Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2013) 37: 114 – 128
© TÜBİTAK
doi:10.3906/mat-1103-10

Paracontact semi-Riemannian submersions

Yılmaz GÜNDÜZALP[1], **Bayram ŞAHİN**[2,*]
[1]Faculty of Education, Dicle University, Diyarbakır, Turkey
[2]Department of Mathematics, İnönü University, 44280, Malatya, Turkey

Received: 04.03.2011 • **Accepted:** 08.10.2011 • **Published Online:** 17.12.2012 • **Printed:** 14.01.2013

Abstract: In this paper, we first define the concept of paracontact semi-Riemannian submersions between almost paracontact metric manifolds, then we provide an example and show that the vertical and horizontal distributions of such submersions are invariant with respect to the almost paracontact structure of the total manifold. The study is focused on fundamental properties and the transference of structures defined on the total manifold. Moreover, we obtain various properties of the O'Neill's tensors for such submersions and find the integrability of the horizontal distribution. We also find necessary and sufficient conditions for a paracontact semi-Riemannian submersion to be totally geodesic. Finally, we obtain curvature relations between the base manifold and the total manifold.

Key words: Almost paracontact metric manifold, semi-Riemannian submersion, paracontact semi-Riemannian submersion

1. Introduction

The theory of Riemannian submersion was introduced by O'Neill and Gray in [11] and [6], respectively. Presently, there is an extensive literature on the Riemannian submersions with different conditions imposed on the total space and on the fibres. Semi-Riemannian submersions were introduced by O'Neill in his book [12]. Later, Riemannian submersions were considered between almost complex manifolds by Watson in [13] under the name of almost Hermitian submersion. He showed that if the total manifold is a Kähler manifold, the base manifold is also a Kähler manifold. Riemannian submersions between almost contact manifolds were studied by Chinea in [3] under the name of almost contact submersions. Since then, Riemannian submersions have been used as an effective tool to describe the structure of a Riemannian manifold equipped with a differentiable structure. For instance, Riemannian submersions have been also considered for quaternionic Kähler manifolds [7],[8] and paraquaternionic Kähler manifolds [8]. On the other hand, in [9] Kaneyuki and Williams defined the almost paracontact structure on pseudo-Riemannian manifold M of dimension (2m+1) and constructed the almost paracomplex structure on $M^{2m+1} \times R$.

In this paper, we define paracontact semi-Riemannian submersions between almost paracontact metric manifolds and study the geometry of such submersions. We observe that paracontact semi-Riemannian submersion has also rich geometric properties.

This paper is organized as follows. In Section 2 we collect basic definitions, some formulas and results for later use. In section 3 we introduce the notion of paracontact semi-Riemannian submersions and give an

*Correspondence: bsahin@inonu.edu.tr
2010 *AMS Mathematics Subject Classification:* 53C15, 53C20, 53C50.

example of paracontact semi-Riemannian submersion. Moreover, we investigate properties of O'Neill's tensors and show that such tensors have nice algebraic properties for paracontact semi-Riemannian submersions. We find the integrability of the horizontal distribution. We also find necessary and sufficient conditions for the fibres of a paracontact semi-Riemannian submersion to be totally geodesic. In section 4, we obtain relations between bisectional curvatures and sectional curvatures of the base manifold, the total manifold and the fibres of a paracontact semi-Riemannian submersion.

2. Preliminaries

Let M be a $(2m+1)$-dimensional differentiable manifold. Let φ be a $(1,1)$-tensor field, ξ a vector field and η a 1-form on M. Then (φ, ξ, η) is called an almost paracontact structure on M if

(i) $\eta(\xi) = 1,\ \varphi^2 = Id - \eta \otimes \xi$.

(ii) the tensor field φ induces an almost paracomplex structure on the distribution $\mathcal{D} = \ker\eta$, that is, the eigendistributions $\mathcal{D}^+, \mathcal{D}^-$ corresponding to the eigenvalues 1, -1 of φ, respectively, have equal dimension m. M is said to be almost paracontact manifold if it is endowed with an almost paracontact structure ([9],[10],[15]).

Let M be an almost paracontact manifold. M will be called an almost paracontact metric manifold if it is additionally endowed with a pseudo-Riemannian metric g of signature $(m+1, m)$ such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \quad X, Y \in \chi(M). \tag{1}$$

For such a manifold, we additionally have $\eta(X) = g(X, \xi)$, $\varphi\xi = 0$, $\eta \circ \varphi = 0$. Moreover, we can define a skew-symmetric 2-form Φ by $\Phi(X, Y) = g(X, \varphi Y)$, which is called the fundamental form corresponding to the structure. Note that $\eta \wedge \Phi$ is, up to a constant factor, the Riemannian volume element of M.

On an almost paracontact manifold, one defines the $(2,1)$-tensor field $N^{(1)}$ by

$$N^{(1)}(X, Y) = [\varphi, \varphi](X, Y) - 2d\eta(X, Y)\xi, \tag{2}$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ given by

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]. \tag{3}$$

If $N^{(1)}$ vanishes identically, then the almost paracontact manifold (structure) is said to be normal ([15]). The normality condition says that the almost paracomplex structure J defined on $M \times R$ by

$$J(X, f\frac{d}{dt}) = (\varphi X + f\xi, \eta(X)\frac{d}{dt}) \tag{4}$$

is integrable.

