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Asymptotics of the ruin probability with claims modeled by α-stable aggregated AR(1) process

Karina PERİLİOĞLU[1,*], **Donata PUPLINSKAITĖ**[1,2]
[1]Faculty of Mathematics and Informatics, Vilnius University, Naugarduko St. 24, LT-03225 Vilnius, Lithuania
[2]Laboratoire de Mathématiques Jean Leray, Université de Nantes, 2 rue de la Houssiniére, 44322 Nantes cedex 3, France

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Abstract: We study the asymptotics of the ruin probability in a discrete time risk insurance model with stationary claims following the aggregated heavy-tailed AR(1) process discussed in Puplinskaitė and Surgailis (2010). The present work is based on the general characterization of the ruin probability with claims modeled by stationary α-stable process in Mikosch and Samorodnitsky (2000). We prove that for the aggregated AR(1) claims' process, the ruin probability decays with exponent $\alpha(1-H)$, where $H \in [1/\alpha, 1)$ is the asymptotic self-similarity index of the claim process, $1 < \alpha < 2$. This result agrees with the decay rate of the ruin probability with claims modeled by increments of linear fractional motion in Mikosch and Samorodnitsky (2000) and also with other characterizations of long memory of the aggregated AR(1) process with infinite variance in Puplinskaitė and Surgailis (2010).

Key words: Ruin probability, dependent α-stable claims, aggregation, random-coefficient AR(1) process, mixed stable moving average, self-similar process, long memory

1. Introduction and the main result

The present note studies the asymptotics of the ruin probability

$$\psi(u) := \mathrm{P}\Big(\sup_{n\ge 1}\big(\sum_{t=1}^{n} Y_t - cn\big) > u\Big), \quad \text{as } u \to \infty, \tag{1}$$

where 'claims' $\{Y_t\}$ form a stationary, α-stable process of a certain type, $1 < \alpha < 2$, obtained by aggregating independent copies of random-coefficient AR(1) heavy-tailed processes. In (1), $c > 0$ is interpreted as a constant deterministic premium rate and u is the initial capital. The above problem was investigated in Mikosch and Samorodnitsky (2000) (see [8]) for stable processes $\{Y_t\}$. Applying large deviations methods for Poisson point processes, they proved the asymptotics $\psi(u) \sim \psi_0(u)$, where $f(u) \sim g(u)$ means that $f(u)/g(u) \to 1$ as $u \to \infty$, and the function ψ_0 is written in terms of the kernel and the control measure of stochastic integral representation of $\{Y_t\}$ (see (15), below, in the special case when $\{Y_t\}$ is a mixed stable moving average). Using the above result, Mikosch and Samorodnitsky ([8]) obtained the 'classical' decay rate $\psi(u) \sim const.\, u^{-(\alpha-1)}$

*Correspondence: karina.laskeviciute@gmail.com
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(see e.g. [4]), for a wide class of weakly dependent symmetric α-stable (SαS) stationary claims, and a markedly different decay rate $\psi(u) \sim const.\, u^{-\alpha(1-H)}$ for increments of fractional SαS motion with self-similarity index $H \in (1/\alpha, 1)$. In view of these findings, Mikosch and Samorodnitsky ([8], p.1817) propose the decay rate of the ruin probability as an alternative characteristic of long-range dependence of a SαS process. See also [1], [2].

The present note complements the results in [8], by obtaining the characteristic decay of the ruin probability when claims are modeled by the mixed SαS process studied in [10]. The latter process arises in the result of aggregation of independent copies of random-coefficient AR(1) processes with heavy-tailed innovations, following the classical scheme of contemporaneous aggregation (see [6]). Aggregation is a common procedure in statistical and econometric modeling and can explain certain empirical 'stylized facts' of financial time series (such as long memory) from simple heterogeneous dynamic models describing the evolution of individual 'agents'. See [3], [12], [13], [14], [5], among others.

