Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2013) 37: 145 – 164
© TÜBİTAK
doi:10.3906/mat-1106-30

Braiding for internal categories in the category of whiskered groupoids and simplicial groups

Erdal ULUALAN*, **Sedat PAK**
Dumlupınar University, Science Faculty, Department of Mathematics, Kütahya, Turkey

Received: 16.06.2011 • **Accepted:** 18.11.2011 • **Published Online:** 17.12.2012 • **Printed:** 14.01.2013

Abstract: In this work, we define the notion of 'braiding' for an internal groupoid in the category of whiskered groupoids and we give a relation between this structure and simplicial groups by using higher order Peiffer elements in the Moore complex of a simplicial group.

Key words: Simplicial groups, crossed modules, groupoids

1. Introduction

Brown and Gilbert [12] have defined a braiding map for a *regular* crossed module over groupoids. They have proved that the category of braided regular crossed modules is equivalent to that of simplicial groups with Moore complex of length 2. Braided monoidal categories were defined by Joyal and Street in [21]. They have also defined crossed semi-modules for monoids with a bracket operation and given an equivalence between the category of braided monoidal categories and the category of crossed semi-modules with bracket operations. For further work about braided monoidal categories see also [8] and [22].

Categorical groups are monoidal groupoids in which every object is invertible, up to isomorphism, with respect to the tensor product (cf. Breen [10] and Joyal-Street [21, 20]). These structures sometimes are equipped with a braiding or a symmetry (cf. [9, 19, 21, 20]). Garzon and Miranda, [19], gave the relation between the category of categorical groups equipped with a braiding and the category of reduced 2-crossed modules by using Brown-Spencer theorem given in [14]. For these categorical notions see also [5, 6, 13].

In order to define the notion of commutativity for a groupoid and to discuss related questions, Brown in [11] has introduced an extra structure called a '*whiskering operation*'. Groupoids with whiskering operations are called 'whiskered groupoids'. To put a braiding on an internal groupoid in the category of whiskered groupoids over the same monoid of objects, we need the notions of left and right multiplications and commutators of two morphisms in a groupoid together with a whiskering operation, similarly to the definition of braiding for a categorical group (cf. [19] and [21]). Brown also in his work [11] has defined these notions for the morphisms in a groupoid by using the whiskering operations.

Thus our aims in this paper are:

(i) to give a definition of 'braiding' for internal groupoids in the category of whiskered groupoids over the same objects set by considering the 'whiskering operations', and

*Correspondence: eulualan@dumlupinar.edu.tr

2000 *AMS Mathematics Subject Classification:* 13D25, 18G30, 18G50, 18G55.

(*ii*) to give a description of the passage from a simplicial group with Moore complex of length 2 to this internal groupoid equipped with the braiding.

2. Preliminaries

In this section, we recall the basic properties of simplicial groups from [18, 23, 24] and the notion of 'braiding' for a monoidal category (cf. [20]).

2.1. Simplicial groups and Moore complexes

A simplicial group $\mathbf{G}$ consists of a family of groups G_n together with face and degeneracy maps $d_i^n : G_n \to G_{n-1}$, $0 \leq i \leq n$ $(n \neq 0)$ and $s_i^n : G_n \to G_{n+1}$, $0 \leq i \leq n$ satisfying the usual simplicial identities given by May [23]. In fact, it can be completely described as a functor $\mathbf{G} : \Delta^{op} \to \mathbf{Grp}$ where Δ is the category of finite ordinals. We will denote the category of simplicial groups by $\mathfrak{SimpGrp}$.

The Moore complex $(\mathbf{NG}, \partial)$ of a simplicial group $\mathbf{G}$ is a chain complex defined by

$$(NG)_n = \ker d_0 \cap \ker d_1 \cap \cdots \cap \ker d_{n-1} \subseteq G_n.$$

The differential ∂_n is the restriction of the missing face operator d_n.

We say that the Moore complex $\mathbf{NG}$ of a simplicial group $\mathbf{G}$ is of *length* k if $NG_n = 1$ for all $n \geq k+1$. We denote the category of simplicial groups with Moore complex of length k by $\mathfrak{SimpGrp}_{\leq k}$. The Moore complex of a simplicial group carries a lot of fine structure and this has been studied, e.g. by Carrasco-Cegarra [15], Arvasi-Porter [2], Conduché [17], Mutlu-Porter [24] and Arvasi-Ulualan [3].

Mutlu and Porter in [24] defined functions $F_{\alpha,\beta}$ which are variants of Carrasco-Cegarra pairing operators (cf. [15]) called *Peiffer Pairings* and they have investigated the image $\partial_n(N_n)$ for $n = 2, 3, 4$, where N_n is a normal subgroup of G_n generated by elements $F_{\alpha,\beta}(x_\alpha, y_\beta)$, and ∂_n is the differential in the Moore complex. They gave a construction of a free simplicial group by using these operators in [25]. For a general construction of these structures over operads, see also [16]. When we construct the relations among simplicial groups and internal groupoids within whiskered groupoids, we use the functions $F_{\alpha,\beta}$.

2.2. Braided categorical groups and crossed modules

Joyal and Street in [21] have defined the notion of braiding for a categorical group. Let A and O be groups and $A \underset{e}{\overset{s,t}{\rightleftarrows}} O$ an internal category in the category of groups. A braiding for this structure (cf. [21], [9], [19]) is a map $\tau_{a,b} : O \times O \to A$ which satisfies the conditions

(*i*) $s\tau_{a,b} = ab$, $t\tau_{a,b} = ba$,

(*ii*) $x : a \to a'$, and $y : b \to b'$ in A, $\tau_{a',b'} \circ xy = yx \circ \tau_{a,b}$,

(*iii*) $\tau_{a,bc} = (I_b\tau_{a,c}) \circ (\tau_{a,b}I_c)$,

(*iv*) $\tau_{ab,c} = (\tau_{a,c}I_b) \circ (I_a\tau_{b,c})$,

(*v*) $\tau_{1,a} = \tau_{a,1} = e(a)$,

for $a, b, c \in O$ and $x, y \in A$.

Crossed modules were introduced by Whitehead [27] as models for connected 2-types. *A crossed module* is a group homomorphism $\partial : M \to P$ together with an action of P on M, written ${}^pm$ for $p \in P$ and $m \in M$, satisfying the conditions $\partial({}^{p}m) = p\partial(m)p^{-1}$ and ${}^{m}\partial m' = mm'm^{-1}$ for all $m, m' \in M, p \in P$. Braided *regular* crossed modules on groupoids were defined by Brown and Gilbert in [12] as models for homotopy connected 3-types. In [1], the relationship between braided crossed modules and reduced simplicial groups was reproved by use of the functions $F_{\alpha,\beta}$. Also in [4], this construction was extended to the 'regularity'. That is, 'a description of the passage from a simplicial group to a braided regular crossed module by use of the functions $F_{\alpha,\beta}$'.