We recall the defining relations of those which will be used in this study. An almost paracontact metric manifold $(M, g, \varphi, \xi, \eta)$ is called

(a) *normal*, if $N_\varphi - 2d\eta \otimes \xi = 0$;

(b) *paracontact*, if $\Phi = d\eta$;

(c) *K-paracontact*, if M is paracontact and ξ Killing;

(d) *para-cosymplectic,* if $\nabla\eta = 0$ and $\nabla\Phi = 0$;

(f) *almost para-cosymplectic,* if $d\eta = 0$ and $d\Phi = 0$;

(g) *weakly para-cosymplectic,* if M is almost para-cosymplectic and $[R(X,Y),\varphi] = R(X,Y)\varphi - \varphi R(X,Y) = 0$;

(h) *para-Sasakian,* if $\Phi = d\eta$ and M is normal;

(j) *quasi-para-Sasakian,* if $d\Phi = 0$ and M is normal ([4],[14],[15]).

We have the following relation between the Levi-Civita connection and fundamental 2-form of M.

Lemma 2.1 ([15]). For an almost paracontact metric manifold (M,φ,ξ,η,g),

$$\begin{aligned} 2g((\nabla_X\varphi)Y,Z) &= -d\Phi(X,Y,Z) - d\Phi(X,\varphi Y,\varphi Z) - N^{(1)}(Y,Z,\varphi X) \\ &+ N^{(2)}(Y,Z)\eta(X) - 2d\eta(\varphi Z,X)\eta(Y) + 2d\eta(\varphi Y,X)\eta(Z). \end{aligned} \tag{5}$$

If M is paracontact, then we have

$$\begin{aligned} 2g((\nabla_X\varphi)Y,Z) &= -N^{(1)}(Y,Z,\varphi X) - 2d\eta(\varphi Z,X)\eta(Y) \\ &+ 2d\eta(\varphi Y,X)\eta(Z). \end{aligned} \tag{6}$$

It is easy to see that if M is an almost paracontact metric manifold, then the following identities are well known:

$$(\nabla_X\varphi)Y = \nabla_X\varphi Y - \varphi(\nabla_X Y), \tag{7}$$

$$(\nabla_X\Phi)(Y,Z) = g(Y,(\nabla_X\varphi)Z), \tag{8}$$

$$(\nabla_X\eta)Y = g(Y,\nabla_X\xi), \tag{9}$$

$$N^2(X,Y) = (\mathcal{L}_{\varphi X}\eta)Y - (\mathcal{L}_{\varphi Y}\eta)X, \tag{10}$$

where $\mathcal{L}$ denotes the Lie derivative.

Let (M,g) and (B,g') be two connected semi-Riemannian manifolds of index $s(0 \le s \le \dim M)$ and $s'(0 \le s' \le \dim B)$ respectively, with $s > s'$. Roughly speaking, a semi-Riemannian submersion is a smooth map $\pi : M \to B$ which is onto and satisfies the following conditions:

(i) $\pi_{*p} : T_pM \to T_{\pi(p)}B$ is onto for all $p \in M$;

(ii) The fibres $\pi^{-1}(p'), p' \in B$, are semi-Riemannian submanifolds of M;

(iii) π_* preserves scalar products of vectors normal to fibres.

The vectors tangent to fibres are called vertical and those normal to fibres are called horizontal. We denote by $\mathcal{V}$ the vertical distribution, by $\mathcal{H}$ the horizontal distribution and by v and h the vertical and horizontal projection. An horizontal vector field X on M is said to be basic if X is π-related to a vector field X' on B. It is clear that every vector field X' on B has a unique horizontal lift X to M and X is basic.

We recall that the sections of $\mathcal{V}$, respectively $\mathcal{H}$, are called the vertical vector fields, respectively horizontal vector fields. A semi-Riemannian submersion $\pi : M \to B$ determines two $(1,2)$ tensor field T and A on M, by the formulas:

$$T(E,F) = T_E F = h\nabla_{vE}vF + v\nabla_{vE}hF \tag{11}$$

and

$$A(E,F) = A_E F = v\nabla_{hE} hF + h\nabla_{hE} vF \tag{12}$$

for any $E,F \in \Gamma(TM)$, where v and h are the vertical and horizontal projections (see [1],[5]). From (11) and (12), one can obtain

$$\nabla_U X = T_U X + h(\nabla_U X); \tag{13}$$

$$\nabla_X U = v(\nabla_X U) + A_X U; \tag{14}$$

$$\nabla_X Y = A_X Y + h(\nabla_X Y), \tag{15}$$

for any $X,Y \in \Gamma(\mathcal{H})$, $U \in \Gamma(\mathcal{V})$. Moreover, if X is basic then $h(\nabla_U X) = h(\nabla_X U) = A_X U$.

We note that for $U,V \in \Gamma(\mathcal{V})$, $T_U V$ coincides with the second fundamental form of the immersion of the fibre submanifolds and for $X,Y \in \Gamma(\mathcal{H})$, $A_X Y = \frac{1}{2}v[X,Y]$ reflecting the complete integrability of the horizontal distribution $\mathcal{H}$. It is known that A is alternating on the horizontal distribution: $A_X Y = -A_Y X$, for $X,Y \in \Gamma(\mathcal{H})$ and T is symmetric on the vertical distribution: $T_U V = T_V U$, for $U,V \in \Gamma(\mathcal{V})$.

We now recall the following result which will be useful for later.

Lemma 2.2 (see [5],[12]). *If $\pi : M \to B$ is a semi-Riemannian submersion and X,Y basic vector fields on M, π-related to X' and Y' on B, then we have the following properties:*

1. *$h[X,Y]$ is a basic vector field and $\pi_* h[X,Y] = [X',Y'] \circ \pi$;*
2. *$h(\nabla_X Y)$ is a basic vector field π-related to $(\nabla'_{X'} Y')$, where ∇ and ∇' are the Levi-Civita connection on M and B;*
3. *$[E,U] \in \Gamma(\mathcal{V})$, $\forall U \in \Gamma(\mathcal{V})$ and $\forall E \in \Gamma(TM)$.*

3. Paracontact semi-Riemannian submersions

In this section, we define the notion of paracontact semi-Riemannian submersion, give an example and study the geometry of such submersions. We now define a (φ,φ')-paraholomorphic map which is similar to the notion of a (φ,φ')-holomorphic map between two almost contact metric manifolds, for (φ,φ')-holomorphic map see: [5].

