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A note on unmixed ideals of Veronese bi-type

Monica LA BARBIERA*
University of Messina, Department of Mathematics Viale Ferdinando Stagno d'Alcontres, 31, 98166 Messina, Italy

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Abstract: We classify the unmixed ideals of Veronese bi-type and in some cases we give a description of their associated prime ideals.

Key words: Unmixed ideals. Veronese bi-type ideals

1. Introduction

Let $R = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ be a polynomial ring in two sets of variables over a field K. In recent papers, monomial ideals of R are introduced and their connection to bipartite complete graphs is studied ([4], [6]). In this paper we study a class of monomial ideals of R, so-called Veronese bi-type ideals. They are an extension of the ideals of Veronese type ([5]) in a polynomial ring in two sets of variables. More precisely, the ideals of Veronese bi-type are monomial ideals of R generated in the same degree: $L_{q,s} = \sum_{k+r=q} I_{k,s}J_{r,s}$, with $k, r \geq 1$, where $I_{k,s}$ is the Veronese-type ideal generated on degree k by the set $\{X_1^{a_{i_1}} \cdots X_n^{a_{i_n}} \mid \sum_{j=1}^{n} a_{i_j} = k, \ \ 0 \leq a_{i_j} \leq s, \ \ s \in \{1, \dots, k\}\}$ and $J_{r,s}$ is the Veronese-type ideal generated on degree r by the set $\{Y_1^{b_{i_1}} \cdots Y_m^{b_{i_m}} \mid \sum_{j=1}^{m} b_{i_j} = r, \ \ 0 \leq b_{i_j} \leq s, \ \ s \in \{1, \dots, r\}\}$ ([2], [3]). For $s = 2$ the Veronese bi-type ideals are the ideals associated to bipartite graphs with loops ([2]).

In this paper some properties of these class of monomial ideals are discussed. In particular, our aim is to classify the unmixed Veronese bi-type ideals.

Establishing whenever an ideal is unmixed in general is a difficult problem because it is necessary to know all its associated prime ideals. In [8] equidimensional and unmixed ideals of Veronese type are characterized. Now we are able to classify the unmixed Veronese bi-type ideals and in some cases we can give a description of the associated prime ideals.

This paper is organized as follows. In Section 1, unmixed ideals of Veronese bi-type are classified and the generalized ideals associated to the walks of special bipartite graphs, described by the Veronese by-type ideals $L_{q,2} = \sum_{k+r=q} I_{k,2}J_{r,2}$, are considered. In Section 2, the toric ideal $I(L_{q,s})$ of the monomial subring $K[L_{q,s}] \subset R$ is studied. Let $L_{q,s} = (f_1, \dots, f_p)$ and $K[L_{q,s}]$ be the K-algebra spanned by $f_1, \dots, f_p$. There is a graded epimorphism of K-algebras: $\varphi : S = K[T_1, \dots, T_p] \rightarrow K[L_{q,s}]$ induced by $\varphi(T_i) = f_i$, where S is a polynomial ring graded by $\deg(T_i) = \deg(f_i)$. Let $I(L_{q,s})$ be the toric ideal of $K[L_{q,s}]$, that is the kernel of φ.

*Correspondence: monicalb@dipmat.unime.it
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We are able to prove that $I(L_{q,s})$ has a quadratic Groebner basis and as a consequence the K-algebra $K[L_{q,s}]$ is Koszul. In order to formulate these results we have to recall the notion of sortability ([5]), and we apply it to the monomial ideals $L_{q,s}$.

2. Unmixed ideals of Veronese bi-type

Let $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$ be the polynomial ring over a field K in two sets of variables with each $\deg X_i = 1$, $\deg Y_j = 1$, for all $i = 1, \ldots, n$, $j = 1, \ldots, m$.

We define the *ideals of Veronese bi-type* of degree q as the monomial ideals of R

$$L_{q,s} = \sum_{r+k=q} I_{k,s} J_{r,s}, \quad r, k \geq 1,$$

where $I_{k,s}$ is the ideal of Veronese-type of degree k in the variables $X_1, \ldots, X_n$ and $J_{r,s}$ is the ideal of Veronese-type of degree r in the variables $Y_1, \ldots, Y_m$.

