Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2013) 37: 95 – 113
© TÜBİTAK
doi:10.3906/mat-1106-8

Scalar curvature and symmetry properties of lightlike submanifolds

Cyriaque ATINDOGBE[1], **Oscar LUNGIAMBUDILA**[2,*], **Joël TOSSA**[3]
[1]Institut de Mathématiques et de Sciences Physiques (IMSP) Université d'Abomey-Calavi, 01 B.P. 613, Porto-Novo, Bénin
[2]Département de Mathématiques et Informatique, Faculté des Sciences Université de Kinshasa, B.P. 190 KINSHASA XI R.D. Congo
[3]Institut de Mathématiques et de Sciences Physiques (IMSP) Université d'Abomey-Calavi, 01 B.P. 613, Porto-Novo, Bénin

Received: 08.06.2011 • **Accepted:** 28.09.2011 • **Published Online:** 17.12.2012 • **Printed:** 14.01.2013

Abstract: In this paper, the induced Ricci tensor and the extrinsic scalar curvature on lightlike submanifolds are obtained. This scalar quantity extend the result given by C. Atindogbe in [1]. An example of extrinsic scalar curvature on one class of 2-degenerate manifolds is provided. We investigate lightlike submanifolds which are locally symmetric, semi-symmetric, Ricci semi-symmetric in indefinite spaces form. In the coisotropic case, we show that, under some conditions, these lightlike submanifolds are totally geodesic.

Key words: Extrinsic scalar curvature, locally symmetric lightlike submanifold, semi-symmetric lightlike submanifold, Ricci semi-symmetric lightlike submanifold

1. Introduction

The scalar curvature is one of the most important concepts in semi-Riemannian geometry and its connected areas such as General Relativity. This scalar quantity, under the geometric point of view, is just the contraction of the symmetric Ricci tensor Ric with a non-degenerate metric g, that is

$$S = g^{\alpha\beta} Ric_{\alpha\beta}. \tag{1.1}$$

In geometry of the lightlike submanifolds, two difficulties arise: since the induced connection is not a Levi-Civita connection (unless M be totally geodesic) the induced Ricci tensor is not symmetric in general. Also, as the induced metric is degenerate, its inverse does not exist and it is not possible to proceed in the usual way by contracting the Ricci tensor to get a scalar quantity.

To overcome these difficulties in degenerate geometry, Duggal considered in [4] one class of globally null manifolds M, warped product of a globally null manifold and a complete Riemannian manifold and shows that its geometry essentially reduces to the Riemannian geometry of a leaf of its screen distribution which is integrable. This information is then used in finding the Ricci tensor and the scalar curvature of M. In [5], Duggal studied one class of lightlike hypersurfaces of genus zero in ambient Lorentzian signature. Any element of this class admit canonical screen distribution that induces a canonical transversal vector bundle and induced symmetric Ricci tensor. Atindogbé in [1] constructed the concept of extrinsic scalar curvature on all lightlike

*Correspondence: lungiaoscar@imsp-uac.org
2000 *AMS Mathematics Subject Classification:* 53B25, 53B30, 53C50.

hypersurfaces, by introducing a symmetrized induced Ricci tensor and using the concept of pseudo-inversion introduced in [2].

In this paper, we give expression of the induced Ricci tensor and extend the concept of extrinsic scalar curvature on r-degenerate submanifolds. We first define the symmetrized induced Ricci tensor on lightlike submanifolds and by using the concept of pseudo-inversion of r-degenerate metrics introduced in [3], summarized in the preliminaries, we overcome the above quoted difficulties in contracting with respect to the r-degenerate metric. We give an example of extrinsic scalar curvature on one class of 2-degenerate manifolds.

The class of semi-Riemannian manifolds satisfying the condition $\nabla R = 0$ is a natural generalization of the class of manifolds of constant curvature, where ∇ is the Levi-Civita connection on semi-Riemannian manifold and R is the corresponding curvature tensor. Such a manifold M is said to be locally symmetric. Locally, at any point p, M admit an isometry σ verifying $\sigma(p) = p$ and $d\sigma(p) = -Id$. A semi-Riemannian manifold is called semi-symmetric if $R(X,Y)\cdot R = 0$, where $R(X,Y)$ is the curvature operator corresponding to the Riemann curvature R. The semi-symmetric manifolds have been classified, in Riemannian case, by Szabó in [11] and [12]. A semi-Riemannian manifold is said to be Ricci semi-symmetric, if it verifies the condition $R(X,Y)\cdot Ric = 0$, where Ric is the Ricci tensor.

In the sequel, in sections 5, 6 and 7, we have considered the lightlike submanifolds $(M, g, S(TM), S(TM^{\perp}))$ of indefinite spaces form $(\overline{M}(c), \overline{g})$ with $Rank(RadTM) = r \leqslant min\{m,n\}$, such that the distribution $RadTM$ is integrable and ∇^t is a metric linear connection on $tr(TM)$. We study symmetry properties of these lightlike submanifolds.

In section 5, we investigate locally symmetric lightlike submanifolds of indefinite spaces form; we show that under the condition $A_{N_j}\xi_j \in \Gamma(S(TM))$, the coisotropic submanifolds of indefinite spaces form with integrable radical distribution are totally geodesic (Theorem 5.2).

In section 6, we give characterization of semi-symmetric lightlike submanifolds of indefinite spaces form (Theorem 6.1). We prove that under conditions $h^l(A_N\xi, X) = 0$ and $g(A_N\xi, A_{N'}\xi) \neq 0$, the coisotropic submanifolds of indefinite spaces form with integrable radical distribution are totally geodesic (Theorem 6.2). In section 7, we give characterization of Ricci semi-symmetric lightlike submanifolds of indefinite spaces form (Theorem 7.1). We show that under condition $Ric(\xi, A_N\xi) \neq 0$, the coisotropic submanifolds of indefinite spaces form with integrable radical distribution are totally geodesic (Theorem 7.2). Note that our characterization results of semi-symmetry and Ricci semi-symmetry properties on r-degenerate submanifolds, in particular on lightlike hypersurfaces, refind again the results given by Sahin [10] in the case of lightlike hypersurfaces of semi-Euclidean spaces; see Corollaries 6.3 and 7.3.

2. Preliminaries

2.1. Lightlike submanifolds of semi-Riemannian manifolds

We follow ([6] and [7]) for the notations and formulas used in this paper. Let $(\overline{M}, \overline{g})$ be an $(m+n)$-dimensional semi-Riemannian manifold of constant index ν, $1 \leqslant \nu < m+n$ and M be a submanifold of $\overline{M}$ of codimension n. We assume that both m and n are $\geqslant 1$. At a point $u \in M$, we define the orthogonal complement $T_uM^{\perp}$ of the tangent space T_uM by

$$T_uM^{\perp} = \{X_u \in T_u\overline{M} : \overline{g}(X_u, Y_u) = 0,\ \forall Y_u \in T_uM\}.$$

We put $RadT_uM = RadT_uM^{\perp} = T_uM \cap T_uM^{\perp}$. The submanifold M of $\overline{M}$ is said to be an r-lightlike submanifold (one can suppose that the index of $\overline{M}$ is $\nu \geqslant r$), if the mapping

$$RadTM : u \in M \longrightarrow RadT_uM$$

defines a smooth distribution on M of rank $r > 0$. We call $RadTM$ the radical distribution on M. In the sequel, an r-lightlike submanifold will simply be called a lightlike submanifold and g is lightlike metric, unless we need to specify r.
Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, that is,

$$TM = RadTM \perp S(TM). \tag{2.1}$$

We consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $Rad(TM)$ in $TM^{\perp}$. Since, for any local frame $\{\xi_i\}$ of $Rad(TM)$, there exists a local frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $S(TM)^{\perp}$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$ and $\overline{g}(N_i, N_j) = 0$, it follows that there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$ (see [6], p144). Let $tr(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\overline{M}_{|M}$. Then

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}), \tag{2.2}$$

$$T\overline{M}_{|M} = TM \oplus tr(TM) = S(TM) \perp (RadTM \oplus ltr(TM)) \perp S(TM^{\perp}). \tag{2.3}$$

Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM/RadTM$ ([9]). Throughout this paper, we will discuss the dependence (or otherwise) of the results on induced objects and refer to ([6] or [7]) for their transformation equations. We say that a submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\overline{M}$ is
(1) proper lightlike if $r < min\{m, n\}$;
(2) coisotropic if $r = n < m$, hence, $S(TM^{\perp}) = \{0\}$;
(3) isotropic if $r = m < n$, hence $S(TM) = \{0\}$;
(4) totally lightlike if $r = m = n$, hence $S(TM) = \{0\} = S(TM^{\perp})$.
The Gauss and Weingarten equations are

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM), \tag{2.4}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^t_X V, \quad \forall X \in \Gamma(TM),\ V \in \Gamma(tr(TM)), \tag{2.5}$$

where $\{\nabla_X Y, A_V X\}$ and $\{h(X, Y), \nabla^t_X V\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$, respectively. Moreover, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad \forall X, Y \in \Gamma(TM), \tag{2.6}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \quad \forall X \in \Gamma(TM),\ N \in \Gamma(ltr(TM)), \tag{2.7}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \quad \forall X \in \Gamma(TM),\ W \in \Gamma(S(TM^{\perp})). \tag{2.8}$$