Definition 3.1 *Let M^{2m+1} and B^{2n+1} be manifolds carrying the almost paracontact metric manifolds structures (φ,ξ,η,g) and (φ',ξ',η',g') respectively. A mapping $\pi : M \to B$ is said to be a (φ,φ')-paraholomorphic map if $\pi_* \circ \varphi = \varphi' \circ \pi_*$.*

By using the above definition, we are ready to give the following notion.

Definition 3.2 *A semi-Riemannian submersion $\pi : M^{2m+1} \to B^{2n+1}$ between the almost paracontact metric manifolds M^{2m+1} and B^{2n+1} is called a paracontact semi-Riemannian submersion if:*

(i) $\pi_*\xi = \xi'$,

(ii) $\pi_* \circ \varphi = \varphi' \circ \pi_*$.

We give an example of a paracontact semi-Riemannian submersion.

Example 3.1 *Consider the following submersion defined by*

$$\begin{aligned} \pi : R_2^5 &\rightarrow R_1^3 \\ (x_1, x_2, y_1, y_2, z) &\rightarrow (\frac{x_1 + x_2}{\sqrt{2}}, \frac{y_1 + y_2}{\sqrt{2}}, z). \end{aligned}$$

Then, the kernel of π_* is

$$\mathcal{V} = \mathrm{Ker}\pi_* = \mathrm{Span}\{V_1 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, V_2 = -\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}\}$$

and the horizontal distribution is spanned by

$$\mathcal{H} = (\mathrm{Ker}\pi_*)^\perp = \mathrm{Span}\{X = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, Y = \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}, \xi = \frac{\partial}{\partial z}\}.$$

Hence, we have

$$g(X, X) = g'(\pi_* X, \pi_* X) = -4z, \quad g(Y, Y) = g'(\pi_* Y, \pi_* Y) = 4z$$

and

$$g(\xi, \xi) = g'(\pi_* \xi, \pi_* \xi) = 1.$$

Thus, π is a semi-Riemannian submersion. Moreover, we can easily obtain that π satisfies

$$\pi_* \xi = \xi'$$

and

$$\pi_* \varphi X = \varphi' \pi_* X, \quad \pi_* \varphi Y = \varphi' \pi_* Y.$$

Thus, π is a paracontact semi-Riemannian submersion.

By using Definition 3.1, we have the following result.

Proposition 3.1 *Let $\pi : M \rightarrow B$ be a paracontact semi-Riemannian submersion from an almost paracontact metric manifold M onto an almost paracontact metric manifold B, and let X be a basic vector field on M, π-related to X' on B. Then, φX is also a basic vector field π-related to $\varphi' X'$.*

The following result can be proved in a standard way.

Proposition 3.2 *Let $\pi : M \rightarrow B$ be a paracontact semi-Riemannian submersion from an almost paracontact metric manifold M onto an almost paracontact metric manifold B. If X, Y are basic vector fields on M, π-related to X', Y' on B, then, we have*

(i) *$h(\nabla_X \varphi)Y$ is the basic vector field π-related to $(\nabla'_{X'} \varphi')Y'$;*

(ii) *$h[X, Y]$ is the basic vector field π-related to $[X', Y']$.*

Next proposition shows that a paracontact semi-Riemannian submersion puts some restrictions on the distributions $\mathcal{V}$ and $\mathcal{H}$.

Proposition 3.3 *Let $\pi : M \to B$ be a paracontact semi-Riemannian submersion from an almost paracontact metric manifold M onto an almost paracontact metric manifold B. Then, the horizontal and vertical distributions are φ-invariant.*

Proof Consider a vertical vector field U; it is known that $\pi_*(\varphi U) = \varphi'(\pi_* U)$. Since U is vertical and π is a semi-Riemannian submersion, we have $\pi_* U = 0$ from which $\pi_*(\varphi U) = 0$ follows and implies that φU is vertical, being in the kernel of π_*. As concerns the horizontal distribution, let X be a horizontal vector field. We have $g(\varphi X, U) = -g(X, \varphi U) = 0$ because φU is vertical and X is horizontal. From $g(\varphi X, U) = 0$ we deduce that φX is orthogonal to U and then φX is horizontal. □

Proposition 3.4 *Let $\pi : M \to B$ be a paracontact semi-Riemannian submersion from an almost paracontact metric manifold M onto an almost paracontact metric manifold B. Then, we have*

(i) $\pi^*\Phi' = \Phi$; holds on the horizontal distribution.

(ii) $\pi^*\eta' = \eta$.

Proof (i) If X and Y are basic vector fields on M π-related to X', Y' on B, then using the definition of a paracontact semi-Riemannian submersion, we have

$$\begin{aligned}(\pi^*\Phi')(X,Y) &= \Phi'(\pi_*X, \pi_*Y) = g'(\pi_*X, \varphi'\pi_*Y) = g'(\pi_*X, \pi_*\varphi Y) \\ &= (\pi^*g')(X, \varphi Y) = g(X, \varphi Y) = \Phi(X,Y)\end{aligned}$$

which gives the proof of assertion (i).

(ii) Let X be basic. Let us consider the case of $\pi_*\eta'$. We have

$$(\pi_*\eta')(X) = \eta'(\pi_*X) = g'(\pi_*X, \xi') = g'(\pi_*X, \pi_*\xi) = (\pi^*g')(X, \xi).$$

Since π is a semi-Riemannian submersion, we have $\pi^*g' = g$ so that

$$(\pi^*g')(X, \xi) = g(X, \xi) = \eta(X)$$

and therefore $(\pi^*\eta')(X) = \eta(X)$ which implies $\pi^*\eta' = \eta$ as claimed. □

Since ξ is horizontal, we have $\eta(U) = 0$ for any vertical vector field U and this implies $\mathcal{V}_p \subset \ker\eta_p$, for any $p \in M$.