$L_{q,s}$ is not trivial for $2 \leq q \leq s(n+m) - 1$, $s \leq q$.

Remark 2.1 *In general, $I_{k,s} \subseteq I_k$, where I_k is the Veronese ideal of degree k generated by all the monomials in the variables $X_1, \ldots, X_n$ of degree k ([6]).*

One has $I_{k,s} = I_k$ for any $k \leq s$. If $s = 1$, $I_{k,1}$ is the square-free Veronese ideal of degree k generated by all the square-free monomials in the variables $X_1, \ldots, X_n$ of degree k. Similar considerations hold for $J_{r,s} \subset K[Y_1, \ldots, Y_m]$.

Example 2.2 *Let $R = K[X_1, X_2; Y_1, Y_2]$ be a polynomial ring.*
1) $L_{2,2} = I_{1,2}J_{1,2} = I_1 J_1 = (X_1Y_1, X_1Y_2, X_2Y_1, X_2Y_2)$;
2) $L_{4,2} = I_{3,2}J_{1,2} + I_{1,2}J_{3,2} + I_{2,2}J_{2,2} = I_{3,2}J_1 + I_1 J_{3,2} + I_2 J_2 = (X_1^2X_2Y_1, X_1^2X_2Y_2, X_1X_2^2Y_1, X_1X_2^2Y_2, X_1Y_1^2Y_2, X_2Y_1^2Y_2, X_1Y_1Y_2^2, X_2Y_1Y_2^2, X_1^2Y_1^2, X_1^2Y_1Y_2, X_1^2Y_2^2, X_2^2Y_1^2, X_2^2Y_2^2, X_2^2Y_1Y_2, X_1X_2Y_1^2, X_1X_2Y_2^2, X_1X_2Y_1Y_2)$.

In this section we classify the unmixed Veronese bi-type ideals. First, we recall some preliminary notions.

Definition 2.3 *Let $G(L_{q,s})$ be the unique minimal set of monic monomial generators of $L_{q,s}$. A vertex cover of $L_{q,s}$ is a subset W of $\{X_1, \ldots, X_n; Y_1, \ldots, Y_m\}$ such that each $u \in G(L_{q,s})$ is divided by some variables of W. Such a vertex cover W is called minimal if no proper subset of W is a vertex cover.*

Denote by $h(L_{q,s})$ the minimal cardinality of the vertex covers of $L_{q,s}$.

Definition 2.4 *A monomial ideal is said to be unmixed if all its minimal vertex covers have the same cardinality.*

Remark 2.5 We recall the one-to-one correspondence between the minimal vertex covers of an ideal and its minimal primes. Hence $\wp$ is a minimal prime ideal of $L_{q,s}$ if and only if $\wp = (\mathcal{A})$ for some minimal vertex cover $\mathcal{A}$ of $L_{q,s}$.

Now we are able to classify the unmixed Veronese bi-type ideals and in some cases we can give a description of the associated prime ideals.

Proposition 2.6 *Let* $R = K[X_1,\ldots,X_n;Y_1,\ldots,Y_m]$. *Let* $2 \le q < s(n+m) - i$ *for* $i = 1,\ldots,s-1$. $L_{q,s}$ *is unmixed if and only if* $n = m$.

Proof By the structure of $G(L_{q,s})$ the minimal vertex covers of $L_{q,s}$ are $W_1 = \{X_1,\ldots,X_n\}$, $W_2 = \{Y_1,\ldots,Y_m\}$. The minimal cardinality of the vertex covers of $L_{q,s}$ is $h(L_{q,s}) = \min\{n,m\}$. Hence all the minimal vertex covers have the same cardinality if and only if $n = m$. □

Example 2.7 $R = K[X_1,X_2;Y_1,Y_2]$
$L_{3,2} = (X_1^2Y_1, X_1^2Y_2, X_1X_2Y_1, X_1X_2Y_2, X_2^2Y_1, X_2^2Y_2, X_1Y_1^2, X_1Y_1Y_2, X_1Y_2^2, X_2Y_1^2, X_2Y_1Y_2, X_2Y_2^2)$.
The minimal vertex covers are: $W_1 = \{X_1,X_2\}$; $W_2 = \{Y_1,Y_2\}$.
$h(L_{3,2}) = |W_1| = |W_2| = 2 \Rightarrow L_{3,2}$ *is unmixed.*