Puplinskaitė and Surgailis (see [9], [10]) discussed aggregation of infinite variance random-coefficient AR(1) processes and long-memory properties of the limiting aggregated process. Let us describe the main results of the last paper. Let $\{X_t, t \in \mathbb{Z}\}$ be a stationary solution of the AR(1) equation

$$X_t = aX_{t-1} + \varepsilon_t, \tag{2}$$

where $\{\varepsilon_t, t \in \mathbb{Z}\}$ are i.i.d. r.v.'s in the domain of the (normal) attraction of an α-stable law, $0 < \alpha < 2$, and where $a \in (-1, 1)$ is a r.v., independent of $\{\varepsilon_t\}$ and satisfying some mild additional condition. Let the $X_{it} = a_i X_{i,t-1} + \varepsilon_{it}$, $i = 1, 2, \ldots$, be independent copies of (2). Then the aggregated process $\big\{N^{-1/\alpha} \sum_{i=1}^N X_{it}, t \in \mathbb{Z}\big\}$ tends, as $N \to \infty$, in the sense of weak convergence of finite-dimensional distributions, to a limiting process $\{\bar{X}_t\}$ written as a stochastic integral

$$\bar{X}_t \ = \ \sum_{s \le t} \int_{(-1,1)} a^{t-s} M_s(\mathrm{d}a), \tag{3}$$

where $\{M_s, s \in \mathbb{Z}\}$ are i.i.d. copies of an α-stable random measure M on $(-1, 1)$ with control measure proportional to the distribution $\Phi(\mathrm{d}x) = \mathrm{P}(a \in \mathrm{d}x)$ of r.v. a (see [10]). In the case when $1 < \alpha < 2$ and the mixing distribution Φ is concentrated in the interval $(0, 1)$ having a density ϕ such that

$$\phi(x) \ \sim \phi_1 \ (1-x)^b \qquad \text{as } x \to 1, \quad \text{for some } \phi_1 > 0, \qquad 0 < b < \alpha - 1, \tag{4}$$

the above authors proved that the aggregated process in (3) has long memory. In particular, it was shown that normalized partial sums of $\{\bar{X}_t\}$ in (3) tend to an α-stable stationary increment process $\{Z(\tau)\}$, which is self-similar with index $H = 1 - (b/\alpha) \in (1/\alpha, 1)$ and is written as a stochastic integral

$$Z(\tau) \quad := \quad \int_{(0,\infty)\times\mathbb{R}} \big(f(x, \tau - s) - f(x, -s)\big)\nu(\mathrm{d}x, \mathrm{d}s), \tag{5}$$

$$f(x,t) \quad := \quad \begin{cases} 1 - \mathrm{e}^{-xt}, & \text{if } x > 0 \text{ and } t > 0, \\ 0, & \text{otherwise,} \end{cases}$$

with respect to an α-stable random measure $\nu(\mathrm{d}x, \mathrm{d}s)$ on $(0, \infty) \times \mathbb{R}$ with control measure $\phi_1 x^{b-\alpha} \mathrm{d}x \mathrm{d}s$. Let us note that (5) is different from the α-stable fractional motion discussed in [8], which arises in a similar context by aggregating AR(1) processes with *common* infinite-variance innovations; see [9]. Under the same assumptions in (4), Puplinskaitė and Surgailis (see [10]) established, further, long memory properties of $\{\bar{X}_t\}$ in (3), namely, a (hyperbolic) decay rate of codifference and the long-range dependence (sample Allen variance) property of Heyde and Yang (see [7]). They also showed that the value $b = \alpha - 1$ separates long memory and short memory in the above aggregation scheme; indeed, in the case $b > \alpha - 1$ the aggregated process has the short-range dependence (sample Allen variance) property and its partial sums tend to an α-stable Lévy process with independent increments (see [10]).

The present paper extends the above-mentioned characterization of long memory in the aggregated AR(1) process of (3) to the ruin probability in (1) with stationary SαS claims $\{\bar{X}_t\}$, $1 < \alpha < 2$.

In the rest of the paper, we assume that $\{\bar{X}_t, t \in \mathbb{Z}\}$ is the mixed moving average in (3), where $M_s(\mathrm{d}a)$ is a SαS random measure with characteristic function $\mathrm{E}\mathrm{e}^{\mathrm{i}\theta M_s(A)} = \mathrm{e}^{-\omega_\alpha |\theta|^\alpha \Phi(A)}$, $\theta \in \mathbb{R}$, where $1 < \alpha < 2$, $\omega_\alpha > 0$ and $A \subset (0,1)$ is any Borel set. This means that all finite-dimensional distributions of $\{\bar{X}_t, t \in \mathbb{Z}\}$ are SαS. In particular,

$$\mathrm{E}\mathrm{e}^{\mathrm{i}\theta \bar{X}_0} = \mathrm{e}^{-\sigma^\alpha |\theta|^\alpha}, \quad \theta \in \mathbb{R}, \quad \text{where} \quad \sigma^\alpha := \omega_\alpha \sum_{k=0}^{\infty} \mathrm{E}|a|^{\alpha k} = \omega_\alpha \mathrm{E}\frac{1}{1-|a|^\alpha}.$$