Denote the projection of TM on $S(TM)$ by P. Then, by using (2.6)-(2.8) and taking into account that $\overline{\nabla}$ is a metric connection, we obtain

$$\overline{g}(h^s(X, Y), W) + \overline{g}(Y, D^l(X, W)) = g(A_W X, Y), \tag{2.9}$$

$$\overline{g}(D^s(X,N),W) = \overline{g}(N, A_W X) \tag{2.10}$$

From the decomposition (2.1) of the tangent bundle of lightlike submanifold, we have

$$\nabla_X PY = \overset{*}{\nabla}_X PY + \overset{*}{h}(X,PY), \quad \forall X,Y \in \Gamma(TM), \tag{2.11}$$

$$\nabla_X \xi = -\overset{*}{A}_\xi X + \overset{*}{\nabla}{}^t_X \xi, \quad \forall X \in \Gamma(TM),\ \xi \in \Gamma(RadTM). \tag{2.12}$$

By using the above equations, we obtain

$$\overline{g}(h^l(X,PY),\xi) = g(\overset{*}{A}_\xi X, PY), \tag{2.13}$$

$$\overline{g}(\overset{*}{h}(X,PY),N) = g(A_N X, PY), \tag{2.14}$$

$$\overline{g}(A_N, N') + \overline{g}(A_{N'}, N) = 0 \tag{2.15}$$

$$\overline{g}(h^l(X,\xi),\xi) = 0, \qquad \overset{*}{A}_\xi \xi = 0. \tag{2.16}$$

In general, the induced connection ∇ on M and the transversal linear connection ∇^t on $tr(TM)$ are not metric connections. Since $\overline{\nabla}$ is a metric connection, by using (2.5) and (2.6) we get

$$(\nabla_X g)(Y,Z) = \overline{g}(h^l(X,Y),Z) + \overline{g}(h^l(X,Z),Y), \quad \forall X,Y,Z \in \Gamma(TM). \tag{2.17}$$

and

$$(\nabla^t_X \overline{g})(V,V') = -\{\overline{g}(A_V X, V') + \overline{g}(A_{V'} X, V)\}. \tag{2.18}$$

However, it is important to note that $\overset{*}{\nabla}$ is a metric connection on $S(TM)$.
We denote the Riemann curvature tensors of $\overline{M}$ and M by $\overline{R}$ and R respectively. The Gauss equation for M is given by

$$\begin{aligned}\overline{R}(X,Y)Z \;=\;& R(X,Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y \\ & - A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y,Z) - (\nabla_Y h^l)(X,Z) \\ & + D^l(X, h^s(Y,Z)) - D^l(Y, h^s(X,Z)) + (\nabla_X h^s)(Y,Z) \\ & - (\nabla_Y h^s)(X,Z) + D^s(X, h^l(Y,Z)) - D^s(Y, h^l(X,Z)).\end{aligned} \tag{2.19}$$

Therefore,

$$\begin{aligned}\overline{R}(X,Y,Z,PU) \;=\;& R(X,Y,Z,PU) + \overline{g}(\overset{*}{h}(Y,PU), h^l(X,Z)) - \overline{g}(\overset{*}{h}(X,PU), h^l(Y,Z)) \\ & + \overline{g}(h^s(Y,PU), h^s(X,Z)) - \overline{g}(h^s(X,PU), h^s(Y,Z))\end{aligned} \tag{2.20}$$

for any $X,Y,Z \in \Gamma(TM)$. Note that for the coisotropic, isotropic and totally lightlike submanifolds, in (2.19), we have $h^s = 0$, $h^l = 0$ and $h^l = h^s = 0$, respectively.

2.2. Pseudo-inversion of r-degenerate metrics

In this section we indicate by case 1, case 2, case 3 and case 4 to mean the r-lightlike submanifolds with $0 < r < min\{m, n\}$ (proper lightlike submanifold), the coisotropic submanifolds, the isotropic submanifolds and totally lightlike submanifolds, respectively. We recall from [3] the following result. Consider on M the local frames $\{\xi_i\}$ and $\{N_i\}$ of sections of $RadTM$ and $ltr(TM)$ satisfying $\overline{g}(N_i, \xi_j) = \delta_{ij}$. Consider on M the 1-forms $\eta_i, i = 1, ..., r$ defined by $\eta_i(\cdot) = \overline{g}(N_i, \cdot)$. Any vector field X on M is expressed on a coordinate neighbourhood $\mathcal{U}$ as

$$X = PX + \sum_{i=1}^{r} \eta_i(X)\xi_i \qquad (case\ 1\ or\ 2) \tag{2.21}$$

$$X = \sum_{i=1}^{m} \eta_i(X)\xi_i \qquad (case\ 3\ or\ 4) \tag{2.22}$$

Now, we define $\flat_g$ as

$$\begin{aligned} \flat_g : \Gamma(TM) &\longrightarrow \Gamma(T^*M), \\ X &\longmapsto X^{\flat_g} \end{aligned}$$

such that for all $Y \in \Gamma(TM)$

$$X^{\flat_g}(Y) = g(X, Y) + \sum_{i=1}^{r} \eta_i(X)\eta_i(Y) \qquad (case\ 1\ or\ 2), \tag{2.23}$$

$$X^{\flat_g}(Y) = \sum_{i=1}^{m} \eta_i(X)\eta_i(Y) \qquad (case\ 3\ or\ 4). \tag{2.24}$$

The map $\flat_g$ is an isomorphism of $\Gamma(TM)$ onto $\Gamma(T^*M)$, its inverse is denoted $\sharp_g$. For $X \in \Gamma(TM)$ (resp., $\omega \in \Gamma(T^*M)$), $X^{\flat_g}$ (resp. $\omega^{\sharp_g}$) is called the dual 1-form of X (resp. the dual vector field of ω) with respect to the degenerate metric g.

We define a $(0, 2)$-tensor $\tilde{g}$ by, for any $X, Y \in \Gamma(TM)$,

$$\tilde{g}(X, Y) = X^{\flat_g}(Y) = g(X, Y) + \sum_{i=1}^{r} \eta_i(X)\eta_i(Y) \qquad (case\ 1\ or\ 2), \tag{2.25}$$

and

$$\tilde{g}(X, Y) = X^{\flat_g}(Y) = \sum_{i=1}^{m} \eta_i(X)\eta_i(Y) \qquad (case\ 3\ or\ 4). \tag{2.26}$$

Clearly, $\tilde{g}$ defines a non-degenerate metric on M. Also, observe that $\tilde{g}$ coincides with g if the latter is non-degenerate. The $(0, 2)$-tensor $\tilde{g}^{-1}$, inverse of $\tilde{g}$ is called the pseudo-inverse of g. Let us consider the local quasi-orthogonal fields of frames $\{\xi_1, ..., \xi_r, X_{r+1}, ..., X_m\}$ and $\{\xi_1, ..., \xi_r\}$ on lightlike submanifold M with respect to the decompositions $TM = S(TM) \perp RadTM$ (case 1 or 2) and $TM = RadTM$ (case 3 or 4). Using relations (2.25) and (2.26), we have

$$\tilde{g}(\xi_i, \xi_j) = \delta_{ij},\ 1 \leqslant i, j \leqslant r \quad and \quad \tilde{g}(X_i, X_j) = g_{ij},\ r+1 \leqslant i, j \leqslant m, \quad (Case1\ or\ 2).$$

$$\tilde{g}(\xi_i, \xi_j) = \delta_{ij}, \quad 1 \leqslant i, j \leqslant m, \quad (Case3\ or\ 4).$$

3. The Ricci tensor of a lightlike submanifold

In this section we study the Ricci tensor Ric of r-degenerate submanifold of a semi-Riemannian manifold $(\overline{M},\overline{g})$. We have the following proposition.

Proposition 3.1 *Let $(M,g,S(TM),S(TM^{\perp}))$ be a lightlike submanifold of a semi-Riemannian manifold $(\overline{M},\overline{g})$, such that the radical distribution $RadTM$ is integrable and ∇^t is a metric linear connection on $tr(TM)$. Then the Ricci tensor of M denoted by Ric is given by, for any $X,Y\in\Gamma(TM)$,*

$$\begin{aligned} Ric(X,Y) &= \overline{R}ic(X,Y)+\sum_{j=1}^{r}h_j^l(X,Y)trA_{N_j}-\sum_{j=1}^{r}g(A_{N_j}X,\overset{*}{A}_{\xi_j}Y)\\ &+\sum_{\alpha=r+1}^{n}h_\alpha^s(X,Y)trA_{W_\alpha}-\sum_{\alpha=r+1}^{n}g(A_{W_\alpha}X,A_{W_\alpha}Y)\\ &-\sum_{j=1}^{r}\eta_j(\overline{R}(\xi_j,Y)X)-\sum_{\alpha=r+1}^{n}\overline{R}(W_\alpha,X,Y,W_\alpha), \end{aligned} \tag{3.1}$$

where $\overline{R}ic$ is the Ricci tensor of $\overline{M}$ and trA_{N_j} (resp., trA_{W_α}) is the trace of the operator A_{N_j} (resp., A_{W_α}).