From the results of the cited works, the category of braided internal categories in the category of groups is equivalent to that of braided crossed modules and the monoid version of this equivalence was given in [20] as we mentioned above. Furthermore, since the category of braided crossed modules is equivalent to that of reduced simplicial groups with Moore complex of length 2, we can say that the category of braided internal categories within groups is also equivalent to that of reduced simplicial groups with Moore complex of length 2 (cf. [26]). We can consider the groupoid cases of these structures. The groupoid case of a braided crossed module is clearly a braided *regular* crossed module of groupoids and the category of these objects is equivalent to that of simplicial groups with Moore complex of length 2 (cf. [12]). So, we can ask what is the groupoid case of a braided categorical groups, or how can the notion of braiding for an internal groupoid in the category of groupoids be defined? To define this structure and to give a relationship between this structure and simplicial groups, we need the notion of whiskered groupoid introduced by Brown in [11].

3. Braiding for internal categories in the category of whiskered groupoids

3.1. Whiskered categories

Let $\mathfrak{C}$ be a small category with set of objects written C_0. The set of arrows of $\mathfrak{C}$ is written C_1. The set of morphisms $x \to y$ from x to y is written $C_1(x,y)$, and x, y are the source and target of such a morphism. The source and target maps are written $s, t : C_1 \to C_0$. We will write the composition of $f : x \longrightarrow y$ and $g : y \longrightarrow z$ as $gf : x \longrightarrow z$, or $g \circ f$. Then we have $s(g \circ f) = s(f)$ and $t(g \circ f) = t(g)$. We write $C_1(x,x)$ as $C_1(x)$.

Brown has defined the notion of 'whiskering' for a category $\mathfrak{C}$ and gave the notions of left and right multiplications on a whiskered category $\mathfrak{C}$. The following definition is due to Brown [11].

Definition 3.1 *A whiskering on a category $\mathfrak{C}$ (whose set of objects is C_0 and set of morphisms is C_1) consists of operations*

$$m_{i,j} : C_i \times C_j \longrightarrow C_{i+j},\ i,j = 0,1,\ i+j \leqslant 1$$

satisfying the following axioms:

Whisk 1. $m_{0,0}$ gives a monoid structure on C_0;

Whisk 2. $m_{0,1} : C_0 \times C_1 \longrightarrow C_1$ is a left action of the monoid C_0 on the category $\mathfrak{C}$ in the sense that, if $x \in C_0$ and $a : u \longrightarrow v$ in C_1, then

$$m_{0,1}(x,a) : m_{0,0}(x,u) \longrightarrow m_{0,0}(x,v)$$

in $\mathfrak{C}$, so that:

$$m_{0,1}(1,a) = a,\ m_{0,1}(m_{0,0}(x,y),a) = m_{0,1}(x, m_{0,1}(y,a))$$

$$m_{0,1}(x, a \circ b) = m_{0,1}(x,a) \circ m_{0,1}(x,b),\ m_{0,1}(x, 1_y) = 1_{xy}.$$

Whisk 3. $m_{1,0} : C_1 \times C_0 \longrightarrow C_1$ is a right action of the monoid C_0 on C_1 with analogous rules.

Whisk 4.

$$m_{0,1}(x, m_{1,0}(a, y)) = m_{1,0}(m_{0,1}(x, a), y),$$

for all $x, y, u, v \in C_0$, $a, b \in C_1$.

A category $\mathfrak{C}$ *together with a whiskering is called a whiskered category.* $\square$

Recall that a *groupoid* is a small category in which every arrow is an isomorphism. That is, for any morphism a there exists a (necessarily unique) morphism a^{-1} such that $a \circ a^{-1} = e_{t(a)}$ and $a^{-1} \circ a = e_{s(a)}$, where $e : C_0 \to C_1$ gives the identity morphism at an object. We write a groupoid as (C_1, C_0), where C_0 is the set of objects and C_1 is the set of morphisms. For any groupoid C, if $C_1(x, y)$ is empty whenever x, y are distinct (that is, if $s = t$), then C is called *totally disconnected*. A groupoid (C_1, C_0) together with the whiskering operations $m_{i,j} : C_i \times C_j \to C_{i+j}$ for $i + j \leqslant 1$ satisfying the conditions (*Whisk 1…Whisk 4*) is called a whiskered groupoid. We denote a whiskered groupoid by $(C_1, C_0, m_{i,j})$.

Let $\partial : M \to P$ be a crossed module. By using the action of P on M, we can consider the semi-direct product group $M \rtimes P$. Then, by taking $C_0 = P$ and $C_1 = M \rtimes P$ we can create a whiskered groupoid as follows: The source and target maps from C_1 to C_0 are given by $s(m, p) = p$ and $t(m, p) = \partial(m)p$ for all $(m, p) \in C_1$. The groupoid composition is given by $(m', p') \circ (m, p) = (m'm, p)$ if $p' = \partial(m)p$. Finally, the whiskering operations $m_{0,1}$ and $m_{1,0}$ are given, respectively, $m_{0,1}(p, (m, p')) = ({}^{p}m, pp')$ and $m_{1,0}((m, p'), p) = (m, p'p)$ for all $m \in M, p, p' \in P$.

Proposition 3.2 *In a whiskered groupoid* $(C_1, C_0, m_{i,j})$, *if the monoid of objects* C_0 *is a group with the multiplication* $m_{0,0}$, *then*

(i) *the set* $K = \{a \in C_1 : t(a) = 1_{C_0}\}$ *is a group with the group operation given for any* $a, b \in K$ *by* $ab = b \circ m_{1,0}(a, s(b))$,

(ii) *the source map* s *from* K *to* C_0 *is a homomorphism of groups,*

(iii) C_0 *acts on* K *by* ${}^{p}a = m_{0,1}(p, m_{1,0}(a, p^{-1}))$ *or* ${}^{p}a = m_{1,0}(m_{0,1}(p, a), p^{-1}))$ *for* $a \in K, p \in C_0$, *and* C_0 *acting on itself by conjugation,*

(iv) *the homomorphism* s *is* C_0*-equivariant relative to the left action of* C_0 *on* K *given above.*

Proof (Sketch)

(i) We leave it to the reader.

(ii) For $a, b \in K$, we have $s(ab) = s(b \circ m_{1,0}(a, s(b))) = s(m_{1,0}(a, s(b)))$. From axiom *Whisk 3.)*, we have

$$m_{1,0}(a, s(b)) : m_{0,0}(s(a), s(b)) \longrightarrow m_{0,0}(t(a), t(b)),$$

and then we obtain $s(m_{1,0}(a, s(b))) = m_{0,0}(s(a), s(b)) = s(a)s(b)$.