In the sequel, we show that base space is a normal if the total space is a normal.

Theorem 3.1. *Let $\pi : M \to B$ be a paracontact semi-Riemannian submersion. If the almost paracontact structure of M is normal, then the almost paracontact structure of B is normal.*

Proof Let X and Y be basic. From (2), we have

$$\pi_*N^{(1)}(X,Y) = \pi_*([\varphi,\varphi](X,Y) - 2d\eta(X,Y)\xi).$$

On the other hand, $\pi_*\varphi = \varphi'\pi_*$ and $\pi_*\xi = \xi'$ imply that

$$\begin{aligned}\pi_*[\varphi,\varphi](X,Y) &= \pi_*\varphi^2[X,Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] \\ &= [\pi_*X, \pi_*Y] - \eta[X,Y]\pi_*\xi + [\pi_*\varphi X, \pi_*\varphi Y] - \varphi'\pi_*[\varphi X, Y] \\ &- \varphi'\pi_*[X, \varphi Y] \\ &= [X', Y'] - g'([X',Y'],\xi')\xi' + [\varphi'X', \varphi'Y'] - \varphi'[\varphi'X', Y'] \\ &- \varphi'[X', \varphi'Y'].\end{aligned}$$

Then, we have

$$\pi_*[\varphi,\varphi](X,Y) = N'(X',Y'). \tag{16}$$

In a similar way, since π is a semi-Riemannian submersion, by using proposition 3.4(ii), we have

$$\pi_* 2d\eta \otimes \xi = 2d\eta' \otimes \xi'. \tag{17}$$

Now, from (16) and (17) we obtain

$$\pi_* N^{(1)}(X,Y) = N'^{(1)}(X',Y') = 0.$$

We now recall that an almost para-Hermitian manifold (M,J,g) is an almost paracomplex manifold (M,J) with a J-invariant semi-Riemannian metric g. The J-invariance of g means that $g(JX,JY) = -g(X,Y)$, for any $X,Y \in \chi(M)$[8]. □

As the fibres of a paracontact semi-Riemannian submersion is an invariant submanifold of M with respect to φ, we have the following.

Proposition 3.5 *Let $\pi : (M^{2m+1},\varphi,\xi,\eta,g) \to (B^{2n+1},\varphi',\xi',\eta',g')$ be a paracontact semi-Riemannian submersion from an almost paracontact metric manifold M onto an almost paracontact metric manifold B. Then, the fibres are almost para-Hermitian manifolds.*

Proof Denoting by F the fibres, it is clear that $\dim F = 2(m-n) = 2r$, where $r = m-n$. On $(F^{2r},\hat{g})$, setting $J = \hat{\varphi}$ and $g|_F = \hat{g}$ we have to show that $(J,\hat{g})$ is an almost para-Hermitian structure. Indeed, by using the definition of an almost paracontact structure we get

$$J^2U = \varphi^2U = U - \eta(U)\xi.$$

Since $\eta(U) = 0$, we have $J^2U = U$. On the other hand,

$$g(JV,JU) = -g(V,J^2U) = -g(V,U), \tag{18}$$

which achieves the proof. □

We now investigate what kind of paracontact structures are defined on the base manifold, when the total manifold has some special paracontact structures.

Proposition 3.6 *Let $\pi : M \to B$ be a paracontact semi-Riemannian submersion. If the total space M is para-cosymplectic, almost para-cosymplectic or quasi-para-Sasakian, then the base space B belongs to the same class.*

Proof Let X, Y and Z be basic vector fields on M π-related to X', Y' and Z' on B. Since M is a para-cosymplectic manifold and π is a paracontact semi-Riemannian submersion, we obtain

$$\begin{aligned} 0 &= (\nabla_X\eta)Y = X\eta(Y) - \eta(\nabla_XY) \\ 0 &= (\nabla'_{X'}\eta')Y' \end{aligned} \tag{19}$$

and

$$\begin{aligned} 0 &= (\nabla_X \Phi)(Y,Z) = X\Phi(Y,Z) - \Phi(\nabla_X Y, Z) - \Phi(\nabla_X Z, Y) \\ 0 &= Xg(Y,\varphi Z) - g(\nabla_X Y, \varphi Z) - g(Y, \varphi \nabla_X Z) \\ 0 &= X'\Phi'(Y',Z') - \Phi'(\nabla'_{X'}Y', Z') - \Phi'(\nabla'_{X'}Z', Y') \\ 0 &= (\nabla'_{X'}\Phi')(Y',Z'). \end{aligned} \tag{20}$$

Thus, from (19) and (20) if the total space M is a para-cosymplectic manifold, then base space B belongs to the same class.

In a similar way, let X, Y and Z be basic. An almost para-cosymplectic manifold M implies $d\Phi(X,Y,Z) = 0$. Then, we have

$$\begin{gathered} X(\Phi(Y,Z)) - Y(\Phi(X,Z)) + Z(\Phi(X,Y)) \\ -\Phi([X,Y],Z) + \Phi([X,Z],Y) - \Phi([Y,Z],X) = 0. \end{gathered}$$

On the other hand, by direct calculations, we obtain

$$\begin{aligned} 0 &= g(\nabla_X Y, \varphi Z) + g(Y, \nabla_X \varphi Z) - g(\nabla_Y X, \varphi Z) - g(X, \nabla_Y \varphi Z) \\ &+ g(\nabla_Z X, \varphi Y) + g(X, \nabla_Z \varphi Y) - g([X,Y], \varphi Z) \\ &+ g([X,Z], \varphi Y) - g([Y,Z], \varphi X). \end{aligned}$$