Proof Suppose that M is an m-dimensional lightlike submanifold of an $(m+n)$-dimensional semi-Riemannian manifold $\overline{M}$ with $Rank(RadTM)=r$ and $r\leqslant min\{m,n\}$. Let R be an induced Riemann curvature tensor on M with respect to $\{S(TM),S(TM^{\perp})\}$. Consider a local quasi-orthonormal frame $\{\xi_i,N_i,E_a,W_\alpha\}$ on $\overline{M}$, where $\{\xi_1,...,\xi_r,E_{r+1},...,E_m\}$ is a local frame field on M with respect to the decomposition (2.1). By definition $Ric(X,Y)=trace(Z\longrightarrow R(Z,X)Y)$, so we have, for any $X,Y\in\Gamma(TM)$

$$\begin{aligned} Ric(X,Y) &= \sum_{i=1}^{r}\tilde{g}^{ii}\tilde{g}(R(\xi_i,X)Y,\xi_i)+\sum_{a=r+1}^{m}\tilde{g}^{aa}\tilde{g}(R(E_a,X)Y,E_a)\\ &= \sum_{i=1}^{r}\overline{g}(R(\xi_i,X)Y,N_i)+\sum_{a=r+1}^{m}\epsilon_a g(R(E_a,X)Y,E_a), \end{aligned} \tag{3.2}$$

where ϵ_a is the causal character of the vector field E_a of the orthonormal frame field $\{E_{r+1},...,E_m\}$ of $S(TM)$. Then, using relation (2.20), we have

$$\begin{aligned} g(R(E_a,X)Y,E_a) &= \overline{g}(\overline{R}(E_a,X)Y,E_a)+\overline{g}(\overset{*}{h}(E_a,E_a),h^l(X,Y))\\ &-\overline{g}(\overset{*}{h}(X,E_a),h^l(E_a,Y))+\overline{g}(h^s(E_a,E_a),h^s(X,Y))\\ &-\overline{g}(h^s(X,E_a),h^s(E_a,Y)). \end{aligned}$$

So,

$$\begin{aligned} g(R(E_a,X)Y,E_a) &= \overline{g}(\overline{R}(E_a,X)Y,E_a)+\sum_{j=1}^{r}h_j^l(X,Y)g(A_{N_j}E_a,E_a)\\ &-\sum_{j=1}^{r}g(\overset{*}{A}_{\xi_j}Y,E_a)g(A_{N_j}X,E_a)+\sum_{\alpha=r+1}^{n}h_\alpha^s(X,Y)g(A_{W_\alpha}E_a,E_a) \end{aligned}$$

$$- \sum_{\alpha=r+1}^{n} g(A_{W_\alpha} Y, E_a) g(A_{W_\alpha} X, E_a). \tag{3.3}$$

Also, using relation (2.19), and since $h^l(X,\xi) = 0$ and A_W are $\Gamma(S(TM))$-valued linear operators, we have

$$\begin{aligned}
\overline{g}(R(\xi_i, X)Y, N_i) &= \overline{g}(\overline{R}(\xi_i, X)Y, N_i) + \sum_{j=1}^{r} h^l_j(X,Y)\overline{g}(A_{N_j}\xi_i, N_i) - \overline{g}(A_{h^l(\xi_i,Y)}X, N_i) \\
&\quad + \sum_{\alpha=r+1}^{n} h^s_\alpha(X,Y)\overline{g}(A_{W_\alpha}\xi_i, N_i) - \overline{g}(A_{h^s(\xi_i,Y)}X, N_i) \\
&= \overline{g}(\overline{R}(\xi_i, X)Y, N_i) + \sum_{j=1}^{r} h^l_j(X,Y)\overline{g}(A_{N_j}\xi_i, N_i) \\
&\quad + \sum_{\alpha=r+1}^{n} h^s_\alpha(X,Y)\overline{g}(A_{W_\alpha}\xi_i, N_i).
\end{aligned} \tag{3.4}$$

Thus, by substituting (3.3) and (3.4) in (3.2), we obtain the result. □

For the cases of coisotropic submanifolds and totally lightlike submanifolds, since $S(TM^\perp) = \{0\}$, using relations (2.15) and (2.18), we obtain that ∇^t is a metric linear connection on $tr(TM)$. Thus, we have the following corollary.

Corollary 3.2 *Let $(M, g, S(TM))$ be a coisotropic submanifold of a semi-Riemannian manifold $(\overline{M}, \overline{g})$, such that the radical distribution $RadTM$ is integrable. Then the Ricci tensor of is given by, for any $X, Y \in \Gamma(TM)$,*

$$\begin{aligned}
Ric(X,Y) &= \overline{R}ic(X,Y) + \sum_{j=1}^{n} h^l_j(X,Y) trA_{N_j} - \sum_{j=1}^{n} g(A_{N_j}X, \overset{*}{A}_{\xi_j} Y) \\
&\quad - \sum_{j=1}^{n} \eta_j(\overline{R}(\xi_j, Y)X).
\end{aligned} \tag{3.5}$$

For the cases of isotropic submanifolds and totally lightlike submanifolds, since h^l vanishes identically on M, we have the following

Corollary 3.3 *Let $(M, g, S(TM))$ be an isotropic submanifold of a semi-Riemannian manifold $(\overline{M}, \overline{g})$, such that ∇^t is a metric linear connection on $tr(TM)$. Then the Ricci tensor of M is given by, for any $X, Y \in \Gamma(TM)$*

$$\begin{aligned}
Ric(X,Y) &= \overline{R}ic(X,Y) + \sum_{\alpha=m+1}^{n} h^s_\alpha(X,Y) trA_{W_\alpha} - \sum_{\alpha=m+1}^{n} g(A_{W_\alpha}X, A_{W_\alpha}Y) \\
&\quad - \sum_{j=1}^{m} \eta_j(\overline{R}(\xi_j, Y)X) - \sum_{\alpha=m+1}^{n} \overline{R}(W_\alpha, X, Y, W_\alpha)
\end{aligned} \tag{3.6}$$

Corollary 3.4 *Let $(M, g, S(TM))$ be a totally lightlike submanifold of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then the Ricci tensor of M is given by, for any $X, Y \in \Gamma(TM)$*

$$Ric(X,Y) = \overline{R}ic(X,Y) - \sum_{j=1}^{m} \eta_j(\overline{R}(\xi_j, Y)X). \tag{3.7}$$

4. Extrinsic scalar curvature

In this section we extend the concept of extrinsic scalar curvature on (M, g), an m-dimensional lightlike submanifold of an $(m+n)$-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ with $Rank(RadTM) = r \leqslant min\{m, n\}$, we suppose that the distribution $RadTM$ is integrable and ∇^t is a metric linear connection on $tr(TM)$. The following range for various induces is used in this section:

$$a', b', ... \in \{1, ..., m\}; i, j, ... \in \{1, ..., r\}; a, b, ... \in \{r+1, ..., m\}; \alpha, \beta, ... \in \{r+1, ..., n\}.$$

We consider the local field of frames $\{\xi_1, ..., \xi_r, \partial_a, N_1, ..., N_r, \partial_\alpha\} \equiv \{\partial_{a'}, N_1, ..., N_r, \partial_\alpha\}$ of $T\overline{M}$ along M, where $\{\partial_{a'}\}_{1 \leqslant a' \leqslant m}$, $\{\partial_a\}_{r+1 \leqslant a \leqslant m}$ and $\{\partial_\alpha\}_{r+1 \leqslant \alpha \leqslant n}$ are local field of frames of TM, $S(TM)$ and $S(TM^\perp)$, respectively.

Recall that the induced Ricci tensor given in (3.1) is not symmetric in general. Then (see [1]) the *symmetrized induced Ricci* tensor on M is defined as the $(0,2)$-tensor Ric^{sym} on M such that, for any $X, Y \in \Gamma(TM)$

$$Ric^{sym}(X,Y) = \frac{1}{2}\{Ric(X,Y) + Ric(Y,X)\} \tag{4.1}$$

In index notation, we have

$$Ric^{sym}_{a'b'} = \frac{1}{2}\{Ric_{a'b'} + Ric_{b'a'}\}. \tag{4.2}$$

By using the pseudo-inverse $\tilde{g}^{-1}$ of r-degenerate metric g and contract the relation (4.2), we obtain the scalar quantity

$$S = \tilde{g}^{a'b'} Ric^{sym}_{a'b'}. \tag{4.3}$$

We define S to be the *extrinsic scalar curvature* of the lightlike submanifold (M, g). Note that this definition is independent of the choice of pair $\{S(TM), S(TM^\perp)\}$.