Proposition 2.8 *Let* $R = K[X_1,\ldots,X_n;Y_1,\ldots,Y_m]$. *If* $q = s(n+m) - i$ *for* $i = 1,\ldots,s-1$, *then* $L_{q,s}$ *is unmixed.*

Proof One has $W_i = \{X_i\}$, $i = 1,\ldots,n$, $W_j = \{Y_j\}$, $j = 1,\ldots,m$, are the minimal vertex covers of $L_{q,s}$ by construction. □

Example 2.9 $R = K[X_1,X_2;Y_1,Y_2]$,
$L_{11,3} = (X_1^3X_2^3Y_1^3Y_2^2, X_1^3X_2^3Y_1^2Y_2^3, X_1^3X_2^2Y_1^3Y_2^3, X_1^2X_2^3Y_1^3Y_2^3)$.
The minimal vertex covers are:
$W_1 = \{X_1\}$; $W_2 = \{X_2\}$; $W_3 = \{Y_1\}$; $W_4 = \{Y_2\}$.
$h(L_{11,3}) = |W_i| = 1$ *for all* $i = 1,2,3,4 \Rightarrow L_{11,3}$ *is unmixed.*

Let $\mathcal{A} \subseteq \{1,2,\ldots,n+m\}$, where $n+m$ is the number of the variables of the polynomial ring R. For a subset $\mathcal{A}$ we denote by $\mathcal{P}_{\mathcal{A}}$ the prime ideal of R generated by the variables whose index is in $\mathcal{A}$.

Theorem 2.10 *Let* $R = K[X_1,\ldots,X_n;Y_1,\ldots,Y_m]$, $L_{q,s} \subset R$.

$$\mathcal{P}_{\mathcal{A}} \in Ass_R(L_{q,s}) \Leftrightarrow |\mathcal{A}| \le i+1,$$

for $i = s(n+m) - q$, $i = 1,\ldots,s-1$.

Proof In the following replace the set of variables $\{X_1,\ldots,X_n\}$ with $\{z_1,\ldots,z_n\}$ and $\{Y_1,\ldots,Y_m\}$ with $\{z_{n+1},\ldots,z_{n+m}\}$.

Assume that $\mathcal{P}_{\mathcal{A}} \in Ass_R(L_{q,s})$. Then there exists a monomial $f \notin L_{q,s}$ such that $L_{q,s} : f = \mathcal{P}_{\mathcal{A}}$. Now we show that we can choose such a monomial f of degree $q-1$ such that $L_{q,s} : f = \mathcal{P}_{\mathcal{A}}$. Suppose that $f \notin L_{q,s}$, $L_{q,s} : f = \mathcal{P}_{\mathcal{A}}$, $\deg(f) \ge q$ and $f = z_1^{a_1}\cdots z_{n+m}^{a_{n+m}}$. Then there exists $j_0 \in \{1,2,\ldots,n+m\}$ such that $a_{j_0} > s$. Since $L_{q,s} : f = \mathcal{P}_{\mathcal{A}}$, we have $z_i f \in L_{q,s}$ for all $i \in \mathcal{A}$ and $z_i f \notin L_{q,s}$ for all $i \notin \mathcal{A}$. Moreover, for all $i \in \mathcal{A}$ there exists a monomial $u_i \in G(L_{q,s})$ such that $u_i | (z_i f)$. Being $f \notin L_{q,s}$, this fact means that, for all $i \in \mathcal{A}$, the variable z_i appears in u_i with exponent $a_i + 1$. Therefore $a_i < s$ for all $i \in \mathcal{A}$. It follows that $j_0 \notin \mathcal{A}$. Now we claim that: I) $\overline{f} = f/z_{j_0} \notin L_{q,s}$ and II) $L_{q,s} : \overline{f} = \mathcal{P}_{\mathcal{A}}$.
The first fact follows from that $f \notin L_{q,s}$ and $a_{j_0} - 1 \ge s$. For the second assertion we proceed as follows. $L_{q,s} : \overline{f} \subseteq L_{q,s} : f$ because $\overline{f}$ divides f. Then $L_{q,s} : \overline{f} \subseteq \mathcal{P}_{\mathcal{A}}$, being $\mathcal{P}_{\mathcal{A}} = L_{q,s} : f$. Moreover, since $a_{j_0} - 1 \ge s$

