell=1}^{p} \binom{p}{\ell} p^\ell N^\ell \equiv p^2 N + \frac{p(p-1)}{2} p^2 N^2 \quad [p^3]\,.$$

   Since $p \neq 2$, p divides $\frac{p(p-1)}{2}$, so $p^2 N \equiv 0[p^3]$ and $N \equiv 0[p]$. Hence the reduction of $M = \mathrm{Id} + pN$ modulo p^2 is the identity: M centralizes $\Omega_{p^2}(T)$, and φ is trivial. □

Consequence *If $p \neq 2$, the restriction map $\rho : \mathrm{Aut}(T) \to \mathrm{Aut}(\Omega_p(T))$ kills no element of finite order. In particular if W is a finite subgroup of $\mathrm{Aut}(T)$ then W embeds into $\mathrm{Aut}(\Omega_p(T)) \simeq \mathrm{GL}_d(\mathbb{F}_p)$. If $p = 2$ then $\ker \rho_{|W} \hookrightarrow (\mathbb{Z}/2\mathbb{Z})^d$.*

Proof. The only non-immediate claim is about the rank of $K = \ker \rho_{|W}$ when $p = 2$. Observe that K has exponent 2, so it is abelian. We go in a direction that will prove fruitful. Taking automorphism groups changes inductive limits to projective limits, so $\mathrm{Aut}(T) \simeq \varprojlim \mathrm{Aut}((\mathbb{Z}/p^n\mathbb{Z})^d) = \varprojlim \mathrm{GL}_d(\mathbb{Z}/p^n\mathbb{Z}) = \mathrm{GL}_d(\mathbb{Z}_p)$. Hence K embeds into $\mathrm{GL}_d(\mathbb{Z}_2)$. Now elements of K are simultaneously diagonalizable over $\bar{\mathbb{Q}}_2$ with eigenvalues ± 1, so

K embeds into $\{\pm 1\}^d$. □

We could use a similar method to get a lazy bound on $\operatorname{rk} W$ for $W \leq \operatorname{Aut}(T)$ an elementary abelian p-group; observe how we are naturally moving towards the p-adic representation. Anyway, embedding into $\mathrm{GL}_d(\mathbb{F}_p)$, i.e. restricting to $\Omega_p(T)$, was too clumsy in the first place. For instance, any element of $\mathrm{GL}_d(\mathbb{F}_p)$ comes from an element of $\mathrm{GL}_d(\mathbb{Z}_p)$, but not necessarily from one of finite order. (The reader may check that $\mathrm{GL}_2(\mathbb{Z}_5)$ has no element of order 5.) Representation-theoretically speaking, embedding into $\mathrm{GL}_d(\mathbb{Z}_p)$ is more appropriate, and this is what we shall now do.

3. Bounding the m-rank

Let φ be an automorphism of order p of a p-torus $T \simeq \mathbb{Z}_{p^\infty}^d$.

Fact 2 (Maschke's Theorem) *Let $T_1 \leq T$ be a φ-invariant subtorus. Then there is a φ-invariant subtorus $T_2 \leq T$ such that $T = T_1 + T_2$ and $T_1 \cap T_2 \leq \Omega_p(T_1)$.*

Proof. There is a subtorus $T_0 \leq T$ with $T = T_1 \oplus T_0$. Let π be the projection on T_1 along T_0 and $\hat{\pi} = \sum_{i=0}^{p-1} \varphi^i \pi \varphi^{-i}$. Then $\hat{\pi}$ is φ-covariant, $\operatorname{im} \hat{\pi} = T_1$, and $\hat{\pi}(t_1) = pt_1$ for $t_1 \in T_1$. Take T_2 to be the maximal subtorus of $\ker \hat{\pi}$. □

Fact 3 (φ, T as above) *If $C_T^\circ(\varphi) = 1$ then $p - 1 | d$ and* $\mathrm{Id} + \varphi + \cdots + \varphi^{p-1} = 0$.

Proof. (This again must be well known.) We may assume $p \neq 2$. Let $\tau \leq T$ be isomorphic to $\mathbb{Z}_{p^\infty}$, and set $\Theta = \sum_{i=0}^{p-1} \varphi^i(\tau)$; Θ is φ-invariant and $\Pr_p(\Theta) \leq p$. So by Maschke's Theorem, we may assume $d \leq p$. As in the proof of the Consequence above, let us view φ as an element of order p of $\mathrm{GL}_d(\mathbb{Z}_p) \leq \mathrm{GL}_d(\mathbb{Q}_p)$. By assumption, 1 is not an eigenvalue.