Now, we give the expressions of the symmetrized Ricci tensor Ric^{sym} and the extrinsic scalar curvature S. Indeed, from relation (3.1), we obtain

$$\begin{aligned} Ric^{sym}(X,Y) &= \overline{R}ic(X,Y) + \sum_{j=1}^{r} h_j^l(X,Y) tr A_{N_j} + \sum_{\alpha=r+1}^{n} h_\alpha^s(X,Y) tr A_{W_\alpha} \\ &- \sum_{\alpha=r+1}^{n} g(A_{W_\alpha}X, A_{W_\alpha}Y) - \sum_{\alpha=r+1}^{n} \overline{R}(W_\alpha, X, Y, W_\alpha) \\ &- \frac{1}{2}\{ \sum_{j=1}^{r} \eta_j(\overline{R}(\xi_j, Y)X) + \sum_{j=1}^{r} \eta_j(\overline{R}(\xi_j, X)Y) \end{aligned}$$

$$+\sum_{j=1}^{r} g(A_{N_j}X, \overset{*}{A}_{\xi_j} Y) + \sum_{j=1}^{r} g(A_{N_j}Y, \overset{*}{A}_{\xi_j} X)\}. \tag{4.4}$$

Using the symmetry of $\overset{*}{A}_{\xi}$ with respect to g, we obtain locally,

$$\begin{aligned} Ric^{sym}_{a'b'} &= \overline{R}ic_{a'b'} + \sum_{j=1}^{r} h^l_j(\partial_{a'}, \partial_{b'}) trA_{N_j} + \sum_{\alpha=r+1}^{n} h^s_\alpha(\partial_{a'}, \partial_{b'}) trA_{W_\alpha} \\ &\quad - \sum_{\alpha=r+1}^{n} g(A_{W_\alpha}\partial_{a'}, A_{W_\alpha}\partial_{b'}) - \sum_{\alpha=r+1}^{n} \overline{R}(W_\alpha, \partial_{a'}, \partial_{b'}, W_\alpha) \\ &\quad -\frac{1}{2}\{\sum_{j=1}^{r} \eta_j(\overline{R}(\xi_j, \partial_{b'})\partial_{a'}) + \sum_{j=1}^{r} \eta_j(\overline{R}(\xi_j, \partial_{a'})\partial_{b'}) \\ &\quad + \sum_{j=1}^{r} g(\overset{*}{A}_{\xi_j} A_{N_j}\partial_{a'}, \partial_{b'}) + \sum_{j=1}^{r} g(\overset{*}{A}_{\xi_j} A_{N_j}\partial_{b'}, \partial_{a'})\}. \end{aligned} \tag{4.5}$$

In the sequel, we define by

$$\sigma^l_j = \frac{1}{m\sqrt{2}} \sum_{a',b'=1}^{m} \tilde{g}^{a'b'} h^l_j(\partial_{a'}, \partial_{b'}) \quad \text{and} \quad \sigma^s_\alpha = \frac{1}{m\sqrt{2}} \sum_{a',b'=1}^{m} \tilde{g}^{a'b'} h^s_\alpha(\partial_{a'}, \partial_{b'}), \tag{4.6}$$

the *lightlike mean curvature function* and the *screen mean curvature function* of M and

$$\overline{\theta} = \sum_{i=1}^{r} \overline{R}ic(N_i, \xi_i) + \sum_{\alpha,\beta=r+1}^{n} \overline{g}^{\alpha\beta}\overline{R}ic(\partial_\alpha, \partial_\beta), \tag{4.7}$$

represent the *transverse energy* in transversal direction $tr(TM)$.

Now, using relation (4.3) by contracting (4.5) with respect to $\tilde{g}^{a'b'}$ we get in Einstein notation, the following expression of the extrinsic scalar curvature on the lightlike submanifold $(M, S(TM), S(TM^\perp))$ given by,

$$\begin{aligned} S &= \overline{S} - \overline{\theta} + \sum_{j=1}^{r} m\sqrt{2}\sigma^l_j trA_{N_j} + \sum_{\alpha=r+1}^{n} m\sqrt{2}\sigma^s_\alpha trA_{W_\alpha} - \sum_{j=1}^{r} tr(\overset{*}{A}_{\xi_j} A_{N_j}) \\ &\quad - \sum_{\alpha=r+1}^{n} \tilde{g}^{a'b'} g(A_{W_\alpha}\partial_{a'}, A_{W_\alpha}\partial_{b'}) - \sum_{\alpha=r+1}^{n} \tilde{g}^{a'b'}\overline{R}(W_\alpha, \partial_{a'}, \partial_{b'}, W_\alpha) \\ &\quad -\frac{1}{2}\sum_{j=1}^{r} \tilde{g}^{a'b'}\{\eta_j(\overline{R}(\xi_j, \partial_{b'})\partial_{a'}) + \eta_j(\overline{R}(\xi_j, \partial_{a'})\partial_{b'})\}, \end{aligned} \tag{4.8}$$

where $\overline{S}$ is the scalar curvature on the ambient manifold $\overline{M}$. For the case of coisotropic submanifold, since $S(TM^\perp) = \{0\}$, the extrinsic scalar curvature on $(M, g, S(TM))$ is given by

$$S = \overline{S} - \overline{\theta} + \sum_{j=1}^{n} m\sqrt{2}\sigma^l_j trA_{N_j} - \sum_{j=1}^{n} tr(\overset{*}{A}_{\xi_j} A_{N_j})$$

$$-\frac{1}{2}\sum_{j=1}^{n}\tilde{g}^{a'b'}\big\{\eta_j(\overline{R}(\xi_j,\partial_{b'})\partial_{a'})+\eta_j(\overline{R}(\xi_j,\partial_{a'})\partial_{b'})\big\}. \tag{4.9}$$

For the ambient manifold $\overline{M}$ with constant sectional curvature c, we obtain the following result.

Theorem 4.1 *Let $(M,g,S(TM),S(TM^{\perp}))$ be a m-dimensional lightlike submanifold of a $(m+n)$-dimensional indefinite space form $(\overline{M}(c),\overline{g})$, such that the radical distribution $RadTM$ is integrable and ∇^t is a metric linear connection on $tr(TM)$. Then*

$$\begin{aligned} S &= (m-1)(m-r)c+\sum_{j=1}^{r}m\sqrt{2}\sigma_j^l trA_{N_j}+\sum_{\alpha=r+1}^{n}m\sqrt{2}\sigma_\alpha^s trA_{W_\alpha} \\ &\quad -\sum_{j=1}^{r}tr(\overset{*}{A}_{\xi_j}A_{N_j})-\sum_{\alpha=r+1}^{n}\tilde{g}^{a'b'}g(A_{W_\alpha}\partial_{a'},A_{W_\alpha}\partial_{b'}) \end{aligned} \tag{4.10}$$

Proof In the ambient manifold $\overline{M}(c)$, we have $\overline{Ric}=(m+n-1)c\overline{g}$, $\eta_j(\overline{R}(\xi_j,X)Y)=cg(X,Y)$ and $\overline{R}(W_\alpha,X,Y,W_\alpha)=cg(X,Y)$, then from (4.5), we obtain

$$\begin{aligned} Ric^{sym}_{a'b'} &= (m-1)cg_{a'b'}+\sum_{j=1}^{r}h_j^l(\partial_{a'},\partial_{b'})trA_{N_j}+\sum_{\alpha=r+1}^{n}h_\alpha^s(\partial_{a'},\partial_{b'})trA_{W_\alpha} \\ &\quad -\sum_{\alpha=r+1}^{n}g(A_{W_\alpha}\partial_{a'},A_{W_\alpha}\partial_{b'})-\frac{1}{2}\sum_{j=1}^{r}\big\{g(\overset{*}{A}_{\xi_j}A_{N_j}\partial_{a'},\partial_{b'})+g(\overset{*}{A}_{\xi_j}A_{N_j}\partial_{b'},\partial_{a'})\big\}. \end{aligned}$$

Thus,

$$\begin{aligned} S &= (m-1)(m-r)c+\sum_{j=1}^{r}m\sqrt{2}\sigma_j^l trA_{N_j}+\sum_{\alpha=r+1}^{n}m\sqrt{2}\sigma_\alpha^s trA_{W_\alpha} \\ &\quad -\sum_{j=1}^{r}tr(\overset{*}{A}_{\xi_j}A_{N_j})-\sum_{\alpha=r+1}^{n}\tilde{g}^{a'b'}g(A_{W_\alpha}\partial_{a'},A_{W_\alpha}\partial_{b'}) \end{aligned}$$

□

For the case of coisotropic submanifold, we have the following result.