(iii) For $p_1, p_2 \in C_0$ and $a \in K$, we have

$$\begin{aligned}
{}^{p_1}({}^{p_2}a) &= {}^{p_1}\left(m_{0,1}(p_2, m_{1,0}(a, p_2^{-1}))\right) \\
&= m_{0,1}(p_1, m_{1,0}(m_{0,1}(p_2, m_{1,0}(a, p_2^{-1})), p_1^{-1})) \\
&= m_{0,1}(p_1, m_{0,1}(p_2, m_{1,0}(m_{1,0}(a, p_2^{-1}), p_1^{-1}))) \quad \text{(due to } \textit{Whisk 4.}\text{)} \\
&= m_{0,1}(p_1, m_{0,1}(p_2, m_{1,0}(a, p_2^{-1}p_1^{-1}))) \quad \text{(due to } \textit{Whisk 2.}\text{)} \\
&= m_{0,1}(p_1p_2, m_{1,0}(a, p_2^{-1}p_1^{-1})) \quad \text{(due to } \textit{Whisk 2.}\text{)} \\
&= {}^{p_1p_2}\, a
\end{aligned}$$

and

$$\begin{aligned}
{}^{(1_{C_0})}a &= m_{0,1}(1, m_{1,0}(a, 1)) \\
&= m_{1,0}(a, 1) \quad \text{(due to } \textit{Whisk 2.}\text{)} \\
&= a. \quad \text{(due to } \textit{Whisk 3.}\text{)}
\end{aligned}$$

For $a, b \in K$ and $p \in C_0$, we have

$$\begin{aligned}
({}^{p}a)({}^{p}b) &= \left[m_{0,1}(p, m_{1,0}(a, p^{-1}))\right]\left[m_{0,1}(p, m_{1,0}(b, p^{-1}))\right] \\
&= m_{0,1}(p, m_{1,0}(b, p^{-1})) \circ m_{1,0}\left(m_{0,1}(p, m_{1,0}(a, p^{-1})), ps(b)p^{-1}\right) \\
&= m_{0,1}(p, m_{1,0}(b, p^{-1})) \circ m_{0,1}\left(p, m_{1,0}(m_{1,0}(a, s(b)), p^{-1})\right) \quad \text{(due to } \textit{Whisk 4.}\text{)} \\
&= m_{0,1}\left(p, m_{1,0}(b \circ m_{1,0}(a, s(b)), p^{-1})\right) \quad \text{(due to } \textit{Whisk 2.}\text{)} \\
&= m_{0,1}(p, m_{1,0}(ab, p^{-1})) \\
&= {}^{p}\,(ab).
\end{aligned}$$

(iv) For $p \in C_0$ and $a \in K$, we have

$$\begin{aligned}
s({}^{p}a) &= s(m_{0,1}(p, m_{1,0}(a, p^{-1}))) \\
&= ps(m_{1,0}(a, p^{-1})) \quad \text{(due to } \textit{Whisk 2.}\text{)} \\
&= ps(a)p^{-1}. \quad \text{(due to } \textit{Whisk 3.}\text{)}
\end{aligned}$$

□

Let **WGp** be the category of whiskered groupoids. Define a subcategory of **WGp** whose objects are whiskered groupoids over the same monoid of objects C_0. We will denote this subcategory by $\mathbf{WGp}/_{\mathbf{C_0}}$. In what follows, $\textbf{Cat-WGp}/_{\mathbf{C_0}}$ will denote the category of internal categories in the category of whiskered groupoids over the same monoid of objects C_0. An object of $\textbf{Cat-WGp}/_{\mathbf{C_0}}$ will be represented by the diagram

$$\begin{tikzcd}
C_1 \arrow[r, shift left=2, "\varepsilon_0, \varepsilon_1"] \arrow[r, shift left=1] \arrow[d, shift right=2, "e'"'] & D_1 \arrow[l, shift left=1, "I"] \arrow[d, shift right=2, "e"'] \\
C_0 \arrow[u, shift right=0] \arrow[u, shift left=2, "s', t'"'] \arrow[r, equal, "id"'] & C_0 \arrow[u, shift right=0] \arrow[u, shift left=2, "s, t"']
\end{tikzcd}$$

where $(C_1 \underset{e'}{\overset{s,t}{\rightleftarrows}} C_0, \circ, m'_{ij})$ and $(D_1 \underset{e}{\overset{s,t}{\rightleftarrows}} C_0, \circ, m_{ij})$ are whiskered groupoids,

$$(C_1 \underset{I}{\overset{\epsilon_0,\epsilon_1}{\rightleftarrows}} D_1 , *)$$

gives a small category, and the maps ϵ_0, ϵ_1 are identities on C_0. A 2-morphism x in the category (C_1, D_1) between the 1-morphisms $a, a' : u \to v \in D_1(u, v)$ is represented by $x : a \Rightarrow a'$, where $\epsilon_0(x) = a$ and $\epsilon_1(x) = a'$, and for 2-morphisms $x, y \in (C_1, D_1)$, $\epsilon_0(x * y) = \epsilon_0(y)$ and $\epsilon_1(x * y) = \epsilon_1(x)$ when $\epsilon_0(x) = \epsilon_1(y)$.

To define the notion of braiding on an object in the category $\mathbf{Cat\text{-}WGp/_{C_0}}$, the notions of left and right multiplications on the whiskered groupoid $(D_1 \underset{e}{\overset{s,t}{\rightleftarrows}} C_0, m_{ij})$ must be defined.

We can take from [11] the left and right multiplications on a whiskered category

$$\mathfrak{C} : (D_1 \underset{e}{\overset{s,t}{\rightleftarrows}} C_0 , m_{ij})$$

for any $a, b \in D_1$, by

$$l(a, b) = m_{0,1}(t(a), b) \circ m_{1,0}(a, s(b))$$

and

$$r(a, b) = m_{1,0}(a, t(b)) \circ m_{0,1}(s(a), b),$$

where $s, t : D_1 \longrightarrow C_0$ are the source and target maps.

If

$$\mathfrak{C} : (D_1 \underset{e}{\overset{s,t}{\rightleftarrows}} C_0 , m_{ij})$$

is a whiskered groupoid, the commutators are defined by

$$[a, b] = r(a, b) \circ l(a, b)^{-1}$$

for $a, b \in D_1$.

Definition 3.3 Let $\mathbf{C}$ be an internal category in the category $\mathbf{WGp/_{C_0}}$ represented by a diagram

$$\begin{tikzcd} C_1 \arrow[r, shift left, "\epsilon_0{,}\epsilon_1"] \arrow[r, shift left=2] \arrow[d, shift left] \arrow[d, shift left=2, "s'{,}t'"] & D_1 \arrow[l, bend left, "I"] \arrow[d, shift right] \arrow[d, shift right=2, "s{,}t"'] \\ C_0 \arrow[u, bend left, "e'"] \arrow[r, equal, "id"'] & C_0 , \arrow[u, bend right, "e"'] \end{tikzcd}$$

as given above. The morphisms ϵ_0, ϵ_1 and I are identity morphisms on the objects set C_0 and they preserve the whiskering structure on the groupoids C_1 and D_1.

Braiding on this internal category is a map

$$\begin{aligned} \tau_{a,b} \quad : \quad (D_1, C_0) \times (D_1, C_0) &\longrightarrow (C_1, C_0) \\ (a,b) &\longmapsto \tau_{a,b} \end{aligned}$$

satisfying the following conditions.