Let $C_\alpha > 0$ be the constant determined from the relation

$$\lim_{u\to\infty} u^\alpha \mathrm{P}(\bar{X}_0 > u) \ = \ \frac{1}{2} C_\alpha \sigma^\alpha. \tag{6}$$

The constant C_α depends only on α and is explicitly written in [11]

$$C_\alpha = \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}.$$

Also define

$$g(z) \ := \ \sup_{w>0} \frac{1-\mathrm{e}^{-w}}{w+z}, \quad z > 0. \tag{7}$$

The function g is continuous in the interval $(0,\infty)$ and satisfies the following conditions

$$\lim_{z\to 0} g(z) = 1, \quad \lim_{z\to\infty} z g(z) = 1. \tag{8}$$

The main result of our paper is the following theorem.

Theorem 1 *Assume that the mixing distribution* $\Phi(A) = \mathrm{P}(a \in A)$ *is absolutely continuous having a density*

$$\phi(a) \ = \ \varphi(a)(1-a)^b, \quad a \in (0,1), \tag{9}$$

where $b > 0$ *and* φ *is integrable on* $(0,1)$ *and has limit* $\lim_{a\to 1}\varphi(a) =: \phi_1 > 0$. *Let* $\psi(u)$ *be the ruin probability in (1) corresponding to* $\{Y_t \equiv \bar{X}_t\}$.

(i) Let $0 < b < \alpha - 1$. *Then*

$$\psi(u) \ \sim \ \frac{C_\alpha K(\alpha,b)}{2c^{H\alpha}} u^{-\alpha(1-H)}, \quad u \to \infty, \tag{10}$$

where $H = 1 - (b/\alpha) \in (1/\alpha, 1)$ *and*

$$K(\alpha, b) := \frac{\phi_1}{\alpha} \int_0^\infty z^{b-1} g^\alpha(z) \mathrm{d}z + \frac{\phi_1}{b} \int_0^\infty z^b g^\alpha(z) \mathrm{d}z. \tag{11}$$

(ii) Let $b > \alpha - 1$. *Then*

$$\psi(u) \quad \sim \quad \frac{C_\alpha K(\alpha, \Phi)}{2c} u^{-(\alpha-1)}, \quad u \to \infty, \tag{12}$$

where

$$K(\alpha, \Phi) := \frac{1}{\alpha - 1} \mathrm{E}\Big[\frac{1}{(1-a)^\alpha}\Big]. \tag{13}$$

In what follows, C stands for a constant whose precise value is unimportant and which may change from line to line.

2. Proof of Theorem 1

The proof of Theorem 1 is based on Theorem 2, below, due to [8], Theorem 2.5. For our purpose, we formulate the above mentioned result in a special case of mixed SαS moving average in (14). For terminology and properties of stochastic integrals with respect to stable random measures, we refer to [11].

Let $\{Y_t\} = \{Y_t, t = 1, 2, \cdots\}$ be a stationary SαS process, $1 < \alpha < 2$, having the form

$$Y_t \quad = \quad \int_{W \times \mathbb{R}} f(v, x - t) M(\mathrm{d}v, \mathrm{d}x), \qquad t = 1, 2, \cdots, \tag{14}$$

where M is a SαS random measure on a measurable product space $W \times \mathbb{R}$ with control measure $\nu \times \mathrm{Leb}$, ν is a $\sigma-$finite measure on W, Leb is the Lebesgue measure, and $f \in L^\alpha(W \times \mathbb{R})$ is a measurable function with $\int_{W \times \mathbb{R}} |f(v, x)|^\alpha \nu(\mathrm{d}v)\mathrm{d}x < \infty$. Introduce

$$m_n \; := \; C_\alpha^{1/\alpha} \Big(\int_{W \times \mathbb{R}} \big| \sum_{t=1}^n f(v, x - t) \big|^\alpha \nu(\mathrm{d}v) \mathrm{d}x \Big)^{1/\alpha}$$