Then, by using $\pi_* \varphi = \varphi' \pi_*$, we get

$$\begin{aligned} &0 = g'(\nabla'_{X'}Y', \varphi' Z') + g'(Y', \nabla'_{X'}\varphi' Z') - g'(\nabla'_{Y'}X', \varphi' Z') - g'(X', \nabla'_{Y'}\varphi' Z') \\ &+g'(\nabla'_{Z'}X', \varphi' Y') + g'(X', \nabla'_{Z'}\varphi' Y') - g'([X',Y'], \varphi' Z') \\ &+g'([X',Z'], \varphi' Y') - g'([Y',Z'], \varphi' X') \\ &0 = X'(\Phi'(Y',Z')) - Y'(\Phi'(X',Z')) + Z'(\Phi'(X',Y')) \\ &-\Phi'([X',Y'],Z') + \Phi'([X',Z'],Y') - \Phi'([Y',Z'],X') \\ &0 = d\Phi'(X',Y',Z'). \end{aligned} \tag{21}$$

In a similar way, we have

$$\begin{aligned} 0 &= 2d\eta(X,Y) = X\eta(Y) - Y\eta(X) - \eta([X,Y]) \\ 0 &= 2d\eta'(X',Y'). \end{aligned} \tag{22}$$

Thus, from (21) and (22) if the total space M is an almost para-cosymplectic manifold, then the space B belongs to the same class.

The rest is proven in the same way. Thus proof is complete. □

We also have the following result which shows that the other structures can be mapped onto the base manifold.

Proposition 3.7 *Let $\pi : M \to B$ be a paracontact semi-Riemannian submersion. If M belongs to any of the classes paracontact, K-paracontact, weakly para-cosymplectic or para-Sasakian, then the base space B belongs to the same class.*

We now check the properties of the tensor fields T and A for a paracontact semi-Riemannian submersion, we will see that such tensors have extra properties for such submersions.

Lemma 3.1 *Let $\pi : M \to B$ be a paracontact semi-Riemannian submersion from a para-cosymplectic manifold M onto an almost paracontact metric manifold B, and let X and Y be horizontal vector fields. Then, we have*

(i) $A_X\varphi Y = \varphi A_X Y$,

(ii) $A_{\varphi X} Y = \varphi A_X Y$.

Proof (i) Let X and Y be horizontal vector fields, and U vertical. Para-cosymplectic manifold M implies that

$$\begin{aligned}(\nabla_X \Phi)(U, Y) &= g((\nabla_X \varphi)Y, U) \\ &= g(\nabla_X \varphi Y - \varphi \nabla_X Y, U) = 0.\end{aligned}$$

Thus, since vertical and horizontal distribution are invariant, from (15) we obtain

$$g(A_X\varphi Y - \varphi A_X Y, U) = 0.$$

Then, we have

$$A_X\varphi Y = \varphi A_X Y.$$

ii) In a similar way, by using (i) we have

$$A_{\varphi X} Y = -A_Y \varphi X = -\varphi A_Y X.$$

Hence, we obtain

$$A_{\varphi X} Y = \varphi A_X Y.$$

□

For the tensor field T we have the following lemma.

Lemma 3.2 *Let $\pi : M \to B$ be a paracontact semi-Riemannian submersion from a para-cosymplectic manifold M onto an almost paracontact metric manifold B, and let U and V be vertical vector fields. Then, we have*

(i) $T_U\varphi V = \varphi T_U V$,

(ii) $T_{\varphi U} V = \varphi T_U V$.

Lemma 3.3 *Let $\pi : M \to B$ be a paracontact semi-Riemannian submersion from a quasi-para-Sasakian manifold M onto an almost paracontact metric manifold B, and let X and Y be horizontal vector fields. Then, we have*

(i) $A_X\varphi Y = \varphi A_X Y$,

(ii) $A_{\varphi X} Y = \varphi A_X Y$.

Proof (i) Let X be a basic vector field on M, Y horizontal vector field and U vertical vector field. For a quasi-para-Sasakian manifold, the relation on $\nabla\varphi$ given in Lemma 2.1(5) becomes:

$$\begin{aligned} g((\nabla_X\varphi)Y, U) &= -d\eta(\varphi U, X)\eta(Y) + d\eta(\varphi Y, X)\eta(U) \\ &= -d\eta(\varphi U, X)\eta(Y). \end{aligned}$$

On the other hand, since $[\varphi U, X] \in \Gamma(\mathcal{V})$, we have

$$d\eta(\varphi U, X)\eta(Y) = (\varphi U\eta(X) - X\eta(\varphi U) - \eta([\varphi U, X]))\eta(Y) = 0.$$

Thus, from (15) we obtain

$$g(A_X\varphi Y - \varphi A_X Y, U) = 0.$$

Then, we have

$$A_X\varphi Y = \varphi A_X Y.$$

ii) In a similar way, by using (i) we have

$$A_{\varphi X}Y = -A_Y\varphi X = -\varphi A_Y X.$$

Hence, we obtain

$$A_{\varphi X}Y = \varphi A_X Y.$$

□

Lemma 3.4 *Let $\pi : M \to B$ be a paracontact semi-Riemannian submersion from a quasi-para-Sasakian manifold M onto an almost paracontact metric manifold B, and let U and V be vertical vector fields. Then, we have*

(i) $T_U\varphi V = \varphi T_U V$,

(ii) $T_{\varphi U}V = \varphi T_U V$.

We now investigate the integrability of the horizontal distribution $\mathcal{H}$.