BW1. For $a, b \in D_1$, $\epsilon_0 \tau_{a,b} = r(a,b)$, and $\epsilon_1 \tau_{a,b} = l(a,b)$. Thus we have

$$\tau_{a,b} : r(a,b) \to l(a,b),$$

and from this axiom we can give the commutator of two morphisms a, b in the groupoid D_1 by

$$[a,b] = (\epsilon_0 \tau_{a,b}) \circ (\epsilon_1 \tau_{a,b})^{-1}.$$

BW2. For $a \in D_1$ and $p \in C_0$, $\tau_{e(p),a} = m'_{0,1}(p, I(a))$, and $\tau_{a,e(p)} = m'_{1,0}(I(a), p)$.

BW3. For $a, b, c \in D_1$, with $t(c) = s(b)$ the following diagram is commutative:

$$\begin{tikzcd}
 & m_{0,1}(t(a),b) \circ [m_{0,1}(t(a),c) \circ m_{1,0}(a,s(c))] & \\
m_{0,1}(t(a)\ \ b \circ c) \circ m_{1,0}(a,s(c)) \arrow[ur, equal, "whisk2."] \arrow[d, dash, "\tau_{a,b\circ c}"'] & & m_{0,1}(t(a),b) \circ [m_{1,0}(a,t(c)) \circ m_{0,1}(s(a),c)] \arrow[ul, "m'_{0,1}(t(a),I(b))\circ\tau_{a,c}"'] \arrow[d, equal, "t(c)=s(b)"] \\
m_{1,0}(a,t(b)) \circ m_{0,1}(s(a)\ \ b \circ c) \arrow[dr, equal, "whisk2."'] & & m_{0,1}(t(a),b) \circ [m_{1,0}(a,s(b)) \circ m_{0,1}(s(a),c)] \\
 & {[m_{1,0}(a,t(b)) \circ m_{0,1}(s(a),b)] \circ m_{0,1}(s(a),c),} \arrow[ur, "\tau_{a,b}\circ m'_{0,1}(s(a),I(c))"'] &
\end{tikzcd}$$

or equivalently,

$$\tau_{a,b\circ c} = [m'_{0,1}(t(a), I(b)) \circ \tau_{a,c}] * [\tau_{a,b} \circ m'_{0,1}(s(a), I(c))].$$

BW4. For $a, b, c \in D_1$ with $t(b) = s(a)$, the following diagram is commutative

$$\begin{tikzcd}
 & m_{0,1}(t(a),c) \circ [m_{1,0}(a,s(c)) \circ m_{1,0}(b,s(c))] & \\
m_{0,1}(t(a),c) \circ m_{1,0}(a \circ b, s(c)) \arrow[ur, equal, "whisk2."] \arrow[d, dash, "\tau_{a\circ b,c}"'] & & {[m_{1,0}(a,t(c)) \circ m_{0,1}(s(a),c)] \circ m_{1,0}(b,s(c))} \arrow[ul, "\tau_{a,c}\circ m'_{1,0}(I(b),s(c))"'] \arrow[d, equal, "t(b)=s(a)"] \\
m_{1,0}(a \circ b, t(c)) \circ m_{0,1}(s(b),c) \arrow[dr, equal, "whisk2."'] & & m_{1,0}(a,t(c)) \circ [m_{0,1}(t(b),c) \circ m_{1,0}(b,s(c))] \\
 & m_{1,0}(a,t(c)) \circ [m_{1,0}(b,t(c)) \circ m_{0,1}(s(b),c)], \arrow[ur, "m'_{1,0}(I(a),t(c))\circ\tau_{b,c}"'] &
\end{tikzcd}$$

or equivalently

$$\tau_{a\circ b,c} = [\tau_{a,c} \circ m'_{1,0}(I(b), s(c))] * [m'_{1,0}(I(a), t(c)) \circ \tau_{b,c}].$$

BW5. For the 2-morphisms $x : a \Rightarrow a'$ and $y : b \Rightarrow b' \in (C_1, D_1)$,

$$\epsilon_i(l(x,y) * \tau_{a,b}) = \epsilon_i(\tau_{a',b'} * r(x,y))$$

for $i = 0, 1$.

An internal category $\mathbf{C}$ together with a braiding is called a braided internal category within whiskered groupoids. □

Example 3.4 *Let*

$$\begin{tikzcd}
C_1 \arrow[r, shift left, "d_0{,}d_1"] \arrow[r] \arrow[r, bend right, "I"'] \arrow[d, shift left, "s'{,}t'"] \arrow[d] & D_1 \arrow[d, shift right, "s{,}t"'] \arrow[d] \\
C_0 \arrow[u, bend left, "e'"] \arrow[r, equal, "id"'] & C_0 \arrow[u, bend right, "e"']
\end{tikzcd}$$

be a braided internal category in the category of whiskered groupoids over the same monoid of objects C_0 together with the braiding $\tau : D_1 \times D_1 \to C_1$. If the monoid of objects C_0 is a trivial monoid, then the left and right actions of C_0 on C_1, D_1 determined by the whiskering operations $m_{i,j}$ are trivial actions, and C_1, D_1 are groups. Then we have $r(a,b) = ab$, $l(a,b) = ba$ for $a, b \in D_1$, and $[a,b] = r(a,b)l(a,b)^{-1} = aba^{-1}b^{-1}$. Thus the braiding axioms above reduce the following conditions:
(i) $\epsilon_0(\tau_{a,b}) = ab$ *and* $\epsilon_1(\tau_{a,b}) = ba$*, that is,* $\tau_{a,b} : ab \to ba$,
(ii) $x : a \to a'$*, and* $y : b \to b'$, $\tau_{a',b'} \circ xy = yx \circ \tau_{a,b}$,
(iii) $\tau_{a,bc} = (I_b\tau_{a,c}) \circ (\tau_{a,b}I_c)$,
(iv) $\tau_{ab,c} = (\tau_{a,c}I_b) \circ (I_a\tau_{b,c})$,
(v) $\tau_{1,a} = \tau_{a,1} = I(a)$,
for $a, b, c \in D_1$ *and* $x, y \in C_1$.

This is the reduced case of braided internal groupoids within whiskered groupoids and gives a braided categorical group as given in [9, 19, 21, 20].

4. Simplicial groups and braided internal categories within whiskered groupoids

In this section we will give a description of the passage from a simplicial group to a braided internal category in the category of whiskered groupoids. First, we recall the semi-direct product groupoids (cf. [12]).

Let C and H be groupoids over the same object set C_0 and H totally disconnected. Suppose that the groupoid C has a left action on the groupoid H. Then, we can define the semi-direct product as follows: Let $h \in H_1(y)$ and $c \in C_1(x,y)$, then, for $x, y \in C_0$

$$(H \tilde{\times} C)(x,y) = H(y) \times C(x,y)$$

is a groupoid and composition is defined by

$$(h, c) \circ (h', c') = (h \circ^c h', c \circ c').$$

Now, we can give a description of the passage from simplicial groups to braided internal categories in the category of whiskered groupoids.