then u_i divides $z_i f/z_{j_0}$ for all $i \in \mathcal{A}$, then $z_i \in L_{q,s} : (f/z_{j_0})$ for all $i \in \mathcal{A}$. Hence $\mathcal{P}_{\mathcal{A}} \subseteq L_{q,s} : (f/x_{j_0})$. It follows the other inclusion $\mathcal{P}_{\mathcal{A}} \subseteq L_{q,s} : \overline{f}$. Hence $\mathcal{P}_{\mathcal{A}} = L_{q,s} : \overline{f}$. After a finite number of these reductions, we find $f \notin L_{q,s}$ of degree $q-1$ such that $\mathcal{P}_{\mathcal{A}} = L_{q,s} : f$. From this fact follows that $fz_i \in L_{q,s}$ for all $i \in \mathcal{A}$ and $fz_i \notin L_{q,s}$ for all $i \notin \mathcal{A}$. In particular $a_i + 1 \leq s$ for all $i \in \mathcal{A}$, and $a_i \leq s$ for all $i \notin \mathcal{A}$. Then $a_i = s$ for all $i \notin \mathcal{A}$. Therefore $f = \prod_{i\in\mathcal{A}} z_i^{a_i} \prod_{i\notin\mathcal{A}} z_i^s$ with $0 \leq a_i < s$ for all $i \in \mathcal{A}$. We have $\deg(\prod_{i\notin\mathcal{A}} z_i^s) = s(n+m-|\mathcal{A}|) = t$. Then we obtain: $s(n+m) \geq (\sum_{i\in\mathcal{A}} a_i + 1) + t = \sum_{i\in\mathcal{A}} a_i + |\mathcal{A}| + t = \sum_{i\in\mathcal{A}} a_i + t + |\mathcal{A}| = \deg(f) + |\mathcal{A}| = q - 1 + |\mathcal{A}|$.

Conversely, let $|\mathcal{A}| \leq i+1$, for $i = s(n+m) - q$, $i = 1, \dots, s-1$, that is $|\mathcal{A}| \leq s(n+m) - q + 1$. Moreover, in these hypotheses one has $s(n+m-|\mathcal{A}|) \leq q-1$. In fact, $s(n+m-|\mathcal{A}|) \leq s(n+m) - i - 1$; then $s|\mathcal{A}| \geq i+1$ that is true for $i = 1, \dots, s-1$. Being $q = s(n+m) - i$ for $i = 1, \dots, s-1$, then by the definition of $L_{q,s}$ it follows that for any monomial $u \in G(L_{q,s})$ there exists an integer $j \in \mathcal{A}$ such that z_j divides u. Therefore $L_{q,s} \subset \mathcal{P}_{\mathcal{A}}$. The condition $s(n+m) \geq q - 1 + |\mathcal{A}|$ implies that $(s-1)|\mathcal{A}| + s(n+m-|\mathcal{A}|) \geq q-1$, which together with $s(n+m-|\mathcal{A}|) \leq q-1$ shows that there exists an integer $c_i < s$, for all $i \in \mathcal{A}$ such that $c_i|\mathcal{A}| + s(n+m-|\mathcal{A}|) = q-1$. Then the monomial $f = \prod_{i\in\mathcal{A}} z_i^{c_i} \prod_{i\notin\mathcal{A}} z_i^s$ has degree $q-1$. Hence $f \notin L_{q,s}$ and as a consequence $\mathcal{P}_{\mathcal{A}} \subseteq L_{q,s} : f$. Now we prove that $\mathcal{P}_{\mathcal{A}} = L_{q,s} : f$. Assume that $\mathcal{P}_{\mathcal{A}}$ is a proper subset of $L_{q,s} : f$. Then there exists a monomial f', in the variables z_i with $i \notin \mathcal{A}$, of degree at least 1 such that $ff' \in L_{q,s}$. This means that there exists a monomial $u = z_1^{a_1} \cdots z_{n+m}^{a_{n+m}} \in G(L_{q,s})$ such that u divides ff'. Therefore $a_i \leq c_i$ for any $i \in \mathcal{A}$ because $f' \in K[z_i | i \notin \mathcal{A}]$. It follows that $q = \deg(u) = \sum_{i=1}^{n+m} a_i \leq \sum_{i\in\mathcal{A}} c_i + s(n+m-|\mathcal{A}|) = \deg(f) = q-1$, which is a contradiction. Hence $\mathcal{P}_{\mathcal{A}}$ is not a proper subset of $L_{q,s} : f$, but $\mathcal{P}_{\mathcal{A}} = L_{q,s} : f$. This equality means that $\mathcal{P}_{\mathcal{A}} \in Ass_R(L_{q,s})$. □