Corollary 4.2 *Let $(M,g,S(TM))$ be a m-dimensional coisotropic submanifold of a $(m+n)$-dimensional indefinite space form $(\overline{M}(c),\overline{g})$, such that the radical distribution $RadTM$ is integrable. Then*

$$S = (m-1)(m-n)c+\sum_{j=1}^{n}m\sqrt{2}\sigma_j^l trA_{N_j}-\sum_{j=1}^{n}tr(\overset{*}{A}_{\xi_j}A_{N_j}) \tag{4.11}$$

Basic example

Let $(x,y)=(x_0,...,x_p,y_0,...,y_p)$ be the usual coordinates on $\mathbb{R}^{2p+2}$. Let $f=f(x_1,...,x_p)$ and $h=h(x_1,...,x_p)$

be the smooth functions on an open subset $\mathcal{O} \subset \mathbb{R}^p$. We define with respect to the natural field of frames $\{\frac{\partial}{\partial x_0}, ..., \frac{\partial}{\partial x_p}, \frac{\partial}{\partial y_0}, ..., \frac{\partial}{\partial y_p}\}$ a 2-degenerate metric $g_{(f,h)}$ on $M = \mathbb{R} \times \mathcal{O} \times \mathbb{R}^{p+1}$ by

$$\begin{aligned} g_{(f,h)} &= \sum_{i=1}^{p} (\frac{\partial f}{\partial x_i} dx_0 dx_i + \frac{\partial h}{\partial x_i} dx_i dy_0) \\ &+ \sum_{i,j=1}^{p} \{(\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} + \frac{\partial h}{\partial x_i}\frac{\partial h}{\partial x_j}) dx_i dx_j + \delta_{ij} dx_i dy_j\}. \end{aligned} \tag{4.12}$$

The 2-degenerate manifold $(M, g_{(f,h)})$ arise as a lightlike submanifold in a $(2p+4)$-dimensional semi-Euclidean space $\overline{M}$. Let $\{u_0, ..., u_p, v_0, ..., v_p, w_1, w_2\}$ be a basis for a space $\overline{M}$. Define an semi-Euclidean metric $\overline{g}$ of signature $(p+2, p+2)$ on $\overline{M}$ by setting

$$\overline{g}(u_i, u_j) = 0 = \overline{g}(v_i, v_j), \quad \overline{g}(u_0, v_j) = 0 = \overline{g}(u_i, v_0), \quad 0 \leqslant i, j \leqslant p.$$

$$\overline{g}(u_0, w_1) = 1 \quad \overline{g}(u_0, w_2) = 0, \quad \overline{g}(u_i, w_1) = 0 = \overline{g}(u_i, w_2), \quad 1 \leqslant i \leqslant p.$$

$$\overline{g}(v_0, w_1) = 0 \quad \overline{g}(v_0, w_2) = 1, \quad \overline{g}(v_i, w_1) = 0 = \overline{g}(v_i, w_2), \quad 1 \leqslant i \leqslant p.$$

$$\overline{g}(u_i, v_j) = \delta_{ij}, \quad \overline{g}(w_i, w_j) = \delta_{ij}, \qquad 1 \leqslant i, j \leqslant p.$$

Consider the application

$$F(x, y) = x_0\ u_0 + ... + x_p\ u_p + y_0\ v_0 + ... + y_p\ v_p + f\ w_1 + h\ w_2. \tag{4.13}$$

$F(x, y)$ defines an embedding of M in $\overline{M}$ and $g_{(f,h)}$ is the induced metric on the embedded submanifold M.

FACT 1. By direct calculation using (4.13), the tangent space TM is defined by

$$\begin{aligned} TM &= \text{Span}\big\{\partial_0^x = u_0,\ \partial_1^x = u_1 + \partial_1^x f\ w_1 + \partial_1^x h\ w_2, ..., \\ &\quad \partial_p^x = u_p + \partial_p^x f\ w_1 + \partial_p^x h\ w_2, \partial_0^y = v_0,\ \partial_1^y = v_1, ..., \partial_p^y = v_p\big\}, \end{aligned} \tag{4.14}$$

where $\partial_i^x = \frac{\partial}{\partial x_i}$ and $\partial_i^y = \frac{\partial}{\partial y_i}$.
The radical distribution $RadTM$ of rank 2 is given by

$$RadTM = \text{Span}\big\{\xi_1 = \partial_0^x - \sum_{i=1}^{p} \partial_i^x f\ \partial_i^y\ ,\ \xi_2 = \partial_0^y - \sum_{i=1}^{p} \partial_i^x h\ \partial_i^y\big\}. \tag{4.15}$$

M is a coisotropic submanifold of a semi-Euclidean space $\overline{M}$. The lightlike transversal vector bundle $ltr(TM)$ of M is given by

$$ltr(TM) = Span\big\{N_1 = w_1 - \frac{1}{2}\xi_1\ ,\ N_2 = w_2 - \frac{1}{2}\xi_2\big\}. \tag{4.16}$$

The corresponding screen distribution $S(TM)$ for the above $ltr(TM)$ is given by

$$S(TM) = \big\{U_1, ..., U_p, V_1, ..., V_p\big\}, \tag{4.17}$$

where

$$U_i = \partial_i^x - \partial_i^x f\ \partial_0^x - \partial_i^x h\ \partial_0^y \quad \text{and} \quad V_i = \partial_i^y.$$

FACT 2. Let's consider on $\overline{M}$ a local field of frames $\{\xi_1, \xi_2, U_i, V_i, N_1, N_2\}_{1\leqslant i\leqslant p}$ such that $\{\xi_1, \xi_2, U_i, V_i\}_{1\leqslant i\leqslant p}$ is a local field of frames on M with respect to the decomposition (2.1). Using the metric $\overline{g}$, the only non-vanishing components of the Levi-Civita connection $\overline{\nabla}$ are, for any i, $1 \leqslant i \leqslant p$,

$$\begin{aligned}
&\overline{\nabla}_{U_i} N_1 = \frac{1}{2}\sum_{j=1}^{p}(\partial_i^x \partial_j^x f)\ V_j, \qquad \overline{\nabla}_{U_i} N_2 = \frac{1}{2}\sum_{j=1}^{p}(\partial_i^x \partial_j^x h)\ V_j,\\
&\overline{\nabla}_{U_i}\xi_1 = -\sum_{j=1}^{p}(\partial_i^x \partial_j^x f)\ V_j, \quad \text{and} \quad \overline{\nabla}_{U_i}\xi_2 = -\sum_{j=1}^{p}(\partial_i^x \partial_j^x h)\ V_j.
\end{aligned} \tag{4.18}$$

Thus, since $h_i^l(X,Y) = -\overline{g}(\overline{\nabla}_X \xi_i, Y)$, we obtain that the non-vanishing components of h^l are

$$h_1^l(U_i, U_j) = \partial_i^x \partial_j^x f, \quad h_2^l(U_i, U_j) = \partial_i^x \partial_j^x h,\ 1 \leqslant i, j \leqslant p. \tag{4.19}$$

Also, by straightforward calculation, using the Gauss equation, we obtain that the only non-vanishing components of induced connection ∇ on M are

$$\begin{aligned}
&\nabla_{U_i} U_j = -(\frac{1}{2}\partial_i^x \partial_j^x f)\ \xi_1 - (\frac{1}{2}\partial_i^x \partial_j^x h)\ \xi_2 - \sum_{k=1}^{p}\big(\partial_i^x \partial_j^x f\ \partial_k^x f + \partial_i^x \partial_j^x h\ \partial_k^x h\big) V_k,\\
&\nabla_{U_i}\xi_1 = -\sum_{j=1}^{p}(\partial_i^x \partial_j^x f)\ V_j, \quad \nabla_{U_i}\xi_2 = -\sum_{j=1}^{p}(\partial_i^x \partial_j^x h)\ V_j, \quad 1 \leqslant i, j \leqslant p.
\end{aligned} \tag{4.20}$$

FACT 3. Using relations (4.18) and (4.20), we obtain that the non vanishing values of operators A_{N_i} and $\overset{*}{A}_{\xi_i}$ are

$$\begin{aligned}
&A_{N_1} U_i = -\frac{1}{2}\sum_{j=1}^{p}(\partial_i^x \partial_j^x f)\ V_j, \qquad A_{N_2} U_i = -\frac{1}{2}\sum_{j=1}^{p}(\partial_i^x \partial_j^x h)\ V_j,\\
&\overset{*}{A}_{\xi_1}\ U_i = \sum_{j=1}^{p}(\partial_i^x \partial_j^x f)\ V_j, \quad \text{and} \quad \overset{*}{A}_{\xi_2}\ U_i = \sum_{j=1}^{p}(\partial_i^x \partial_j^x h)\ V_j,\ 1 \leqslant i \leqslant P.
\end{aligned} \tag{4.21}$$

Thus, we infer that

$$A_{N_1} = -\frac{1}{2}\ \overset{*}{A}_{\xi_1} \quad \text{and} \quad A_{N_2} = -\frac{1}{2}\ \overset{*}{A}_{\xi_2} \tag{4.22}$$

FACT 4. Note the local field of frames on M by $\{\partial_{a'}\}_{1\leqslant a'\leqslant 2p+2} \equiv \{\xi_1, \xi_2, U_i, V_i\}_{1\leqslant i\leqslant p}$. By direct calculation, using relations (4.21) we have

$$\begin{aligned}
tr(\overset{*}{A}_{\xi_1} A_{N_1}) &= \sum_{a',b'=1}^{2p+2} \tilde{g}^{a'b'}_{(f,h)} \tilde{g}_{(f,h)}(\overset{*}{A}_{\xi_1} A_{N_1}\partial_{a'}, \partial_{b'}) \\
&= \sum_{i,j=1}^{p} \tilde{g}^{-1}_{(f,h)}(U_i, U_j) g_{(f,h)}(\overset{*}{A}_{\xi_1} A_{N_1} U_i, U_j) \\
&= -\frac{1}{2} \sum_{i,j,k=1}^{p} \tilde{g}^{-1}_{(f,h)}(U_i, U_j)(\partial^x_i \partial^x_k f) g_{(f,h)}(\overset{*}{A}_{\xi_1} V_k, U_j) \\
&= 0.
\end{aligned}$$