Theorem 3.2 *Let $\pi : M \to B$ be a paracontact semi-Riemannian submersion from an almost para-cosymplectic manifold M onto an almost paracontact metric manifold B. Then, the horizontal distribution is integrable.*

Proof Let X and Y be basic vector fields. It suffices to prove that $v([X,Y]) = 0$, for basic vector fields on M. Since M is an almost para-cosymplectic manifold, it implies $d\Phi(X,Y,V) = 0$, for any vertical vector V. Then, one obtains

$$\begin{aligned} &X(\Phi(Y,V)) - Y(\Phi(X,V)) + V(\Phi(X,Y)) \\ &-\Phi([X,Y],V) + \Phi([X,V],Y) - \Phi([Y,V],X) = 0. \end{aligned}$$

Since $[X,V],[Y,V]$ are vertical and the two distributions are φ-invariant, the last two and the first two terms vanish. Thus, one gets

$$g([X,Y],\varphi V) = V(g(X,\varphi Y)).$$

On the other hand, if X is basic then $h(\nabla_V X) = h(\nabla_X V) = A_X V,$ thus we have

$$\begin{aligned} V(g(X,\varphi Y)) &= g(\nabla_V X,\varphi Y) + g(\nabla_V \varphi Y, X) \\ &= g(A_X V,\varphi Y) + g(A_{\varphi Y} V, X). \end{aligned}$$

Since, A is skew-symmetric and alternating operator, we get $V(g(X,\varphi Y)) = 0.$ This proves the assertion. □

Since for a quasi-para-Sasakian manifold $d\Phi = 0,$ applying Theorem 3.2, we have the following results.

Corollary 3.1 *Let $\pi : M \to B$ be a paracontact semi-Riemannian submersion from a quasi-para-Sasakian manifold M onto an almost paracontact metric manifold B. Then, the horizontal distribution is integrable.*

Corollary 3.2 *Let $\pi : M \to B$ be a paracontact semi-Riemannian submersion from a para-cosymplectic manifold M onto an almost paracontact metric manifold B. Then, the horizontal distribution is integrable.*

Theorem 3.3 *Let $\pi : M \to B$ be a paracontact semi-Riemannian submersion from an almost para-cosymplectic manifold M onto an almost paracontact metric manifold B. If X horizontal vector field is an infinitesimal automorphism of φ-tensor field, then fibres are totally geodesic.*

Proof Let W and V be vertical vector fields on $M,$ X horizontal. Since M is an almost para-cosymplectic manifold, it implies $d\Phi = 0.$ Then, we obtain:

$$\begin{aligned} d\Phi(W,\varphi V, X) = W(\Phi(\varphi V, X)) - \varphi V(\Phi(W,X)) + X(\Phi(W,\varphi V)) \\ -\Phi([W,\varphi V],X) + \Phi([W,X],\varphi V) - \Phi([\varphi V, X], W) = 0. \end{aligned}$$

Since $[W,\varphi V]$ is vertical and the two distributions are φ-invariant, the first two terms vanish. Thus, one gets

$$X(\Phi(W,\varphi V)) + \Phi([W,X],\varphi V) - \Phi([\varphi V, X], W) = 0.$$

Thus, we have

$$\begin{aligned} 0 &= Xg(W,V) + g([W,X],V) - g([\varphi V, X],\varphi W) \\ 0 &= g(\nabla_X V, W) + g(\nabla_W X, V) - g(\varphi[X,\varphi V], W) \\ 0 &= g([X,V] + \nabla_V X, W) + g(\nabla_W X, V) - g(\varphi[X,\varphi V], W). \end{aligned}$$

On the other hand, if X horizontal vector field is an infinitesimal automorphism of φ-tensor field, then we have

$$[X,\varphi V] = \varphi[X,V] \Rightarrow [X,V] = \varphi[X,\varphi V].$$

Thus, we obtain

$$g(T_V X, W) + g(T_W X, V) = 0.$$

Since, T is skew-symmetric and alternating operator, we get $g(T_V W, X) = 0.$ This proves the assertion. □

From Theorem 3.3, we have the following result.

Corollary 3.3 *Let $\pi : M \to B$ be a paracontact semi-Riemannian submersion from a quasi-para-Sasakian manifold M onto an almost paracontact metric manifold B. If X horizontal vector field is an infinitesimal automorphism of φ-tensor field, then fibres are totally geodesic.*

4. Curvature relations for paracontact semi-Riemannian submersions

We begin this section relating the φ-paraholomorphic bisectional and sectional curvatures of the total space, the base and the fibres of a paracontact semi-Riemannian submersions.

Let us recall the sectional curvature of semi-Riemannian manifolds for non-degenerate planes. Let M be a semi-Riemannian manifold and P a non-degenerate tangent plane to M at p. The number

$$K(U,V)=\frac{g(R(U,V)U,V)}{g(U,U)g(V,V)-g(U,V)^2}$$

is independent of the choice of basis U,V for P and is called the sectional curvature.

Let π be a paracontact semi-Riemannian submersion between almost paracontact metric manifolds M and N. We denote Riemannian curvatures of M,N and any fibre $\pi^{-1}(x)$ by R,R' and $\hat{R}$, respectively. For $X,Y,Z,W\in\Gamma(\mathcal{H})$, we have $R^*(X,Y,Z,W)=R'(\pi_*X,\pi_*Y,\pi_*Z,\pi_*W)\circ\pi$.

Let $\pi:M\to N$ be a paracontact semi-Riemannian submersion, where M and N are almost paracontact metric manifolds with structures (φ,ξ,η,g) and (φ',ξ',η',g'), respectively. We denote by B the φ-paraholomorphic bisectional curvature, defined for any pair of nonzero nonlightlike vectors X and Y on M orthogonal to ξ by the formula

$$B(X,Y)=\frac{R(X,\varphi X,Y,\varphi Y)}{\|X\|^2\|Y\|^2}.$$

We note that if X is a nonlightlike vector field, then φX is also a nonlightlike vector field.