Let $\mathbf{G}$ be a simplicial group with Moore complex $(\mathbf{NG}, \partial)$. From this Moore complex, we will construct a braided internal category within whiskered groupoids over the same monoid of objects C_0 denoting it by the diagram

$$\begin{tikzcd}
C_1 \arrow[r, shift left, "d_0{,}d_1"] \arrow[r, shift right] \arrow[r, bend right, "I"'] & D_1 \\
C_0 \arrow[u, bend left, "e'"] \arrow[u, shift left] \arrow[u, shift right, "s'{,}t'"'] \arrow[r, equal, "id"'] & C_0. \arrow[u, shift left] \arrow[u, shift right, "s{,}t"] \arrow[u, bend right, "e"']
\end{tikzcd}$$

Let $C_0 = NG_0 = G_0$. Using the action of G_0 via s_0, we define the semi-direct product $D_1 = NG_1 \rtimes s_0NG_0$. Notice that s_0 is a section of d_0 and NG_1 is the kernel of d_0 the group G_1 is the semi-direct product $G_1 = D_1 = NG_1 \rtimes G_0$. For $(g,p) \in D_1$, we define the source, target and identity maps by $s(g,p) = p$, $t(g,p) = d_1(g)p$ and $e(p) = (1,p)$, respectively, and where $d_1 = d_1^1 = \partial_1$ the differential of the Moore complex restricted to NG_1. Thus we have

$$\bullet\, p \xrightarrow{(g,p)} \bullet\, \partial_1(g)p$$

is a morphism in D_1. The groupoid composition on D_1 can be given by

$$(g',p') \circ (g,p) = (g'g,p)$$

when $s(g',p') = p' = t(g,p) = (\partial_1 g)p$. Then we have $s((g',p') \circ (g,p)) = s((g'g,p)) = p = s(g,p)$ and $t((g',p') \circ (g,p)) = t((g'g,p)) = \partial_1(g')\partial_1(g)p = \partial_1(g')p' = t(g',p')$. Furthermore, the inverse a^{-1} of the morphism $a = (g,p) : p \to d_1gp$ can be defined by $a^{-1} = (g^{-1}, d_1gp) : d_1gp \to p$. Thus we have $a \circ a^{-1} = (g,p) \circ (g^{-1}, d_1gp) = (1, d_1gp) = e(d_1gp) = e(t(a))$ and $a^{-1} \circ a = (g^{-1}, d_1gp) \circ (g,p) = (1,p) = e(p) = e(s(a))$. Thus we have the following proposition.

Proposition 4.1 *The groupoid*

$$\mathfrak{D} : (D_1 \underset{e}{\overset{s,t}{\rightleftarrows}} C_0)$$

constructed above is a whiskered groupoid together with the operations on $\mathfrak{D}$ *given by*

$$\begin{array}{rl}
m_{0,1}: & C_0 \times D_1 \longrightarrow D_1 \\
 & (p,(g,q)) \longmapsto (s_0pgs_0p^{-1}, pq), \\
m_{1,0}: & D_1 \times C_0 \longrightarrow D_1 \\
 & ((g,q),p) \longmapsto (g,qp)
\end{array}$$

for $p,q \in C_0$ *and* $(g,q) \in D_1$*, and*

$$m_{0,0} : C_0 \times C_0 \longrightarrow C_0$$

is the group operation on NG_0*.*

The left and right multiplications l, r on the whiskered groupoid

$$\mathfrak{D} : (D_1 \underset{e}{\overset{s,t}{\rightleftarrows}} C_0, m_{ij})$$

for the morphisms $a = (g, p) : p \to d_1(g)p$ and $b = (h, q) : q \to d_1(h)q$ in D_1 can be given by

$$\begin{aligned} l(a, b) &= m_{0,1}(t(a), b) \circ m_{1,0}(a, s(b)) \\ &= m_{0,1}(\partial_1 gp, (h, q)) \circ m_{1,0}((g, p), q) \\ &= (s_0 d_1 g s_0 p h s_0 p^{-1} s_0 d_1 g^{-1}, d_1 gpq) \circ (g, pq) \\ &= (s_0 d_1 g s_0 p h s_0 p^{-1}(s_0 d_1 g^{-1})g, pq), \end{aligned}$$

and

$$\begin{aligned} r(a, b) &= m_{1,0}(a, t(b)) \circ m_{0,1}(s(a), b) \\ &= m_{1,0}((g, p), d_1 hq) \circ m_{0,1}(p, (h, q)) \\ &= (g, pd_1 hq) \circ (s_0 p h s_0 p^{-1}, pq) \\ &= (g s_0 p h s_0 p^{-1}, pq). \end{aligned}$$

Now, we continue the construction. Using the action of NG_0 on $NG_2/\partial_3(NG_3 \cap D_3) = NG_2'$ via $s_0 s_0 = s_1 s_0$, we can construct

$$C_1' = NG_2' \rtimes NG_0$$

together with the source, target and identity maps given by

$$s(\overline{l}, p) = p = t(\overline{l}, p),\ e(p) = (1, p)$$

for $\overline{l} = l(\partial_3(NG_3 \cap D_3)) \in NG_2'$ and $p \in NG_0$. The groupoid composition on C_1' can be given by $(\overline{l_1}, p) \circ (\overline{l_2}, p) = (\overline{l_1 l_2}, p)$ for $l_1, l_2 \in NG_2$. Thus we have the following result.

Proposition 4.2 *The diagram*

$$\mathfrak{C} : (C_1' \underset{e}{\overset{s,t}{\rightleftarrows}} C_0)$$

becomes a whiskered groupoid together with the maps $\sigma_{0,0} = m_{0,0}$

$$\sigma_{0,1} : C_0 \times C_1' \to C_1'$$

given by $(p, (\overline{l}, q)) \mapsto (\overline{(s_1 s_0 p) l (s_1 s_0 p^{-1})}, pq)$ *and*

$$\sigma_{1,0} : C_1' \times C_0 \to C_1'$$

given by $((\overline{l}, q), p) \longmapsto (\overline{l}, qp)$, *for* $(\overline{l}, q) \in C_1'$ *and* $p \in C_0$.

Thus far we have constructed two whiskered groupoids

$$\mathfrak{C} : (C_1' \underset{e}{\overset{s,t}{\rightleftarrows}} C_0, \sigma_{ij})$$

and

$$\mathfrak{D} : (D_1 \underset{e}{\overset{s,t}{\rightleftarrows}} C_0, m_{ij}),$$

over the same objects set C_0 where the groupoid $\mathfrak{C}$ is a totally disconnected groupoid.