and a function $\psi_0 : (0, \infty) \to (0, \infty)$ by

$$\begin{aligned} \psi_0(u) \quad := \quad & \frac{C_\alpha}{2} \int_{W \times \mathbb{R}} \sup_{n \ge 1} \frac{\big(\sum_{t=1}^n f(v, x - t)\big)_+^\alpha}{(u + nc)^\alpha} \nu(\mathrm{d}v) \mathrm{d}x \\ + \quad & \frac{C_\alpha}{2} \int_{W \times \mathbb{R}} \sup_{n \ge 1} \frac{\big(\sum_{t=1}^n f(v, x - t)\big)_-^\alpha}{(u + nc)^\alpha} \nu(\mathrm{d}v) \mathrm{d}x, \end{aligned} \tag{15}$$

where $x_+ := \max(x, 0), x_- := \max(-x, 0)$ and where the constant C_α is the same as in (6).

Theorem 2 *(see [8]). Let* $\{Y_t\}$ *be given as in (14). Assume that* $m_n = O(n^\beta)$ *for some* $\beta \in (0, 1)$. *Then*

$$\psi(u) \;\sim\; \psi_0(u) \qquad (u \to \infty).$$

Proof of Theorem 1. In order to use Theorem 2, we first rewrite the process in (3) in the form of (14):

$$\bar{X}_t \quad = \quad \int_{(0,1)\times\mathbb{R}} f(a, t-x) M(\mathrm{d}a, \mathrm{d}x), \tag{16}$$

where

$$f(a,x) := a^{[x]}\mathbf{1}(x \ge 0) = \begin{cases} a^{[x]}, & x \ge 0, \\ 0, & x < 0, \end{cases} \qquad (a,x) \in (0,1)\times\mathbb{R},$$

and $M(\mathrm{d}a, \mathrm{d}x)$ is a SαS random measure on $(0,1)\times\mathbb{R}$ with control measure $\Phi \times$ Leb; $[x]$ is the integer part of $x \in \mathbb{R}$.

Condition $m_n = O(n^\beta)$ of Theorem 2 for the process in (3) is verified in [10], (A.12), with $\beta = H = 1-(b/\alpha) \in (1/\alpha, 1)$. Therefore it suffices to show (10) with $\psi(u)$ replaced by $\psi_0(u)$ as defined in (15). We have

$$\begin{aligned} \psi_0(u) \quad &= \quad \frac{C_\alpha}{2} \int_{(0,1)\times\mathbb{R}} \sup_{n\ge 1} \frac{\big(\sum_{t=1}^n a^{[t-x]}\mathbf{1}(t \ge x)\big)^\alpha}{(u+nc)^\alpha} \Phi(\mathrm{d}a)\mathrm{d}x \\ &= \quad \frac{C_\alpha}{2}\Big(\mathrm{E} \sum_{x=-\infty}^{0} \sup_{n\ge 1} \frac{\big(\sum_{t=1}^n a^{t-x}\big)^\alpha}{(u+nc)^\alpha} \; + \; \mathrm{E}\sum_{x=1}^{\infty} \sup_{n \ge x} \frac{\big(\sum_{t=x}^n a^{t-x}\big)^\alpha}{(u+nc)^\alpha}\Big) \\ &=: \quad \frac{C_\alpha}{2}\big(I_1 + I_2\big). \end{aligned} \tag{17}$$

Consider first the expectation

$$\begin{aligned} I_2 \quad &= \quad \mathrm{E}\sum_{x=1}^{\infty} \frac{1}{(1-a)^\alpha} \sup_{k\ge 1}\Big(\frac{1-a^k}{u+(k-1+x)c}\Big)^\alpha \\ &= \quad c^{-\alpha}\int_0^1 y^{-\alpha}\phi(1-y)\,\mathrm{d}y \sum_{x=1}^{\infty}\sup_{k\ge 1}\Big(\frac{1-(1-y)^k}{(u/c)+k-1+x}\Big)^\alpha \\ &= \quad c^{-\alpha}\Big\{\int_0^\epsilon y^{-\alpha}\phi(1-y)\,\mathrm{d}y \sum_{x=1}^{\infty}\sup_{k\ge 1}\Big(\frac{1-(1-y)^k}{(u/c)+k-1+x}\Big)^\alpha \\ &+ \quad \int_\epsilon^1 y^{-\alpha}\phi(1-y)\,\mathrm{d}y \sum_{x=1}^{\infty}\sup_{k\ge 1}\Big(\frac{1-(1-y)^k}{(u/c)+k-1+x}\Big)^\alpha\Big\} \\ &=: \quad c^{-\alpha}\big\{I_{21} + I_{22}\big\}. \end{aligned} \tag{18}$$