The minimal polynomial μ of φ over $\mathbb{Q}_p$ divides $X^p - 1 = (X-1)(1 + X + \cdots + X^{p-1})$, so it divides $1 + X + \cdots + X^{p-1}$. The latter is irreducible over $\mathbb{Z}_p$ by Eisenstein's criterion, so $\mu = 1 + X + \cdots + X^{p-1}$. But μ divides the characteristic polynomial which has degree d. So $p - 1 \leq d \leq p$. Over $\bar{\mathbb{Q}}_p$, φ has $p-1$ eigenvalues, which sum to -1. So if $d = p$, one of them, say j, occurs twice: hence $1 + \operatorname{Tr} \varphi = j \in \mathbb{Q}_p$, against $p \neq 2$. So $d = p - 1$. □

Lemma 2 *For $W \leq \operatorname{Aut} \mathbb{Z}_{p^\infty}^d$ an elementary abelian p-group,* $\operatorname{rk} W \leq \frac{1}{p-1} d$.

Proof. $E = \mathbb{Q}_p^d$ is a sum of W-irreducible subspaces $\oplus_{i \in I} E_i \bigoplus \oplus_{j \in J} F_j$ with E_i's the W-trivial lines. Since W is abelian, it acts W-covariantly. Let $\rho_j : W \to \operatorname{Aut}_W(F_j)$ be the restriction map, with (non-trivial) image W_j and kernel K_j. Each $\operatorname{End}_W F_j$ is a skew-field by Schur's Lemma, so the abelian group W_j of exponent p has order p. As $C_W(E) = 1$, $W \hookrightarrow \prod_{j \in J} W/K_j$, and $\operatorname{rk} W \leq \# J$. By Fact 3, $\dim F_j \geq p - 1$, whence $\# J \leq \frac{d}{p-1}$. □

Corollary 1 *Let G be a connected, $U_p^\perp$ group. Then $m_p(G) \leq \frac{p}{p-1} \Pr_p(G)$.*

Proof. For $V \leq S$ an elementary abelian subgroup of a Sylow p-subgroup S, write $V = (V \cap S^\circ) \oplus W$. By [3, Corollary 3.1], $C_S(S^\circ) = S^\circ$; use Lemma 2. □

4. Maximal abelian p-subgroups

Thesis [2, Theorem 1.2] *Let G be a connected, $U_p^\perp$ group with $m_p(G) \geq 3$. Then any maximal elementary abelian p-subgroup $V < G$ has p-rank at least 3.*

The flaw in [2] lies at the bottom of page 172. On the very last line, "commutation with v" need not in general be "a map from $\Omega_1(T)/A$ to A". Observe that in [2] Theorem 6.4 relies on Corollary 4.2, which relies on Theorem 1.2.

Counter-Example *In* $\mathrm{PSL}_5(\mathbb{C})$ *let Θ be the usual torus and σ be the Weyl element naturally associated with the 5-cycle* (12345)*. Let $\theta \in C_\Theta(\sigma) \setminus \{1\}$. Then $\langle \theta, \sigma \rangle$ does not extend to an elementary abelian 5-group of rank 3.*

Proof. The actual computations will take place in $\mathrm{SL}_5(\mathbb{C})$. Let $\lambda = e^{\frac{2i\pi}{5}} \in \mathbb{C}$; then $Z(\mathrm{SL}_5(\mathbb{C})) = \{\lambda^k \operatorname{Id}\}$. The matrix $s = (\delta_{j,i+1}) \in \mathrm{SL}_5(\mathbb{C})$ (equality modulo 5) reduces modulo $Z(\mathrm{SL}_5(\mathbb{C}))$ to $\sigma \in \mathrm{PSL}_5(\mathbb{C})$; conjugation by s rotates coefficients of a matrix $(m_{i,j}) \in \mathrm{SL}_5(\mathbb{C})$ along the 5 (complete) diagonals. So given $\theta \in \Theta \leq \mathrm{PSL}_5(\mathbb{C})$ and a diagonal matrix $t \in \mathrm{SL}_5(\mathbb{C})$ representing it, one sees that $[\sigma, \theta] = 1$ iff $t_{i,i} = \lambda^{k+\ell i}$ for some integers k and ℓ; thus $C_\Theta(\sigma)$ has order 5. Fix $\theta \in C_\Theta(\sigma) \setminus \{1\}$. Conjugation by t on $(m_{i,j})$ multiplies $m_{i,j}$ by $\lambda^{\ell(j-i)}$. So $C(\theta) = \Theta \rtimes \langle \sigma \rangle$, and $\langle \theta, \sigma \rangle$ is maximal. □

The following merely serves the purpose of exposing an important method.