The groupoid action of $(g,q) \in D_1$ on $(\bar{l},q) \in C_1'$ can be given by

$$^{(g,q)}(\bar{l},q) = (\overline{s_1 g l s_1 g^{-1}}, d_1 gq).$$

By using this groupoid action of $\mathfrak{D}$ on $\mathfrak{C}$, we can define the semi-direct product groupoid

$$C_1(x,y) = (C_1' \tilde{\times} D_1)(x,y) = C_1'(y) \times D_1(x,y)$$

for $x, y \in C_0$, on the object set C_0, together with the vertical composition given by

$$((\bar{l}, d_1 gq),(g,q)) \circ ((\overline{l'}, d_1 g'q'),(g',q')) = ((\overline{l s_1 g l' s_1 g^{-1}}, d_1 gq),(gg',q'))$$

when $q = (d_1 g')q'$. The source and target maps $s', t' : C_1 \to C_0$ are defined by $s'((\bar{l}, d_1 gq),(g,q)) = q$ and $t'((\bar{l}, d_1 gq),(g,q)) = d_1 gq$ for any 2-morphism $((\bar{l}, d_1 gq),(g,q))$ in C_1. Thus, for any $x = ((\bar{l}, d_1 gq),(g,q))$ and $y = ((\overline{l'}, d_1 g'q'),(g',q'))$ in C_1 with $s'(x) = q = d_1 g'q' = t'(y)$, we have $s'(x \circ y) = q' = s'(y)$ and $t'(x \circ y) = d_1 g d_1 g' q' = d_1 gq = t'(x)$.

Furthermore, the diagram

$$C_1 \underset{I}{\overset{\varepsilon_0,\varepsilon_1}{\rightleftarrows}} D_1,$$

together with the maps

$$\epsilon_0((\bar{l}, d_1 gq),(g,q)) = (g,q) : q \to d_1 gq$$

$$\epsilon_1((\bar{l}, d_1 gq),(g,q)) = (d_2 lg, q) : q \to d_1 d_2 l d_1 gq = d_1 gq$$

and

$$I(g,q) = ((1, d_1 gq),(g,q))$$

for $((\bar{l}, d_1 gq),(g,q)) \in C_1$, gives an internal category in the category of whiskered groupoids over the same objects set C_0. Thus $((\bar{l}, d_1 gq),(g,q)) \in C_1$ is a 2-morphism from the 1-morphism (g,q) to the 1-morphism $(d_2 lg, q)$. The horizontal composition is given by

$$x * y = ((\bar{l}, d_1 gq),(g,q)) * ((\overline{l'}, d_1 g'q),(g',q)) = ((\overline{ll'}, d_1 g'q),(g',q))$$

when $(g,q) = (d_2 l' g', q)$, and hence $d_1 gq = d_1 g'q$. We thus obtain $\epsilon_0(x * y) = (g',q) = \epsilon_0(y)$ and $\epsilon_1(x * y) = (d_2 l d_2 l' g', q) = (d_2 lg, q) = \epsilon_1(x)$ and $\epsilon_0 I = \epsilon_1 I = id_{D_1}$.

Proposition 4.3 *The semi-direct product groupoid over the objects set* C_0

$$C_1 = C_1' \tilde{\times} D_1 \underset{e}{\overset{s,t}{\rightleftarrows}} C_0$$

is a whiskered groupoid together with the maps $m'_{0,1}: C_0 \times C_1 \to C_1$ *given by*

$$(p, ((\bar{l}, d_1gq), (g,q))) \mapsto ((\overline{s_1s_0pls_1s_0p^{-1}}, pd_1gq), (s_0pgs_0p^{-1}, pq))$$

and $m'_{1,0}: C_1 \times C_0 \to C_1$ *given by*

$$(((\bar{l}, d_1gq), (g,q)), p) \mapsto ((\bar{l}, d_1gqp), (g, qp))$$

for $((\bar{l}, d_1gq), (g,q)) \in C_1$ *and* $p \in C_0$*.*

Thus far we have an internal category in the category of whiskered groupoids over the same monoid of objects C_0 as

$$\mathbf{C} = \left(\begin{tikzcd} C_1 \arrow[r, shift left, "\varepsilon_0{,}\varepsilon_1"] \arrow[r, shift right] \arrow[d, shift left, "s'{,}t'"] \arrow[d, shift right] & D_1 \arrow[l, bend left, "I"] \arrow[d, shift left] \arrow[d, shift right, "s{,}t"'] \\ C_0 \arrow[u, bend left, "e"] \arrow[r, equal, "id"'] & C_0. \arrow[u, bend right, "e"'] \end{tikzcd} \right).$$

Proposition 4.4 *The braiding*

$$\tau : (D_1, C_0) \times (D_1, C_0) \to (C_1, C_0)$$

can be given by

$$\tau_{a,b} = ((\overline{s_0gs_1s_0ps_1hs_1s_0p^{-1}s_0g^{-1}s_1gs_1s_0ps_1h^{-1}s_1s_0p^{-1}s_1g^{-1}}, d_1gpd_1hq), (gs_0phs_0p^{-1}, pq))$$

for $a = (g,p)$, $b = (h,q) \in D_1$*.*

Proof We now show that all axioms of braiding given in Definition 3.3 are satisfied. We display the elements omitting the overlines in our calculation to save from complication.

BW1. For $a = (g,p), b = (h,q) \in D_1$, we have

$$\begin{aligned} \epsilon_0\tau_{a,b} &= (gs_0phs_0p^{-1}, pq) \\ &= r(a,b), \end{aligned}$$

and

$$\begin{aligned} \epsilon_1\tau_{a,b} &= (d_2(s_0gs_1s_0ps_1hs_1s_0p^{-1}s_0g^{-1}s_1gs_1s_0ps_1h^{-1}s_1s_0p^{-1}s_1g^{-1})gs_0phs_0^{-1}, pq) \\ &= (s_0d_1gs_0phs_0p^{-1}s_0d_1g^{-1}gs_0ph^{-1}s_0p^{-1}g^{-1}gs_0phs_0p^{-1}, pq) \\ &= (s_0d_1gs_0phs_0p^{-1}s_0d_1g^{-1}g, pq) \\ &= l(a,b). \end{aligned}$$

BW2. For $p \in C_0$, and $a = (h,q) \in D_1$, we have

$$\begin{aligned} \tau_{e(p),a} &= ((s_0(1)s_1s_0ps_1hs_1s_0p^{-1}s_0(1)^{-1}s_1(1)s_1s_0ps_1h^{-1}s_1s_0p^{-1}s_1(1)^{-1}, pd_1hq), (1s_0phs_0p^{-1}, pq)) \\ &= ((1, pd_1hq), (s_0phs_0p^{-1}, pq)) \\ &= m'_{0,1}(p, ((1, d_1hq), (h,q))) \\ &= m'_{0,1}(p, I(a)) \end{aligned}$$

and

$$\begin{aligned}\tau_{a,e(p)} &= ((s_0hs_1s_0ps_1s_0p^{-1}s_0h^{-1}s_1hs_1s_0ps_1s_0p^{-1}s_1h^{-1}, d_1hqp), (h, qp))\\ &= ((1, d_1hqp), (h, qp))\\ &= m'_{1,0}((((1, d_1hq), (h, q)), p)\\ &= m'_{1,0}(I(a), p).\end{aligned}$$

BW3. It is easily checked from [24] that

$$s_0(x_0)x_1s_0(x_0)^{-1} = (s_1s_0d_1(x_0))x_1(s_1s_0d_1(x_0))^{-1} \qquad (*)$$

for $x_0 \in NG_1$ and $x_1 \in NG_2$, so the action ${}^{\partial_1(x_0)}x_1$ is that via s_0.