Clearly, in view of (9), we can replace $\phi(1-y)$ by $\phi_1 y^b$ in the integral I_{21}. For notational simplicity, assume that $\phi(1-y) = \phi_1 y^b$, $0 < y < \epsilon$. Then $u^b I_{21}$ can be rewritten as

$$
\begin{aligned}
u^b I_{21} &= \phi_1 u^b \int_0^{\epsilon} y^{b-\alpha}\,\mathrm{d}y \sum_{x=1}^{\infty} \sup_{k\ge 1} \Big(\frac{1-(1-y)^k}{(u/c)+k-1+x}\Big)^{\alpha} \\
&= \phi_1 u^b \int_0^{\epsilon} y^{b}\,\mathrm{d}y \sum_{x=1}^{\infty} \sup_{k\ge 1} \Big(\frac{1-(1-y)^k}{y((u/c)+x-1)+yk}\Big)^{\alpha} \\
&= \phi_1 u^b \int_0^{\epsilon((u/c)+x-1)} \frac{z^b}{((u/c)+x-1)^b}\,\mathrm{d}\Big(\frac{z}{(u/c)+x-1}\Big) \sum_{x=1}^{\infty} \sup_{k\ge 1} \Big(\frac{1-\big(1-\frac{z}{(u/c)+x-1}\big)^k}{z+\frac{zk}{(u/c)+x-1}}\Big)^{\alpha} \\
&= \phi_1 \sum_{x=1}^{\infty} \frac{u^b}{((u/c)+x-1)^{b+1}} \int_0^{\epsilon((u/c)+x-1)} z^b (g_{u,x}(z))^{\alpha}\mathrm{d}z, \qquad (19)
\end{aligned}
$$

where

$$
g_{u,x}(z) \;:=\; \sup_{k\ge 1} \frac{1-\big(1-\frac{z}{(u/c)+x-1}\big)^k}{z+\frac{zk}{(u/c)+x-1}} \mathbf{1}(0 < z < \epsilon((u/c)+x-1)). \qquad (20)
$$

According to Lemma 3 below, the function $g_{u,x}(z)$ tends to $g(z)$ in (7) as $u \to \infty$, and satisfies condition (25); therefore, by dominated convergence theorem, the integral in (19) tends to $\int_0^{\infty} z^b g^{\alpha}(z)\mathrm{d}z < \infty$ uniformly in $x \ge 1$. We also have that

$$
\sum_{x=1}^{\infty} \frac{u^b}{((u/c)+x-1)^{b+1}} = \sum_{x=0}^{\infty} \frac{1}{u} \frac{1}{\big((1/c)+(x/u)\big)^{b+1}} \to \int_0^{\infty} \frac{\mathrm{d}x}{((1/c)+x)^{b+1}} = \frac{c^b}{b}.
$$

Whence and from (19) we obtain that

$$
\lim_{u\to\infty} u^b I_{21} = \frac{\phi_1 c^b}{b} \int_0^{\infty} z^b g^{\alpha}(z)\mathrm{d}z. \qquad (21)
$$

On the other hand,

$$
\begin{aligned}
|I_{22}| &\le C\mathrm{E}\Big[(1-a)^{-\alpha}\mathbf{1}(0<a<1-\epsilon) \sum_{x=1}^{\infty} \sup_{k\ge 1} \Big(\frac{1-a^k}{(u/c)+k-1+x}\Big)^{\alpha}\Big] \\
&\le C \sum_{x=1}^{\infty} \Big(\frac{1}{(u/c)+x}\Big)^{\alpha} = O(u^{-(\alpha-1)}),
\end{aligned}
$$

implying $\lim_{u\to\infty} u^b I_{22} = 0$ thanks to condition $b < \alpha - 1$.