Example 2.11 $R = K[X_1, X_2; Y_1, Y_2]$,
$L_{15,4} = (X_1^4 X_2^4 Y_1^4 Y_2^3, X_1^4 X_2^4 Y_1^3 Y_2^4, X_1^4 X_2^3 Y_1^4 Y_2^4, X_1^3 X_2^4 Y_1^4 Y_2^4)$.
By Theorem 2.10 $Ass_R(L_{15,4}) = \{(X_1), (X_2), (Y_1), (Y_2), (X_1, X_2), (X_1, Y_1), (X_1, Y_2), (X_2, Y_1), (X_2, Y_2), (Y_1, Y_2)\}$.

As an application, we observe that the ideals of Veronese bi-type can be associated to graphs with loops. In fact for $s = 2$, the ideals $L_{q,s}$ are associated to the walks of length $q-1$ of the strong quasi-bipartite graphs with loops ([2]).

Definition 2.12 *A graph G with loops is a strong quasi-bipartite if all vertices of V_1 are joined to all vertices of V_2 and for each vertex of V there is a loop.*

Definition 2.13 *Let G be a strong quasi-bipartite graph on the vertex set $V = \{v_1, \dots, v_n\}$. A walk of length q in G is an alternating sequence $w = \{v_{i_0}, l_{i_1}, v_{i_1}, l_{i_2}, \dots, v_{i_{q-1}}, l_{i_q}, v_{i_q}\}$, where v_{i_j} is a vertex of G and $l_{i_j} = \{v_{i_{j-1}}, v_{i_j}\}$ is the edge joining $v_{i_{j-1}}$ and v_{i_j} or a loop if $v_{i_{j-1}} = v_{i_j}$, $1 \leq i_1 \leq i_2 \leq \ldots \leq i_q \leq n$.*

Example 2.14 *Let G be a strong quasi-bipartite graph on vertices $\{x_1, x_2; y_1, y_2\}$. A walk of length 2 is*

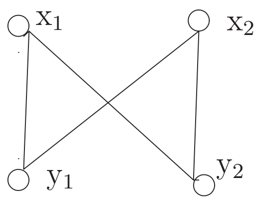
$$w = \{x_1, l_1, x_1, l_2, y_1\},$$

where $l_1 = \{x_1, x_1\}$ is the loop on x_1 and $l_2 = \{x_1, y_1\}$ is the edge joining x_1 and y_1.

Let G be a strong quasi-bipartite graph on vertex set $\{x_1, \ldots, x_n; y_1, \ldots, y_m\}$. The *generalized ideal* $I_q(G)$ associated to G is the ideal of the polynomial ring $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$ generated by the monomials of degree q corresponding to the walks of length $q-1$. Hence the generalized ideal $I_q(G)$ is generated by all monomials of degree $q \geq 3$ corresponding to the walks of length $q-1$ and the variables in each generator of $I_q(G)$ have at most degree 2. Therefore:

$$I_q(G) = L_{q,2} = \sum_{k+r=q} I_{k,2}J_{r,2}, \quad for \quad q \geq 3 \quad ([2]).$$

Example 2.15 *Let $R = K[X_1, X_2; Y_1, Y_2]$ be a polynomial ring over a field K and G be the strong quasi-bipartite graph on vertices x_1, x_2, y_1, y_2:*