Now, for $a = (g, p), b = (h, q)$ and $c = (k, m) \in D_1$ with $t(c) = d_1km = q = s(b)$, we have

$$\begin{aligned}m'_{0,1}(t(a), I(b)) &= m'_{0,1}(d_1gp, ((1, d_1hq), (h, q))\\ &= ((1, d_1gpd_1hq), (s_0d_1gs_0phs_0p^{-1}s_0d_1g^{-1}, d_1gpq))\end{aligned}$$

and

$$\tau_{a,c} = ((s_0gs_1s_0ps_1ks_1s_0p^{-1}s_0g^{-1}s_1gs_1s_0ps_1k^{-1}s_1s_0p^{-1}s_1g^{-1}, d_1gpd_1km), (gs_0pks_0p^{-1}, pm)).$$

Thus we obtain

$$\begin{aligned}&m'_{0,1}(t(a), I(b)) \circ \tau_{a,c}\\ &= ((s_1s_0d_1g(s_1s_0ps_1hs_1s_0p^{-1})s_1s_0d_1g^{-1}s_0gs_1s_0ps_1ks_1s_0p^{-1}s_0g^{-1}s_1g\\ &\qquad s_1s_0ps_1ks_1s_0p^{-1}s_1g^{-1}(s_1s_0d_1g(s_1s_0ps_1h^{-1}s_1s_0p^{-1})s_1s_0d_1g^{-1}), d_1gpd_1hq),\\ &\qquad\qquad (s_0d_1gs_0phs_0p^{-1}s_0d_1g^{-1}gs_0pks_0p^{-1}, pm))\\ &= ((s_0gs_1s_0ps_1hs_1ks_1s_0p^{-1}s_0g^{-1}s_1gs_1s_0ps_1k^{-1}s_1s_0p^{-1}s_1g^{-1}s_0g\\ &\qquad s_1s_0ps_1h^{-1}s_1s_0p^{-1}s_0g^{-1}, d_1gpd_1hq), (s_0d_1gs_0phs_0p^{-1}s_0d_1g^{-1}gs_0pks_0p^{-1}, pm) \quad (\text{ since } (*)).\end{aligned}$$

On the other hand, we obtain

$$m'_{0,1}(s(a), I(c)) = ((1, pd_1km), (s_0pks_0p^{-1}, pm))$$

and

$$\tau_{a,b} = ((s_0gs_1s_0ps_1hs_1s_0p^{-1}s_0g^{-1}s_1gs_1s_0ps_1h^{-1}s_1s_0p^{-1}s_1g^{-1}, d_1gpd_1hq), (gs_0phs_0p^{-1}, pq)).$$

Thus,

$$\begin{aligned}&\tau_{a,b} \circ m'_{0,1}(s(a), I(c))\\ &\qquad = (((s_0gs_1s_0ps_1hs_1s_0p^{-1}s_0g^{-1}s_1gs_1s_0ps_1h^{-1}s_1s_0p^{-1}s_1g^{-1}, d_1gpd_1hq), (gs_0phks_0p^{-1}, pm)).\end{aligned}$$

Therefore we obtain

$$[m'_{0,1}(t(a),I(b))\circ\tau_{a,c}]*[\tau_{a,b}\circ m'_{0,1}(s(a),I(c))]$$
$$=((s_0gs_1s_0ps_1(hk)s_1s_0p^{-1}s_0g^{-1}s_1gs_1s_0ps_1(hk)^{-1}s_1s_0p^{-1}$$
$$s_1g^{-1},d_1gpd_1hd_1km),(gs_0p(hk)s_0p^{-1},pm)) \quad (\text{since } q=d_1km)$$
$$=\tau_{(g,p),(hk,m)}$$
$$=\tau_{a,b\circ c}.$$

BW4. For $a=(g,p),b=(h,q)$ and $c=(k,m)\in D_1$ with $t(b)=d_1hq=p=s(a)$, we have

$$\tau_{a,c}=((s_0gs_1s_0ps_1ks_1s_0p^{-1}s_0g^{-1}s_1gs_1s_0ps_1k^{-1}s_1s_0p^{-1}s_1g^{-1},d_1gpd_1km),(gs_0pks_0p^{-1},pm))$$

and

$$m'_{1,0}(I(b),s(c))=((1,d_1hqm),(h,qm)).$$

Thus we obtain

$$\tau_{a,c}\circ m'_{1,0}(I(b),s(c))=((s_0gs_1s_0ps_1ks_1s_0p^{-1}s_0g^{-1}s_1gs_1s_0ps_1k^{-1}s_1s_0p^{-1}s_1g^{-1},d_1gpd_1km),(gs_0pks_0p^{-1}h,qm)).$$

On the other hand, we have

$$m'_{1,0}(I(a),t(c))=((1,d_1gpd_1km),(g,pd_1km))$$

and

$$\tau_{b,c}=((s_0hs_1s_0qs_1ks_1s_0q^{-1}s_0h^{-1}s_1hs_1s_0qs_1k^{-1}s_1s_0q^{-1}s_1h^{-1},d_1hqd_1km),(hs_0qks_0q^{-1},qm)).$$

Thus we have

$$m'_{1,0}(I(a),t(c))\circ\tau_{b,c}$$
$$=((s_1gs_0hs_1s_0qs_1ks_1s_0q^{-1}s_0h^{-1}s_1hs_1s_0qs_1k^{-1}s_1s_0q^{-1}s_1h^{-1}s_1g^{-1},d_1gpd_1km),(ghs_0qks_0q^{-1},qm)).$$

Therefore, we obtain

$$[\tau_{a,c}\circ m'_{1,0}(I(b),s(c))]*[m'_{1,0}(I(a),t(c))\circ\tau_{b,c}]$$
$$=((s_0gs_1s_0ps_1ks_1s_0p^{-1}s_0g^{-1}s_1gs_1s_0ps_1k^{-1}s_1s_0p^{-1}s_0hs_1s_0qs_1ks_1s_0q^{-1}$$
$$s_0h^{-1}s_1hs_1s_0qs_1k^{-1}s_1s_0q^{-1}s_1h^{-1}s_1g^{-1},d_1gpd_1km),(ghs_0qks_0q^{-1},qm))$$
$$=((s_0gs_1s_0d_1h(s_1s_0qs_1ks_1s_0q^{-1})s_1s_0d_1h^{-1}s_0g^{-1}s_1g$$
$$s_1s_0d_1h(s_1s_0qs_1k^{-1}s_1s_0q^{-1})s_1s_0d_1h^{-1}s_0hs_1s_0qs_1ks_1s_0q^{-1}$$
$$s_0h^{-1}s_1hs_1s_0qs_1k^{-1}s_1s_0q^{-1}s_1h^{-1}s_1g^{-1},d_1gd_1hqd_1km),(ghs_0qks_0q^{-1}qm)) \quad (\text{since } p=d_1hq)$$