Consider the term I_1 in (17):

$$\begin{aligned}
I_1 &= \mathrm{E}\sum_{x=-\infty}^{0}\sup_{n\geq 1}\frac{\big(\sum_{t=1}^{n}a^{t-x}\big)^{\alpha}}{(u+nc)^{\alpha}} \\
&= \mathrm{E}\sum_{x=-\infty}^{0}\sup_{n\geq 1}\frac{a^{(1-x)\alpha}(1-a^n)^{\alpha}}{(1-a)^{\alpha}(u+nc)^{\alpha}} \\
&= c^{-\alpha}\int_0^1 \mathrm{d}y\, y^{-\alpha}\phi(1-y)\sum_{x=-\infty}^{0}(1-y)^{(1-x)\alpha}\sup_{n\geq 1}\Big(\frac{1-(1-y)^n}{(u/c)+n}\Big)^{\alpha} \\
&= c^{-\alpha}\Big\{\int_0^{\epsilon} \mathrm{d}y\, y^{-\alpha}\phi(1-y)\sum_{x=-\infty}^{0}(1-y)^{(1-x)\alpha}\sup_{n\geq 1}\Big(\frac{1-(1-y)^n}{(u/c)+n}\Big)^{\alpha} \\
&+ \int_{\epsilon}^1 \mathrm{d}y\, y^{-\alpha}\phi(1-y)\sum_{x=-\infty}^{0}(1-y)^{(1-x)\alpha}\sup_{n\geq 1}\Big(\frac{1-(1-y)^n}{(u/c)+n}\Big)^{\alpha}\Big\} \\
&=: c^{-\alpha}\big\{I_{11}+I_{12}\big\}.
\end{aligned}$$

For notational simplicity, assume that $\phi(1-y)=\phi_1 y^b$, $0<y<\epsilon$. Then $u^b I_{11}$ can be rewritten as

$$\begin{aligned}
u^b I_{11} &= u^b\phi_1\int_0^{\epsilon}\mathrm{d}y\, y^{b-\alpha}\sum_{x=-\infty}^{0}(1-y)^{(1-x)\alpha}\sup_{n\geq 1}\Big(\frac{1-(1-y)^n}{(u/c)+n}\Big)^{\alpha} \\
&= u^b\phi_1\int_0^{\epsilon}\mathrm{d}y\, y^{b}\frac{(1-y)^{\alpha}}{1-(1-y)^{\alpha}}\sup_{n\geq 1}\Big(\frac{1-(1-y)^n}{(yu/c)+yn}\Big)^{\alpha} \\
&= c^b\phi_1\int_0^{\epsilon u/c}\mathrm{d}z\,\big(\frac{c}{u}\big)\frac{(1-cz/u)^{\alpha}}{1-(1-cz/u)^{\alpha}}z^b\sup_{n\geq 1}\Big(\frac{1-(1-cz/u)^n}{z+czn/u}\Big)^{\alpha} \\
&= c^b\phi_1\int_0^{\epsilon u/c}\mathrm{d}z\,\big(\frac{cz}{u}\big)\frac{(1-cz/u)^{\alpha}}{1-(1-cz/u)^{\alpha}}z^{b-1}(g_{u,1}(z))^{\alpha}.
\end{aligned}$$

Using Lemma 3 below, and the facts that $\lim_{x\to 0}x(1-x)^{\alpha}/(1-(1-x)^{\alpha})=1/\alpha$ and $0\leq x(1-x)^{\alpha}/(1-(1-x)^{\alpha})\leq 1/\alpha$ for all $x\in(0,1]$, we have that

$$\lim_{u\to\infty}u^b I_{11} = \frac{\phi_1 c^b}{\alpha}\int_0^{\infty}z^{b-1}g^{\alpha}(z)\mathrm{d}z. \tag{22}$$

Next,

$$\begin{aligned}
I_{12} &= \mathrm{E}\Big[(1-a)^{-\alpha}\mathbf{1}(0<a<1-\epsilon)\sum_{x=-\infty}^{0}a^{(1-x)\alpha}\sup_{n\geq 1}\Big(\frac{1-a^n}{(u/c)+n}\Big)^{\alpha}\Big] \\
&\leq c^{\alpha}\mathrm{E}\Big[(1-a)^{-\alpha}\mathbf{1}(0<a<1-\epsilon)\frac{a^{\alpha}}{1-a^{\alpha}}\Big]u^{-\alpha} \\
&= Cu^{-\alpha}.
\end{aligned}$$

Since $b<\alpha-1$, we have $\lim_{u\to\infty}u^b I_{12}=0$. This proves part (i).