$I_3(G) = I_1J_2 + I_2J_1 = (X_1Y_1Y_2, X_2Y_1Y_2, X_1Y_1^2, X_2Y_1^2, X_1Y_2^2, X_2Y_2^2, X_1X_2Y_1, X_1X_2Y_2, X_1^2Y_1, X_1^2Y_2, X_2^2Y_1, X_2^2Y_2)$.
$I_4(G) = I_{3,2}J_1 + I_1J_{3,2} + I_2J_2 = (X_1^2X_2Y_1, X_1^2X_2Y_2, X_1X_2^2Y_1, X_1X_2^2Y_2, X_1Y_1^2Y_2, X_2Y_1^2Y_2, X_1Y_1Y_2^2, X_2Y_1Y_2^2, X_1^2Y_1^2, X_1^2Y_1Y_2, X_1^2Y_2^2, X_2^2Y_1^2, X_2^2Y_2^2, X_2^2Y_1Y_2, X_1X_2Y_1^2, X_1X_2Y_2^2, X_1X_2Y_1Y_2)$.

The following result classifies the ideals $I_q(G)$ that are unmixed.

Proposition 2.16 *Let $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$.*
1) If $2 \leq q < 2(n+m)-1$ then $L_{q,2}$ is unmixed if and only if $n = m$.
2) If $q = 2(n+m)-1$, then $L_{q,2}$ is unmixed.

Proof By Propositions 2.6 and 2.8. □

We consider the case $q \geq 3$. In fact, for $q = s = 2$, the ideal $L_{2,2} = (\{X_iY_j | i = 1, \ldots, n, \;\; j = 1, \ldots, m\})$ doesn't describe the edge ideal $I(G) = I_2(G)$ of a strong quasi-bipartite graph, but it is the edge ideal of a complete bipartite graph (with no loops) on the vertex set $\{x_1, \ldots, x_n; y_1, \ldots, y_m\}$. Then $L_{2,2}$ satisfies the characterization of unmixed bipartite graphs given in [7].

3. The toric ideal of $K[L_{q,s}]$

Let $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$ and $L_{q,s} = (f_1, \ldots, f_p)$ be the ideal of Veronese bi-type. The monomial subring of R spanned by $F = \{f_1, \ldots, f_p\}$ is the K-algebra $K[L_{q,s}] = K[F] = K[f_1, \ldots, f_p]$. Note that $K[F]$ is a graded algebra with the grading $K[F]_i = K[F] \cap R_i$. There is a graded epimorphism of K-algebras: $\varphi : S = K[T_1, \ldots, T_p] \to K[L_{q,s}]$ induced by $\varphi(T_i) = f_i$, where S is a polynomial ring graded by $\deg(T_i) = \deg(f_i)$. Note that the map φ is given by $\varphi(h(T_1, \ldots, T_p)) = h(f_1, \ldots, f_p)$ for all $h \in S$.

Let $I(L_{q,s})$ be the kernel of the K-algebra epimorphism, called the *toric ideal* of $K[L_{q,s}]$. It is known that the toric ideal of a monomial K-algebra is a graded prime ideal generated by a finite set of binomials.

Now we prove that $I(L_{q,s})$ has a quadratic Groebner basis. In order to formulate this result we have to recall the notion of sortability, introduced in [5], and we apply it to the monomial ideal $L_{q,s}$.

Let $A = K[z_1, \dots, z_t]$ be a polynomial ring and L be a monomial ideal of A generated in degree q. Let B be the set of the exponent vectors of the monomials of $G(L)$. If $u = (u_1, \dots, u_t), v = (v_1, \dots, v_t) \in B$, then $\underline{z}^u = \prod_i z_i^{u_i}, \underline{z}^v = \prod_i z_i^{v_i} \in L$. We write $\underline{z}^u \underline{z}^v = z_{i_1} \cdots z_{i_{2q}}$ with $i_1 \leq i_2 \leq \dots \leq i_{2q}$. Then we set $\underline{z}^{u'} = \prod_{j=1}^q z_{2j-1}$ and $\underline{z}^{v'} = \prod_{j=1}^q z_{2j}$. This defines a map

$$sort : B \times B \to M_q \times M_q, \quad (u, v) \longmapsto (u', v'),$$

where M_q is the set of all integer vectors $(a_1, \dots, a_t)$ such that $\sum_{i=1}^t a_i = q$.