Likewise, we have $tr(\overset{*}{A}_{\xi_2} A_{N_2}) = 0$.
Also,

$$\begin{aligned}
tr A_{N_1} &= \sum_{a',b'=1}^{2p+2} \tilde{g}^{a'b'}_{(f,h)} \tilde{g}_{(f,h)}(A_{N_1}\partial_{a'}, \partial_{b'}) \\
&= \sum_{i,j=1}^{p} \tilde{g}^{-1}_{(f,h)}(U_i, U_j) g_{(f,h)}(A_{N_1} U_i, U_j) \\
&= -\frac{1}{2} \sum_{i,j,k=1}^{p} \tilde{g}^{-1}_{(f,h)}(U_i, U_j)(\partial^x_i \partial^x_k f) g_{(f,h)}(V_k, U_j) \\
&= -\frac{1}{2} \sum_{i,j=1}^{p} \tilde{g}^{-1}_{(f,h)}(U_i, U_j)\partial^x_i \partial^x_j f.
\end{aligned}$$

Likewise, we have

$$tr A_{N_2} = -\frac{1}{2} \sum_{i,j=1}^{p} \tilde{g}^{-1}_{(f,h)}(U_i, U_j)\partial^x_i \partial^x_j h.$$

By using (4.6) and (4.19), we have

$$\sigma^l_1 = \frac{1}{(2p+2)\sqrt{2}} \sum_{i,j=1}^{p} \tilde{g}^{-1}_{(f,h)}(U_i, U_j) h^l_1(U_i, U_j) = \frac{1}{(2p+2)\sqrt{2}} \sum_{i,j=1}^{p} \tilde{g}^{-1}_{(f,h)}(U_i, U_j)\partial^x_i \partial^x_j f$$

$$\text{and} \quad \sigma^l_2 = \frac{1}{(2p+2)\sqrt{2}} \sum_{i,j=1}^{p} \tilde{g}^{-1}_{(f,h)}(U_i, U_j)\partial^x_i \partial^x_j h.$$

Thus, by using (4.11), we obtain finally

$$S = -\frac{1}{2}\{\big(\sum_{i,j=1}^{p} \tilde{g}^{ij}_{(f,h)}\partial^x_i \partial^x_j f\big)^2 + \big(\sum_{i,j=1}^{p} \tilde{g}^{ij}_{(f,h)}\partial^x_i \partial^x_j h\big)^2\}, \tag{4.23}$$

where $\tilde{g}^{ij}_{(f,h)} = \tilde{g}^{-1}_{(f,h)}(U_i, U_j)$.

5. Locally symmetric lightlike submanifolds

In this section we study the lightlike submanifolds verifying local symmetry property. We establish that, under certain conditions, the locally symmetric coisotropic submanifolds of indefinite spaces form are totally geodesic.

A lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$ of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is said to be locally symmetric, if and only if for any $X, Y, Z, T, V_1 \in \Gamma(TM)$ and $N \in \Gamma(ltr(TM))$ the following hold (see [8])

$$g((\nabla_{V_1} R)(X,Y)Z, PT) = 0 \quad \text{and} \quad \overline{g}((\nabla_{V_1} R)(X,Y)Z, N) = 0. \tag{5.1}$$

This condition is equivalent to $(\nabla_{V_1} R)(X,Y)Z = 0$.

Let's consider the lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$ of a semi-Riemannian manifold $(\overline{M}(c), \overline{g})$ with a constant sectional curvature c. The induced Riemann curvature tensor on M is given by, for any $X, Y, Z \in \Gamma(TM)$,

$$\begin{aligned} R(X,Y)Z &= c\{g(Y,Z)X - g(X,Z)Y\} \\ &\quad + A_{h^l(Y,Z)}X - A_{h^l(X,Z)}Y + A_{h^s(Y,Z)}X - A_{h^s(X,Z)}Y. \end{aligned} \tag{5.2}$$

By straightforward calculation, using (5.2) and (2.17), we have, for any $V_1, X, Y, Z \in \Gamma(TM)$,

$$\begin{aligned} (\nabla_{V_1} R)(X,Y)Z &= c\overline{g}(h^l(V_1,Y),Z)X + c\overline{g}(h^l(V_1,Z),Y)X - c\overline{g}(h^l(V_1,X),Z)Y \\ &\quad - c\overline{g}(h^l(V_1,Z),X)Y + (\nabla_{V_1} A)_{h^l(Y,Z)}X + (\nabla_{V_1} A)_{h^s(Y,Z)}X \\ &\quad - (\nabla_{V_1} A)_{h^l(X,Z)}Y - (\nabla_{V_1} A)_{h^s(X,Z)}Y + A_{(\nabla_{V_1} h^l)(Y,Z)}X \\ &\quad + A_{(\nabla_{V_1} h^s)(Y,Z)}X - A_{(\nabla_{V_1} h^l)(X,Z)}Y - A_{(\nabla_{V_1} h^s)(X,Z)}Y. \end{aligned} \tag{5.3}$$

In the following, we investigate the effect of local symmetry condition on geometry of lightlike submanifolds of indefinite spaces form.

Theorem 5.1 *Let $(M, g, S(TM), S(TM^{\perp}))$ be a locally symmetric lightlike submanifold of an indefinite space form $(\overline{M}(c), \overline{g})$, such that the radical distribution $RadTM$ is integrable and ∇^t is a metric linear connection on $tr(TM)$. Suppose that $RadTM = Span\{\xi_i\}$, $ltr(TM) = Span\{N_i\}$ and the following conditions are verified*

$$A_{N_j}\xi_j \in \Gamma(S(TM)) \quad \text{and} \quad h^s(X,\xi) = 0, \quad \forall \xi \in \Gamma(RadTM), \ X \in \Gamma(TM).$$

Then the lightlike second fundamental form h^l vanishes identically on M.

Proof By taking $Y = Z = \xi_j$ into (5.3) and using assumptions, we obtain that

$$\begin{aligned} \overline{g}((\nabla_{V_1} R)(X,\xi_j)\xi_j, N_j) &= -c\overline{g}(h^l(V_1,X),\xi_j) + \overline{g}(A_{h^l(X,\nabla_{V_1}\xi_j)}\xi_j, N_j) \\ &= -c\overline{g}(h^l(V_1,X),\xi_j) - \overline{g}(A_{N_j}\xi_j, h^l(X,\nabla_{V_1}\xi_j)) \\ &= -c\overline{g}(h^l(V_1,X),\xi_j). \end{aligned} \tag{5.4}$$

If M is locally symmetric, we infer from (5.4) that $h^l(V_1, X) = 0, \ \forall V_1, X \in \Gamma(TM)$. □

For the coisotropic manifolds, we have the following theorem.

Theorem 5.2 *Let $(M, g, S(TM))$ be a coisotropic submanifold of an indefinite space form $(\overline{M}(c), \overline{g})$, such that the radical distribution $RadTM = TM^{\perp}$ is integrable. Suppose that $TM^{\perp} = Span\{\xi_i\}$, $ltr(TM) = Span\{N_i\}$ and $\forall j = 1, ..., n$, $A_{N_j}\xi_j \in \Gamma(S(TM))$. Then M is locally symmetric if and only if it is totally geodesic.*

Proof If M is totally geodesic, since $R = \overline{R}_{|TM}$, we obtain that

$$(\nabla_{V_1} R)(X,Y)Z = (\overline{\nabla}_{V_1}\overline{R})(X,Y)Z = 0,$$

for any $V_1, X, Y, Z \in \Gamma(TM)$. The converse is obtained by virtue of Theorem 5.1. □

Corollary 5.3 *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite space form $(\overline{M}(c), \overline{g})$. Then M is locally symmetric if and only if it is totally geodesic.*

6. Semi-symmetric lightlike submanifolds

In this section we deal with semi-symmetric submanifolds in semi-Riemannian manifolds of constant sectional curvature. We consider curvature operator on a smooth manifold defined by

$$R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}. \tag{6.1}$$

A lightlike submanifold $(M, g, S(TM), S(TM^\perp))$ of a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is said to be semi-symmetric if the following condition is satisfied (see [10])

$$(R(V_1,V_2)\cdot R)(X,Y,Z,W) = 0 \quad \forall V_1, V_2, X, Y, Z, W \in \Gamma(TM) \tag{6.2}$$

where R is the induced Riemann curvature on M. This is equivalent to

$$-R(R(V_1,V_2)X,Y,Z,W) - \ldots - R(X,Y,Z,R(V_1,V_2)W) = 0.$$

In general the condition (6.2) is not equivalent to $(R(V_1,V_2)\cdot R)(X,Y)Z = 0$ as in the non-degenerate setting. Indeed, by direct calculation we have for any $V_1, V_2, X, Y, Z, W \in \Gamma(TM)$,

$$\begin{aligned}(R(V_1,V_2)\cdot R)(X,Y,Z,W) = \\ g((R(V_1,V_2)\cdot R)(X,Y)Z, W) + (R(V_1,V_2).g)(R(X,Y)Z,W).\end{aligned} \tag{6.3}$$

Now, let's consider $(M, g, S(TM), S(TM^\perp)$, an m-dimensional lightlike submanifold of an $(m+n)$-dimensional indefinite space form $(\overline{M}(c), \overline{g})$ with $Rank(RadTM) = r \leqslant min\{m,n\}$ and suppose that the distribution $RadTM$ is integrable and ∇^t is a metric linear connection on $tr(TM)$. Since the Ricci tensor on $\overline{M}(c)$ is given by $\overline{R}ic(X,Y) = (m+n-1)c\overline{g}(X,Y)$, from relation (3.1), we obtain that, for any $X, Y \in \Gamma(TM)$,

$$\begin{aligned}Ric(X,Y) &= (m-1)cg(X,Y) + \sum_{j=1}^{r} h^l_j(X,Y)trA_{N_j} - \sum_{j=1}^{r} g(A_{N_j}X, \overset{*}{A}_{\xi_j} Y) \\ &\quad + \sum_{\alpha=r+1}^{n} h^s_\alpha(X,Y)trA_{W_\alpha} - \sum_{\alpha=r+1}^{n} g(A_{W_\alpha}X, A_{W_\alpha}Y).\end{aligned} \tag{6.4}$$

In the following, we investigate the effect of semi-symmetry condition on geometry of lightlike submanifolds of indefinite spaces form.