(ii) We use Theorem 2 as in part (i). Condition $m_n = O(n^\beta)$ is proved in [10], (A.13), with $\beta = 1/\alpha \in (0,1)$. Therefore it suffices to show (10) for $\psi_0(u)$. Consider the expectation I_2 in (17). Then

$$u^{\alpha-1} I_2 \;=\; u^{\alpha-1} c^{-\alpha} \mathrm{E}\Big[\frac{1}{(1-a)^\alpha} \sum_{x=1}^\infty \frac{1}{((u/c)+x-1)^\alpha}\, q_u^\alpha(a,x)\Big],$$

where

$$q_u(a,x) := \sup_{k\ge 1} \frac{1-a^k}{1+\frac{k}{(u/c)+x-1}}.$$

Note $0 \le q_u(a,x) \le 1$ and $q_u(a,x) \to 1\,(u\to\infty)$ for any $0<a<1$, $x\ge 1$ fixed. Indeed,

$$q_u(a,x) - 1 \;=\; \sup_{k\ge 1} \frac{-a^k - \frac{k}{(u/c)+x-1}}{1+\frac{k}{(u/c)+x-1}} \;=\; -\inf_{k\ge 1} \frac{a^k + \frac{k}{(u/c)+x-1}}{1+\frac{k}{(u/c)+x-1}} \;\to\; 0$$

follows by taking, e.g., $k = [\log u]$ in the last infimum. Therefore by the dominated convergence theorem

$$\begin{aligned}\lim_{u\to\infty} u^{\alpha-1} I_2 \;&=\; c^{-\alpha} \lim_{u\to\infty} \mathrm{E}\Big[\frac{1}{(1-a)^\alpha} \sum_{x=1}^\infty \frac{u^{\alpha-1}}{((u/c)+x-1)^\alpha}\Big] \\ &=\; \frac{1}{c(\alpha-1)} \mathrm{E}\big[\frac{1}{(1-a)^\alpha}\big] \;=\; c^{-1} K(\alpha,\Phi), \end{aligned} \tag{23}$$

where we used the fact that the last expectation is finite.

Next, consider

$$I_1 \;=\; \mathrm{E}\Big[\frac{a^\alpha}{(1-a^\alpha)(1-a)^\alpha} \Big(\sup_{n\ge 1} \frac{1-a^n}{u+nc}\Big)^\alpha\Big].$$

We claim that $I_1 = o(u^{-(\alpha-1)})$ and therefore part (ii) follows from the limit in (23). To prove the last claim, split the expectation $I_1 = I_{11} + I_{12}$ according to whether $0<a<1-\epsilon$ or $1-\epsilon<a<1$ holds, similarly to (18). It is clear that $I_{11} = O(u^{-\alpha}) = o(u^{-(\alpha-1)})$. Therefore it suffices to estimate I_{12} only. Then using (26), below, and the inequality $|1-(1-y)^\alpha| > Cy$, $0<y<\epsilon$, we obtain

$$\begin{aligned} I_{12} \;&\le\; C \int_0^\epsilon \frac{y^{b-\alpha}\mathrm{d}y}{1-(1-y)^\alpha} \Big(\sup_{n\ge 1} \frac{1-(1-y)^n}{u+nc}\Big)^\alpha \\ &\le\; C \int_0^\epsilon y^{b-1}\mathrm{d}y \Big(\sup_{n\ge 1} \frac{1-(1-y)^n}{y(u/c)+ny}\Big)^\alpha \\ &\le\; C \int_0^\epsilon y^{b-1}\mathrm{d}y \Big(\sup_{n\ge 1} \frac{1-\mathrm{e}^{-ny}}{y(u/c)+ny}\Big)^\alpha \\ &\le\; C \int_0^\epsilon y^{b-1} g^\alpha(yu/c)\mathrm{d}y \\ &\le\; C \int_0^\epsilon \frac{y^{b-1}}{(1+yu)^\alpha}\mathrm{d}y \\ &=\; C u^{-b} \int_0^{\epsilon u} \frac{z^{b-1}}{(1+z)^\alpha}\mathrm{d}z, \end{aligned}$$

where the last inequality follows from (8). If $\alpha > b$, the last integral is bounded and hence $I_{12} = O(u^{-b}) = o(u^{-(\alpha-1)})$. On the other hand, if $b \ge \alpha$, we easily obtain $I_{21} = O(u^{-\alpha}\log(u)) = o(u^{-(\alpha-1)})$. This concludes the proof of Theorem 1.