The set B is called *sortable* if $\mathrm{Im}(sort) \subseteq B \times B$.

The ideal L is called *sortable* if the set of exponent vectors of the monomials of $G(L)$ is sortable. In other words, let $\underline{z}^u, \underline{z}^v \in L$, then L is said sortable if $\underline{z}^{u'}, \underline{z}^{v'} \in L$, where $(u', v') = sort(u, v)$.

Theorem 3.1 *Let $R = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$. $L_{q,s}$ is sortable.*

Proof Let $L_{q,s} = (\{X_1^{a_1} \cdots X_n^{a_n} Y_1^{b_1} \cdots Y_m^{b_m} \,|\, \sum_{i=1}^n a_i + \sum_{j=1}^m b_j = q, \ \ 0 \leq a_i, b_j \leq s\})$ and B be the set of the exponent vectors of the monomials of $G(L_{q,s})$.

Let $f_i = X_1^{a_1} \cdots X_n^{a_n} Y_1^{b_1} \cdots Y_m^{b_m}, f_j = X_1^{c_1} \cdots X_n^{c_n} Y_1^{d_1} \cdots Y_m^{d_m} \in G(L_{q,s})$, then $u = (a_1, \dots, a_n; b_1, \dots, b_m)$, $v = (c_1, \dots, c_n; d_1, \dots, d_m) \in B$. One has that $f_i f_j = \underbrace{X_1 \cdots X_1}_{a_1+c_1-times} \cdots \underbrace{X_n \cdots X_n}_{a_n+c_n-times} \underbrace{Y_1 \cdots Y_1}_{b_1+d_1-times} \cdots \underbrace{Y_m \cdots Y_m}_{b_m+d_m-times}$ is a monomial of degree $2q$. If one replaces the set of variables $\{X_1, \dots, X_n\}$ with $\{z_1, \dots, z_n\}$ and $\{Y_1, \dots, Y_m\}$ with $\{z_{n+1}, \dots, z_{n+m}\}$, then $f_i f_j = z_{i_1} \cdots z_{i_{2q}}$ with $i_1 \leq \dots \leq i_{2q}$. Then we consider $f_i' = \underline{z}^{u'} = \prod_{l=1}^q z_{2l-1}$ and $f_j' = \underline{z}^{v'} = \prod_{l=1}^q z_{2l}$. We must prove that $f_i', f_j' \in L_{q,s}$. We have that f_i' is of degree q and we write $f_i' = \prod_{l=1}^q z_{2l-1} = X_1^{a_1'} \cdots X_n^{a_n'} Y_1^{b_1'} \cdots Y_m^{b_m'}$. If $a_i + c_i$ is even then $a_i' = \frac{a_i+c_i}{2} \leq s$ and if $a_i + c_i$ is odd then $a_i' = \frac{a_i+c_i+1}{2} < s$. Similarly, if $b_j + d_j$ is even then $b_j' = \frac{b_j+d_j}{2} \leq s$. If $b_j + d_j$ is odd then $b_j' = \frac{b_j+d_j+1}{2} < s$. Moreover, because f_i' is of degree q and there exist $a_i' \neq 0$, $b_j' \neq 0$ with $0 \leq a_i', b_j' \leq q$ for all i, j, then $X_1^{a_1'} \cdots X_n^{a_n'} \neq X_i^q$ and $Y_1^{b_1'} \cdots Y_m^{b_m'} \neq Y_j^q$. It follows that $X_1^{a_1'} \cdots X_n^{a_n'} \in I_{k,s}$ and $Y_1^{b_1'} \cdots Y_m^{b_m'} \in J_{r,s}$ with $k + r = q$. Hence $f_i' \in L_{q,s}$. In the same way the argument holds for f_j'. Hence $L_{q,s}$ is sortable. □

Corollary 3.2 *Let $R = K[X_1, \dots, X_n; Y_1, \dots, Y_m]$ and $L_{q,s} \subset R$. Then:*

1) $I(L_{q,s})$ has a quadratic Groebner basis.

2) $K[L_{q,s}]$ is Koszul.

Proof 1) By Theorem 3.1 and [1](Lemma 5.2).

2) The conclusion follows by 1). □

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