The φ-paraholomorphic sectional curvature is $H(X)=B(X,X)$ for any nonzero nonlightlike vector X orthogonal to ξ. We denote by B' and H' the φ-paraholomorphic bisectional and φ-paraholomorphic sectional curvature of B, respectively. Similarly, $\hat{B}$ and $\hat{H}$ denote the bisectional and the sectional paraholomorphic curvatures of a fibre.

The following is a translation of the results of Gray [6] and O'Neill [11] to the present situation:

Proposition 4.1. *Let $\pi:M\to N$ a paracontact semi-Riemannian submersion from an almost paracontact metric manifold M onto an almost paracontact metric manifold N. Let U and V be nonzero nonlightlike unit vertical vectors, and X and Y nonzero nonlightlike unit horizontal vectors orthogonal to ξ. Then we have*

$$\begin{aligned}
(a)B(U,V) &= \hat{B}(U,V)-\epsilon_U\epsilon_V[g(T_UV,T_{\varphi U}\varphi V)-g(T_{\varphi U}V,T_U\varphi V)];\\
(b)B(X,U) &= \epsilon_U\epsilon_X[g((\nabla_UA)_X\varphi X,\varphi U)-g((\nabla_{\varphi U}A)_X\varphi X,U)\\
&+ g(A_XU,A_{\varphi X}\varphi U)-g(A_X\varphi U,A_{\varphi X}U)\\
&- g(T_UX,T_{\varphi U}\varphi X)+g(T_{\varphi U}X,T_U\varphi X)];\\
(c)B(X,Y) &= B'(X',Y')\circ\pi-\epsilon_X\epsilon_Y[2g(A_X\varphi X,A_Y\varphi Y)\\
&- g(A_{\varphi X}Y,A_X\varphi Y)+g(A_XY,A_{\varphi X}\varphi Y)],
\end{aligned}$$

where $\epsilon_U=g(U,U)\in\{\pm1\}$, $\epsilon_V=g(V,V)\in\{\pm1\}$, $\epsilon_X=g(X,X)\in\{\pm1\}$ and $\epsilon_Y=g(Y,Y)\in\{\pm1\}$.

Using Proposition 4.1, we have the following result.

Proposition 4.2 *Let $\pi:M\to N$ a paracontact semi-Riemannian submersion from an almost paracontact metric manifold M onto an almost paracontact metric manifold N. Let U be nonzero nonlightlike unit vertical vector, and X nonzero nonlightlike unit horizontal vector orthogonal to ξ. Then, one has*

(a) $H(U) = \hat{H}(U) + \|T_U \varphi U\|^2 - g(T_{\varphi U} \varphi U, T_U U)$;

(b) $H(X) = H'(X') \circ \pi - 3\|A_X \varphi X\|^2$.

If the total manifold is a quasi-para-Sasakian manifold, then we have the following result for curvature relations between M, N and $\pi^{-1}(x)$.

Theorem 4.1 *Let $\pi : M \to N$ be a paracontact semi-Riemannian submersion from a quasi-para-Sasakian manifold M onto an almost paracontact metric manifold N. Let U and V be nonzero nonlightlike unit vertical vectors, and X and Y nonzero nonlightlike unit horizontal vectors orthogonal to ξ. Then, we have:*

(a) $B(U, V) = \hat{B}(U, V) - \epsilon_U \epsilon_V [2\|T_U V\|^2 - 2\eta(T_U V)^2]$;

(b) $B(X, Y) = B'(X', Y') \circ \pi$.

Proof (a) From Proposition 4.1(a), we have

$$B(U, V) = \hat{B}(U, V) - \epsilon_U \epsilon_V [g(T_U V, T_{\varphi U} \varphi V) - g(T_{\varphi U} V, T_U \varphi V)].$$

Using Lemma 3.4, we get

$$\begin{aligned} g(T_U \varphi V, T_{\varphi U} V) &= g(\varphi T_U V, \varphi T_U V) \\ &= -g(T_U V, T_U V) + \eta(T_U V)\eta(T_U V) \\ &= -\|T_U V\|^2 + \eta(T_U V)^2. \end{aligned} \tag{23}$$

Using again Lemma 3.4, we get

$$\begin{aligned} g(T_{\varphi U} \varphi V, T_U V) &= g(\varphi^2 T_U V, T_U V) \\ &= g(T_U V - \eta(T_U V)\xi, T_U V) \\ &= \|T_U V\|^2 - \eta(T_U V)^2. \end{aligned} \tag{24}$$

From (23) and (24), we have (a).

(b) Since horizontal distribution is integrable, we have $A = 0$. Thus (b) is clear from Proposition 4.1.(c). □

A a result of Theorem 4.1, we have the following result.

Corollary 4.1. *Let $\pi : M \to N$ be a paracontact semi-Riemannian submersion from a quasi-para-Sasakian manifold M onto an almost paracontact metric manifold N. Let U be nonzero nonlightlike unit vertical vector, and X nonzero nonlightlike unit horizontal vector orthogonal to ξ. Then, one has*

(a) $H(U) = \hat{H}(U) - 2\|T_U U\|^2 + 2\eta(T_U U)^2$;

(b) $H(X) = H'(X') \circ \pi$.

For a para-cosymplectic manifold M, we have the following relations.

Theorem 4.2 *Let $\pi : M \to N$ be a paracontact semi-Riemannian submersion from a para-cosymplectic manifold M onto an almost paracontact metric manifold N. Let U and V be nonzero nonlightlike unit vertical vectors, and X and Y nonzero nonlightlike unit horizontal vectors orthogonal to ξ. Then, we have:*

(a) $B(U,V) = \hat{B}(U,V) - \epsilon_U\epsilon_V[2\|T_UV\|^2 - 2\eta(T_UV)^2]$;

(b) $B(X,Y) = B'(X',Y')\circ\pi$;

(c) $B(X,U) = -\epsilon_U\epsilon_X 2\|T_UX\|^2$.