Lemma 3 *Let $g(z)$, $g_{u,x}(z)$ be defined at (7), (20), respectively. Then*

$$\lim_{u\to\infty} g_{u,x}(z) \;=\; g(z) \qquad (\forall\, z>0,\ \forall\, x\ge 1), \tag{24}$$

$$g_{u,x}(z) \;\le\; Cg(z), \qquad (\forall\, z>0,\ \forall\, u\ge 1,\ \forall\, x\ge 1), \tag{25}$$

where the constant C is independent of u, x, z. The function $g(z)$ satisfies (8).

Proof Let $\tau_k(y) := (1-(1-y)^k)/(1-\mathrm{e}^{-ky})$, $0<y<1$, $k=1,2,\cdots$. Let us first prove the elementary inequality: for any $0<\epsilon<1$ there exists a constant $C>0$, independent of $0<\epsilon<1$, $k\ge 1$ and such that

$$|\tau_k(y)-1| \;\le C(\epsilon + k^{-1}), \quad \forall\, 0<y<\epsilon,\ \forall\, k=1,2,\ldots. \tag{26}$$

Indeed, let $0<y\le 1/(2k)$. Since $1-\mathrm{e}^{-x}\ge x/2$, $0<x<1/2$ so

$$|\tau_k(y)-1| \le 2\frac{|\mathrm{e}^{-ky}-(1-y)^k|}{ky} \le C\frac{k|\mathrm{e}^{-y}-1+y|}{ky} \le Cy \le C/k.$$

Next, let $1/(2k)<y<\epsilon<1$. Then $1-\mathrm{e}^{-ky}\ge 1-\mathrm{e}^{-1/2}>0$ and $\log(1-y)\le -y(1-\epsilon)$. Therefore

$$\begin{aligned} |\tau_k(y)-1| &\le C|\mathrm{e}^{-ky}-(1-y)^k| \le C \sup_{k\ge 1,\,1/2<x\le\epsilon k} |\mathrm{e}^{k\log(1-\frac{x}{k})}-\mathrm{e}^{-x}| \\ &\le C\sup_{x>1/2}\left(\mathrm{e}^{-x(1-\epsilon)}-\mathrm{e}^{-x}\right) \;\le C\epsilon, \end{aligned}$$

since $\sup_{x\ge 1/2} x\mathrm{e}^{-x(1-\epsilon)}<\infty$. This proves (26).

Using (26) we can write

$$\begin{aligned} g_{u,x}(z) \;&:=\; \sup_{k\ge 1}\tau_k\Big(\frac{z}{(u/c)+x-1}\Big)\frac{1-\mathrm{e}^{-\frac{zk}{(u/c)+x-1}}}{z+\frac{zk}{(u/c)+x-1}}\mathbf{1}(0<z<\epsilon((u/c)+x-1)) \\ &\le\; C\sup_{k\ge 1}\frac{1-\mathrm{e}^{-\frac{zk}{(u/c)+x-1}}}{z+\frac{zk}{(u/c)+x-1}} \;\le\; Cg(z), \end{aligned} \tag{27}$$

thus proving the bound in (25). The convergence (24) follows similarly from (27) and (26).

To show (8), note that $\omega\mapsto\frac{1-\mathrm{e}^{-\omega}}{z+\omega}$ increases on the interval $(0,\omega_*)$ and decreases on (ω_*,∞), where $\omega_*=\omega_*(z)>0$ is the unique solution of $\omega+z+1=\mathrm{e}^{\omega}$. Thus, $g(z)=\frac{1}{z+1+\omega_*}$. It is clear that $\omega_*\to 0\,(z\to 0)$ and therefore $\lim_{z\to 0} g(z)=1$. Moreover, $\omega_*\to\infty\,(z\to\infty)$ and $\omega_*\le\log(1+z)$, implying $\lim_{z\to\infty} zg(z)=\lim_{z\to\infty}\frac{z}{z+1+\omega_*}=1$. Lemma 3 is proved. □

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