Theorem 6.1 *Let $(M, g, S(TM), S(TM^{\perp}))$ be a semi-symmetric lightlike submanifold of an indefinite space form $(\overline{M}(c), \overline{g})$, such that the radical distribution $RadTM$ is integrable and ∇^t is a metric linear connection on $tr(TM)$. Suppose that the following conditions are verified:*
(1) $h^s(X, \xi) = 0$ and $h^l(A_V\xi, X) = 0$, $\forall \xi \in \Gamma(RadTM),\ X \in \Gamma(TM),\ V \in \Gamma(tr(TM))$,
(2) $g(A_N\xi, A_{N'}\xi) \neq 0$, $\forall \xi \in \Gamma(RadTM) - \{0\},\ N, N' \in \Gamma(ltr(TM)) - \{0\}$.
Then the lightlike second fundamental form h^l vanishes identically on M.

Proof Let's suppose that M is an m-dimensional lightlike submanifold of an $(m+n)$-dimensional space form $\overline{M}(c)$ with $Rank(RadTM) = r \leqslant min\{m, n\}$. By straightforward calculation, using (5.2), we have, for any $X, Y, Z, T \in \Gamma(TM),\ \xi \in \Gamma(RadTM)$,

$$\begin{aligned}(R(\xi, X)\cdot R)(\xi, Y, Z, T) \;=\;& -\sum_{j=1}^{r} h_j^l(A_{h^l(X,Y)}\xi, Z)g(A_{N_j}\xi, T) - \sum_{\alpha=1}^{n} h_\alpha^s(A_{h^l(X,Y)}\xi, Z)g(A_{N_j}\xi, PT)\\ & -\sum_{j=1}^{r} h_j^l(A_{h^s(X,Y)}\xi, Z)g(A_{N_j}\xi, T) - \sum_{\alpha=1}^{n} h_\alpha^s(A_{h^s(X,Y)}\xi, Z)g(A_{N_j}\xi, PT)\\ & -\sum_{j=1}^{r} h_j^l(Y, A_{h^l(X,Z)}\xi)g(A_{N_j}\xi, T) - \sum_{\alpha=1}^{n} h_\alpha^s(Y, A_{h^l(X,Z)}\xi)g(A_{N_j}\xi, PT)\\ & -\sum_{j=1}^{r} h_j^l(Y, A_{h^s(X,Z)}\xi)g(A_{N_j}\xi, T) - \sum_{\alpha=1}^{n} h_\alpha^s(Y, A_{h^s(X,Z)}\xi)g(A_{N_j}\xi, PT)\\ & -g(A_{h^l(Y,Z)}\xi, A_{h^l(X,T)}\xi) - g(A_{h^s(Y,Z)}\xi, PA_{h^l(X,T)}\xi)\\ & -g(PA_{h^l(Y,Z)}\xi, A_{h^s(X,T)}\xi) - g(A_{h^s(Y,Z)}\xi, A_{h^s(X,T)}\xi) \qquad (6.5)\end{aligned}$$

In virtue of assumption, using (2.9) and since $h = h^l + h^s$, we obtain

$$\begin{aligned}(R(\xi, X)\cdot R)(\xi, Y, Z, T) \;=\;& -\sum_{j=1}^{r} h_j^l(A_{h(X,Y)}\xi, Z)g(A_{N_j}\xi, T)\\ & -\sum_{j=1}^{r} h_j^l(Y, A_{h(X,Z)}\xi)g(A_{N_j}\xi, T) - g(A_{h^l(Y,Z)}\xi, A_{h^l(X,T)}\xi)\\ =\;& -g(A_{h^l(Y,Z)}\xi, A_{h^l(X,T)}\xi). \qquad (6.6)\end{aligned}$$

Thus, if M is semi-symmetric, by taking $X = Z$ and $Y = T$ into (6.6), we obtain
$g(A_{h^l(X,Y)}\xi, A_{h^l(X,Y)}\xi) = 0$, that is $h^l(X, Y) = 0,\ \ \forall X, Y \in \Gamma(TM)$. □
For the coisotropic submanifolds, we have the following result.

Theorem 6.2 *Let $(M, g, S(TM))$ be a coisotropic submanifold of an indefinite space form $(\overline{M}(c), \overline{g})$, such that the radical distribution $RadTM = TM^{\perp}$ is integrable. Suppose that the following conditions are verified*
(1) $h^l(A_N\xi, X) = 0$, $\forall \xi \in \Gamma(TM^{\perp}),\ X \in \Gamma(TM),\ N \in \Gamma(ltr(TM))$;
(2) $g(A_N\xi, A_{N'}\xi) \neq 0$, $\forall \xi \in \Gamma(TM^{\perp}) - \{0\},\ N, N' \in \Gamma(ltr(TM)) - \{0\}$.
Then M is semi-symmetric if and only if it is totally geodesic.

Proof If M is totally geodesic, since $R = \overline{R}_{|TM}$, we obtain, for any $V_1, V_2, X, Y, Z, T \in \Gamma(TM)$, $(R(V_1, V_2)\cdot R)(X, Y, Z, T) = (\overline{R}(V_1, V_2)\cdot \overline{R})(X, Y, Z, T) = 0$. The converse is obtained in virtue of Theorem 6.1. □

For the lightlike hypersurfaces, since $h^l_j(A_N\xi, X) = B(A_N\xi, X) = -Ric(\xi, X)$, where B is the local second fundamental form of M. So, we obtain the following result.

Corollary 6.3 *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite space form $(\overline{M}(c), \overline{g})$, such that the following conditions are verified:*
(1) $Ric(\xi, X) = 0$, $\forall \xi \in \Gamma(TM^{\perp})$, $X \in \Gamma(TM)$;
(2) $A_N\xi$ is a non-null vector field.
Then M is semi-symmetric if and only if it is totally geodesic.

7. Ricci semi-symmetric lightlike submanifolds

In this section, we study Ricci semi-symmetric lightlike submanifolds of semi-Riemannian manifolds of constant sectional curvature. We prove that Ricci semi-symmetric coisotropic submanifolds are totally geodesic under some condition.

A lightlike submanifold M of a semi-Riemannian manifold $\overline{M}$ is said to be Ricci semi-symmetric if the following is satisfied

$$(R(V_1, V_2) \cdot Ric)(X, Y) = 0, \quad \forall V_1, V_2, X, Y \in \Gamma(TM), \tag{7.1}$$

where R and Ric are induced Riemann curvature and Ricci tensor on M, respectively. The latter condition is equivalent to

$$-Ric(R(V_1, V_2)X, Y) - Ric(X, R(V_1, V_2)Y) = 0$$

Now, let's consider M, an m-dimensional lightlike submanifold of an $(m+n)$-dimensional indefinite space form $\overline{M}(c)$ with $Rank(RadTM) = r \leqslant min\{m, n\}$ and suppose that the distribution $RadTM$ is integrable and ∇^t is a metric linear connection on $tr(TM)$. Since $h^l(X, \xi) = 0$, $\forall \xi \in \Gamma(RadTM)$ and for any $W \in \Gamma(S(TM^{\perp}))$, A_W is $S(TM)$-valued, by straightforward calculation, using (5.2) and (6.4), we obtain that, for any $\xi \in \Gamma(RadTM)$, $X, Y \in \Gamma(TM)$,

$$\begin{aligned} Ric(R(\xi, X)\xi, Y) = {} & (m-1)cg(A_{h^s(X,\xi)}\xi, Y) + \sum_{j=1}^{r} h^l_j(A_{h^s(X,\xi)}\xi, Y)trA_{N_j} \\ & - \sum_{j=1}^{r} g(A_{N_j}A_{h^s(X,\xi)}\xi, \overset{*}{A}_{\xi_j} Y) + \sum_{\alpha=r+1}^{n} h^s_\alpha(A_{h^s(X,\xi)}\xi, Y)trA_{W_\alpha} \\ & - \sum_{\alpha=r+1}^{n} g(A_{W_\alpha}A_{h^s(X,\xi)}\xi, A_{W_\alpha}Y) - (m-1)cg(A_{h^s(\xi,\xi)}X, Y) \\ & - \sum_{j=1}^{r} h^l_j(A_{h^s(\xi,\xi)}X, Y)trA_{N_j} + \sum_{j=1}^{r} g(A_{N_j}A_{h^s(\xi,\xi)}X, \overset{*}{A}_{\xi_j} Y) \\ & - \sum_{\alpha=r+1}^{n} h^s_\alpha(A_{h^s(\xi,\xi)}X, Y)trA_{W_\alpha} + \sum_{\alpha=r+1}^{n} g(A_{W_\alpha}A_{h^s(\xi,\xi)}X, A_{W_\alpha}Y). \end{aligned} \tag{7.2}$$