Proof. (a) From Proposition 4.1(a), we have

$$B(U,V) = \hat{B}(U,V) - \epsilon_U\epsilon_V[g(T_UV, T_{\varphi U}\varphi V) - g(T_{\varphi U}V, T_U\varphi V)].$$

Using Lemma 3.2, we get

$$\begin{aligned} g(T_U\varphi V, T_{\varphi U}V) &= g(\varphi T_UV, \varphi T_UV) \\ &= -g(T_UV, T_UV) + \eta(T_UV)\eta(T_UV) \\ &= -\|T_UV\|^2 + \eta(T_UV)^2. \end{aligned} \tag{25}$$

Using again Lemma 3.2, we get

$$\begin{aligned} g(T_{\varphi U}\varphi V, T_UV) &= g(\varphi^2 T_UV, T_UV) \\ &= g(T_UV - \eta(T_UV)\xi, T_UV) \\ &= \|T_UV\|^2 - \eta(T_UV)^2. \end{aligned} \tag{26}$$

The above equations imply

$$B(U,V) = \hat{B}(U,V) - \epsilon_U\epsilon_V[2\|T_UV\|^2 - 2\eta(T_UV)^2].$$

(b) Since M is a para-cosymplectic manifold and the distribution $\mathcal{H}$ is integrable we have $A = 0$. Then using Proposition 4.1(c), we have

$$B(X,Y) = B'(X',Y')\circ\pi.$$

(c) Since M is a para-cosymplectic manifold M and $A = 0$, due to Corollary 3.2 we have

$$B(X,U) = -\epsilon_U\epsilon_X g(T_UX, T_{\varphi U}\varphi X) + \epsilon_U\epsilon_X g(T_{\varphi U}X, T_U\varphi X).$$

On the other hand, using Lemma 3.2, we have

$$\begin{aligned} g(T_UX, T_{\varphi U}\varphi X) &= g(T_UX, \varphi^2 T_UX) \\ &= g(T_UX, T_UX - \eta(T_UX)\xi) \\ &= g(T_UX, T_UX) \end{aligned} \tag{27}$$

and

$$\begin{aligned} g(T_{\varphi U}X, T_U\varphi X) &= g(\varphi T_UX, \varphi T_UX) \\ &= -g(T_UX, T_UX) + \eta(T_UX)\eta(T_UX) \\ &= -g(T_UX, T_UX). \end{aligned} \tag{28}$$

From (29) and (30), we have $B(X,U) = -\epsilon_U \epsilon_X 2\|T_U X\|^2$.

Corollary 4.2 *Let $\pi : M \to N$ be a paracontact semi-Riemannian submersion from a para-cosymplectic manifold M onto an almost paracontact metric manifold N. Let U be nonzero nonlightlike unit vertical vectors, and X nonzero nonlightlike unit horizontal vectors orthogonal to ξ. Then, we have:*

(a) $H(U) = \hat{H}(U) - 2\|T_U U\|^2 + 2\eta(T_U U)^2]$;

(b) $H(X) = H'(X') \circ \pi$.

Acknowledgment

This work is supported by the Scientific and Technological Council of Turkey (TÜBİTAK) with number (TBAG-109T125). The authors are grateful to the referees for their valuable comments and suggestions.

References

[1] Baditoiu, G., Ianuş, S.: Semi-Riemannian submersions from real and complex pseudo-hyperbolic spaces. Diff. Geom. and Appl. 16, 79–94 (2002).

[2] Caldarella, A.V.: On paraquaternionic submersions between paraquaternionic Kähler manifolds. Acta Applicandae Mathematicae 112, 1–14 (2010).

[3] Chinea, D.: Almost contact metric submersions. Rend. Circ. Mat. Palermo, II Ser. 34, 89–104 (1985).

[4] Dacko, P., Olszak, Z.: On weakly para-cosymplectic manifolds of dimension 3. Journal of Geometry and Physics 57, 561–570 (2007).

[5] Falcitelli, M., Ianuş, S., Pastore, A.M.: Riemannian Submersions and Related Topics, World Scientific 2004.

[6] Gray, A.: Pseudo-Riemannian almost product manifolds and submersions. J. Math. Mech. 16, 715–737 (1967).

[7] Ianuş, S., Mazzocco, R., Vilcu, G.E.: Riemannian submersions from quaternionic manifolds. Acta Appl. Math. 104, 83–89 (2008).

[8] Ianuş, S., Marchiafava, S., Vilcu, G.E.: Paraquaternionic CR-submanifolds of paraquaternionic Kähler Manifols and semi-Riemannian submersions. Central European Journal of Mathematics 8, 735–753 (2010).

[9] Kaneyuki, S., Williams, F.L.: Almost paracontact and paraHodge structures on manifolds. Nagoya Math. J. 99, 173–187 (1985).

[10] Montano, B.C.: Bi-Legendrian structures and paracontact geometry. International Journal of Geometric Methods in Modern Physics 6, 487–504 (2009).

[11] O'Neill, B.: The fundamental equations of a submersion, Michigan Math. J. 13, 459 469 (1966).

[12] O'Neill, B.: Semi-Riemannian Geometry with Application to Relativity, New York. Academic Press 1983.

[13] Watson, B.: Almost Hermitian submersions. J. Diff. Geom. 11, 147–165 (1976).

[14] Welyczko, J.: On Legendre curves in 3-dimensional normal almost paracontact metric manifolds. Result.Math. 54, 377–387 (2009).

[15] Zamkovoy, S.: Canonical connections on paracontact manifolds. Ann. Glob. Anal. Geom. 36, 37–60 (2009).