Observation *Let G be a connected, $U_p^\perp$ group, and $S \leq G$ a Sylow p-subgroup. Then S is connected iff abelian iff nilpotent.*

Proof. Only one claim is non-trivial; we prove it by induction on the Morley rank (read: dimension) of G. Suppose S nilpotent; let $\omega \in S$. Then by nilpotence, $\tau = C^\circ_{S^\circ}(\omega) \neq 1$. By [3, Corollary 3.1], ω lies in any maximal p-torus of $C^\circ(\omega)$, so $\omega \in C^\circ(\tau)$. Hence $\langle S^\circ, \omega \rangle \leq C^\circ(\tau)$. If $C^\circ(\tau) < G$ we are done by induction. Otherwise τ is central and we can factor by $Z^\circ(G)$, pursuing by induction. □

I shall now bring my reader some comfort.

Lemma 3 *The thesis of* [2, Theorem 1.2] *holds for $p = 2$, and so does* [2, Corollary 6.5].

Proof. Suppose $m_2(G) \geq 3$; clearly $\mathrm{Pr}_2(G) \geq 2$. Let i, j be 2 commuting involutions; by torality [3, Theorem 3] there is a Sylow 2-subgroup S with $i \in S^\circ$ and $j \in S$.

Suppose $\mathrm{Pr}_2(G) \geq 3$. If $j \in S^\circ$ we are done. If not, consider the map $\varphi(k) = [j, k] : \Omega_2(S^\circ) \to \Omega_2(S^\circ)$. Then $\operatorname{im} \varphi \leq \ker \varphi$ and $\operatorname{rk} \operatorname{im} \varphi + \operatorname{rk} \ker \varphi \geq 3$, so $\operatorname{rk} \ker \varphi \geq 2$ and we are done. From now on, suppose $\mathrm{Pr}_2(G) = 2$ (so $m_2(G) \leq 4$) and let $V = \Omega_2(S^\circ)$.

Assume first that $j \in S \backslash S^\circ$. If j inverts S° then $\langle i, j \rangle \leq \langle V, j \rangle$: we are done. Otherwise $\tau = C^\circ_{S^\circ}(j) \neq 1$. If $i \notin \tau$ then $\langle i, j \rangle \leq \langle i, \Omega_2(\tau), j \rangle$: we are done. So assume $i \in \tau \leq C^\circ(j)$. By torality [3, Theorem 3 and Corollary 3.1], i lies in a 2-torus of $C^\circ(j)$ and j lies in *any* 2-torus of $C^\circ(j)$, so i and j are cotoral.

So assume that $j \in S^\circ$, that is $V = \langle i, j \rangle$. By assumption there is an elementary abelian 2-subgroup of rank 3: $A = \langle r, s, t \rangle \leq S$; clearly $A \cap S^\circ \neq 1$, say $r \in V$. If s or t is in V then $\langle i, j \rangle = V \leq A$: we are done. Suppose that s and t (hence st as well) lie in $S \setminus S^\circ$. Since $|\operatorname{Aut}(V)| = 6$, one of s, t, st must centralize $V = \langle i, j \rangle$: we are done again. □

Here is a final word on counter-examples.

Lemma 4 *Let G be a counter-example to* [2, Theorem 1.2]*. Then* $\Pr_p(G) = p - 1$*. In particular,* [2, Theorem 1.2] *also holds for* $p = 3$*.*

Proof. By Lemma 3, $p \geq 3$. As $m_p(G) \geq 3$, one sees with Corollary 1 that $\Pr_p(G) \geq 2$. Equality can hold only for $p = 3$; as there is an elementary 3-group of rank 3, there is an automorphism of order 3 fixing $\Omega_3(\mathbb{Z}_{3^\infty}^2)$, against Fact 1: equality cannot hold.

Hence $\Pr_p(G) \geq 3$. Let $V = \langle \alpha, \omega \rangle$ be a maximal abelian p-group and $S \geq V$ a Sylow p-subgroup. By torality we may assume $\alpha \in S^\circ$, so $\omega \in S \setminus S^\circ$. If $C^\circ_{S^\circ}(\omega) \neq 1$ then by maximality, $\alpha \in C^\circ_{S^\circ}(\omega) \leq C^\circ(\omega)$, and as in the proof of Lemma 3, α and ω are cotoral, a contradiction. Hence $C^\circ_{S^\circ}(\omega) = 1$. Let $\varphi \in \operatorname{End} \Omega_p(S^\circ)$ map x to $[x, \omega]$; writing ω as an automorphism, $\varphi(x) = \omega(x) - x$ and $\varphi^n(x) = \sum_{i=0}^n (-1)^i \binom{n}{i} \omega^i(x)$. As $(-1)^i \binom{p-1}{i} \equiv 1\ [p]$, $\varphi^{p-1} = \operatorname{Id} + \omega + \cdots + \omega^{p-1}$. But $C^\circ_{S^\circ}(\omega) = 1$, so Fact 3 applied to ω implies $\varphi^{p-1} = 0$. Since $\ker \varphi = C_{\Omega_p(S^\circ)}(\omega) = \langle \alpha \rangle$, one has $\operatorname{rk} \Omega_p(S^\circ) \leq p - 1$. □

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