Also,

$$Ric(\xi, R(\xi, X)Y) = cg(X, Y)\{\sum_{\alpha=r+1}^{n} h^s_\alpha(\xi, \xi)trA_{W_\alpha} - \sum_{\alpha=r+1}^{n} g(A_{W_\alpha}\xi, A_{W_\alpha}\xi)\}$$

$$
\begin{aligned}
&-\sum_{j=1}^{r} g(A_{N_j}\xi, \overset{*}{A}_{\xi_j}\, A_{h^l(X,Y)}\xi) + \sum_{\alpha=r+1}^{n} h^s_\alpha(\xi, A_{h^l(X,Y)}\xi) trA_{W_\alpha}\\
&-\sum_{\alpha=r+1}^{n} g(A_{W_\alpha}\xi, A_{W_\alpha}A_{h^l(X,Y)}\xi) - \sum_{j=1}^{r} g(A_{N_j}\xi, \overset{*}{A}_{\xi_j}\, A_{h^s(X,Y)}\xi)\\
&+\sum_{\alpha=r+1}^{n} h^s_\alpha(\xi, A_{h^s(X,Y)}\xi) trA_{W_\alpha} - \sum_{\alpha=r+1}^{n} g(A_{W_\alpha}\xi, A_{W_\alpha}A_{h^s(X,Y)}\xi)\\
&+\sum_{j=1}^{r} g(A_{N_j}\xi, \overset{*}{A}_{\xi_j}\, A_{h^s(\xi,Y)}X) - \sum_{\alpha=r+1}^{n} h^s_\alpha(\xi, A_{h^s(\xi,Y)}X) trA_{W_\alpha}\\
&+\sum_{\alpha=r+1}^{n} g(A_{W_\alpha}\xi, A_{W_\alpha}A_{h^s(\xi,Y)}X). \qquad (7.3)
\end{aligned}
$$

In the following theorem, we give result which shows the effect of Ricci semi-symmetric condition on the geometry of lightlike submanifolds of indefinite spaces form.

Theorem 7.1 *Let $(M, g, S(TM), S(TM^\perp))$ be a Ricci semi-symmetric lightlike submanifold of an indefinite space form $(\overline{M}(c), \overline{g})$, such that the radical distribution $RadTM$ is integrable and ∇^t is a metric linear connection on $tr(TM)$.*
If $h^s(X,\xi) = 0$, $\forall \xi \in \Gamma(RadTM), X \in \Gamma(TM)$, then at least one of the following holds:

(i) h^l vanishes identically on M,
(ii) $Ric(\xi, A_N\xi) = 0$, for any $\xi \in \Gamma(RadTM)$, $N \in \Gamma(ltr(TM))$,
where Ric is the induced Ricci tensor on M.

Proof Let's suppose that M is an m-dimensional Ricci semi-symmetric lightlike submanifold of an $(m+n)$-dimensional space form $\overline{M}(c)$ with $Rank(RadTM) = r \leqslant min\{m, n\}$. Since $h^s(X,\xi) = 0$, by using relations (2.9), (7.2) and (7.3), we have, for any $X, Y \in \Gamma(TM)$,

$$
\begin{aligned}
(R(\xi,X)\cdot Ric)(\xi,Y) &= -Ric(R(\xi,X)\xi, Y) - Ric(\xi, R(\xi,X)Y)\\
&= cg(X,Y)\sum_{\alpha=r+1}^{n} g(A_{W_\alpha}\xi, A_{W_\alpha}\xi) + \sum_{j=1}^{r} g(A_{N_j}\xi, \overset{*}{A}_{\xi_j}\, A_{h^l(X,Y)}\xi)\\
&\quad -\sum_{\alpha=r+1}^{n} h^s_\alpha(\xi, A_{h^l(X,Y)}\xi) trA_{W_\alpha} + \sum_{\alpha=r+1}^{n} g(A_{W_\alpha}\xi, A_{W_\alpha}A_{h^l(X,Y)}\xi)\\
&\quad +\sum_{j=1}^{r} g(\overset{*}{A}_{\xi_j}\, A_{N_j}\xi, A_{h^s(X,Y)}\xi) - \sum_{\alpha=r+1}^{n} h^s_\alpha(\xi, A_{h^s(X,Y)}\xi) trA_{W_\alpha}\\
&\quad +\sum_{\alpha=r+1}^{n} g(A_{W_\alpha}\xi, A_{W_\alpha}A_{h^s(X,Y)}\xi)\\
&= \sum_{j=1}^{r} g(A_{N_j}\xi, \overset{*}{A}_{\xi_j}\, A_{h^l(X,Y)}\xi). \qquad (7.4)
\end{aligned}
$$

From (7.4), using relations (2.13) and (6.4), we obtain,

$$0=\sum_{j=1}^{r}h_j^l(A_{N_j}\xi, A_{h^l(X,Y)}\xi)=\sum_{i=1}^{r}\sum_{j=1}^{r}h_j^l(A_{N_j}\xi, A_{N_i}\xi)h_i^l(X,Y)$$

$$=-\sum_{i=1}^{r}Ric(\xi, A_{N_i}\xi)h_i^l(X,Y)=-\overline{g}(h^l(X,Y), \sum_{i=1}^{r}Ric(\xi, A_{N_i}\xi)\xi_i).$$

Since $\overline{g}$ is non-degenerate, we infer that $h^l=0$ or $Ric(\xi, A_N\xi)=0$. □

For the coisotropic submanifold, we have the following result.

Theorem 7.2 *Let $(M, g, S(TM))$ be a coisotropic submanifold of an indefinite space form $(\overline{M}(c), \overline{g})$, such that the radical distribution $RadTM = TM^{\perp}$ is integrable and $Ric(\xi, A_N\xi)\neq 0$, for any $\xi \in \Gamma(TM^{\perp})$, $N\in\Gamma(ltr(TM))$. Then M is Ricci semi-symmetric if and only if it is totally geodesic.*

Proof If M is totally geodesic, since $R=\overline{R}_{|TM}$, we obtain $(R(V_1,V_2)\cdot Ric)(X,Y)=(\overline{R}(V_1,V_2)\cdot\overline{R}ic)(X,Y)=0$, for any $V_1, V_2, X, Y\in\Gamma(TM)$. The converse is obtained by virtue of Theorem 7.1. □

For the lightlike hypersurface, we have the following.

Corollary 7.3 *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite space form $(\overline{M}(c), \overline{g})$, such that $Ric(\xi, A_N\xi)\neq 0$, for any $\xi\in\Gamma(TM^{\perp})$, $N\in\Gamma(ltr(TM))$. Then M is Ricci semi-symmetric if and only if it is totally geodesic.*

References

[1] Atindogbe, C.: *Scalar curvature on lightlike hypersurfaces.* Applied Sciences, 11, 9–18 (2009).

[2] Atindogbe, C., Ezin, J.P., Tossa J.: *Pseudo-inversion of degenerate metrics.* Int. J. of Mathematical Sciences, 55, 3479–3501 (2003).

[3] Atindogbe, C., Lungiambudila O., Tossa, J.: *Lightlike Osserman submanifolds of semi-Riemannian manifolds.* Afrika Matematika, 22, 129–151 (2011).

[4] Duggal, K.L.: *Constant scalar curvature and warped product globally null manifolds.* J. Geom. Phys. 43 (4), 327–340 (2002).

[5] Duggal, K.L.: *On scalar curvature in lightlike geometry.* J. Geom. Phy., 57, 473–481 (2007).

[6] Duggal, K.L., Bejancu, A.: *Lightlike submanifolds of semi-Riemannian manifolds and applications.* Kluwer Academic Publishers, Amsterdam, 1996.

[7] Duggal, K.L., Sahin, B.: *Differential geometry of lightlike submanifolds.* Birkhäuser Verlag AG, Germany, 2010.

[8] Günes, R., Sahin B., Kiliç, E.: *On lightlike hypersurfaces of semi-Riemannian space form.* Turk J. Math 27, 283–297 (2003).

[9] Kupeli, D.N.: *Singular Semi-Riemannian Geometry.* vol. 366 of Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.

[10] Sahin, B.: *Lightlike hypersurfaces of semi-Euclidean spaces satisfying curvature conditions of semi-symmetry.* Turk J. Math 31, 139–162 (2007).

[11] Szabo, Z.I.: *Structure theorem on Riemannian spaces satisfying $R(X,Y)\cdot R=0$, I: The local version.* J. Differential Geom. 17, 531–582 (1982).

[12] Szabo, Z.I.: *Structure theorem on Riemannian spaces satisfying $R(X,Y)\cdot R=0$, II: The global version.* Geom. Dedicata 19, 65–108 (1985).