$$= ((s_0gs_0h(s_1s_0qs_1ks_1s_0q^{-1})s_0h^{-1}s_0g^{-1}s_1gs_0h(s_1s_0qs_1k^{-1}s_1s_0q^{-1})s_0h^{-1}$$
$$s_0h(s_1s_0qs_1ks_1s_0q^{-1})s_0h^{-1}s_1hs_1s_0qs_1k^{-1}s_1s_0q^{-1}$$
$$s_1h^{-1}s_1g^{-1}, d_1gd_1hqd_1km), (ghs_0qks_0q^{-1}, qm)) \quad \text{(since } (*))$$
$$= ((s_0gs_0hs_1s_0qs_1ks_1s_0q^{-1}s_0h^{-1}s_0g^{-1}s_1gs_1h$$
$$s_1s_0qs_1k^{-1}s_1s_0q^{-1}s_1h^{-1}s_1g^{-1}, d_1(gh)qd_1km), (ghs_0qks_0q^{-1}, qm))$$
$$= \tau_{(gh,q),(k,m)}$$
$$= \tau_{a\circ b,c}.$$

BW5. Let

$$x = ((l,p),(g,q)) : a = (g,q) \Rightarrow (d_2lg,q) = a'$$

and

$$y = ((l',p'),(g',q')) : b = (g',q') \Rightarrow (d_2l'g',q') = b'$$

be 2-morphisms in C_1 with $p = d_1gq$ and $p' = d_1g'q'$. We obtain

$$l(x,y) = m'_{0,1}(t'(x),y) \circ m'_{1,0}(x,s'(y))$$
$$= ((s_1s_0pl's_1g's_1s_0p^{-1}ls_1s_0ps_1(g')^{-1}s_1s_0p^{-1}, pp'), (s_0pg's_0p^{-1}g, qq'))$$

and

$$r(x,y) = m'_{1,0}(x,t'(y)) \circ m'_{0,1}(s'(x),y)$$
$$= ((ls_1gs_1s_0ql's_1s_0q^{-1}s_1g^{-1}, pp'), (gs_0qg's_0q^{-1}, qq')).$$

On the other hand, we obtain

$$\tau_{a,b} = ((s_0gs_1s_0qs_1g's_1s_0q^{-1}s_0g^{-1}s_1gs_1s_0qs_1(g')^{-1}s_1s_0q^{-1}, d_1gqd_1g'q'), (gs_0qg's_0q^{-1}, qq'))$$

and

$$\tau_{a',b'} = ((s_0d_2ls_0gs_1s_0qs_1d_2l's_1g's_1s_0q^{-1}s_0g^{-1}s_0d_2l^{-1}$$
$$s_1d_2ls_1gs_1s_0qs_1(g')^{-1}s_1d_2(l')^{-1}s_1s_0q^{-1}s_1g^{-1}s_1d_2l^{-1}, d_1gqd_1g'q'), (d_2lgs_0qd_2lg's_0q^{-1}, qq')).$$

Thus we have

$$\tau_{a',b'} * r(x,y) = ((s_0d_2(ls_1g)s_1d_2(s_1s_0ql's_1g's_1s_0q^{-1})s_0d_2(ls_1g)^{-1}$$
$$s_1d_2(ls_1g)s_1d_2(s_1s_0ql's_1g's_1s_0q^{-1})^{-1}s_1d_2(ls_1g)^{-1}$$
$$(ls_1g)s_1s_0ql's_1s_0q^{-1}s_1g^{-1}, pp'), (gs_0qg's_0q^{-1}, qq'))$$

and

$$l(x,y) * \tau_{a,b} = ((s_1s_0pl's_1g's_1s_0p^{-1}ls_1gs_1s_0qs_1(g')^{-1}s_1s_0q^{-1}s_1g^{-1}, pp'), (gs_0qg's_0q^{-1}, qq'))$$

From the definitions of ϵ_0 and ϵ_1, we obtain

$$\begin{aligned}\epsilon_0(\tau_{a',b'} * r(x,y)) &= (gs_0qg's_0q^{-1}, qq')\\ &= r(a,b)\\ &= \epsilon_0(l(x,y) * \tau_{a,b})\end{aligned}$$

and

$$\begin{aligned}\epsilon_1(\tau_{a',b'} * r(x,y)) = (&s_0d_1gs_0qd_2l'g's_0q^{-1}s_0d_1g^{-1}d_2lgs_0q(g')^{-1}d_2(l')^{-1}s_0q^{-1}g^{-1}d_2l^{-1}d_2lg\\ &s_0qd_2l's_0q^{-1}g^{-1}gs_0qg's_0q^{-1}, qq')\\ = (&s_0(d_1gq)d_2l'g's_0(d_1gq)^{-1}d_2lg, qq')\\ = \;&l(a',b')\\ = (&d_2(s_1s_0pl's_1g's_1s_0p^{-1}ls_1gs_1s_0qs_1(g')^{-1}s_1s_0q^{-1}s_1g^{-1})gs_0qg's_0q^{-1}, qq')\\ = \;&\epsilon_1(l(x,y) * \tau_{a,b}).\end{aligned}$$

□

Therefore we obtained a braided internal category in the category of whiskered groupoids from a simplicial group.

Notice that, in general, there is no the equality $\tau_{a',b'} * r(x,y) = l(x,y) * \tau_{a,b}$. To have this equality, we give the following result.

Proposition 4.5 *Let $x : a \Rightarrow a'$ and $y : b \Rightarrow b'$ be 2-morphisms in $\mathbf{C}$. If the Moore complex of the simplicial group $\mathbf{G}$ is of length 2, and $a, a', b, b' \in D_1(p,p)$ for any $p \in C_0$, then $x, y \in C_1(p,p)$ and $\tau_{a',b'} * r(x,y) = l(x,y) * \tau_{a,b}$.*

Proof Let

$$x = ((l,p),(g,p)) : a = (g,p) \Rightarrow (d_2lg, p) = a'$$

and

$$y = ((l',p),(g',p)) : b = (g',p) \Rightarrow (d_2l'g', p) = b'$$

be 2-morphisms in C_1. If the 1-morphism $a = (g,p)$ is a morphism from p to p in D_1, we must have $g \in \ker d_1$. That is, $a = (g,p)$ is a morphism from p to p in D_1 for any $p \in C_0$ if $g \in \ker d_1$. Then, we have also $a' = (d_2lg, p) : p \to d_1d_2(l)d_1(g)p = p$. Thus, if $g \in \ker d_1$, we have $s'(x) = p = t'(x)$, that is $x \in C_1(p,p)$. Similarly, the morphisms b, b' are from p to p in D_1 if $g' \in \ker d_1$. So, we have $s'(y) = p = t'(y)$, that is $y \in C_1(p,p)$ if $g' \in \ker d_1$.

Therefore, if $g, g' \in \ker d_1$, we have

$$l(x,y) * \tau_{a,b} = ((s_1s_0pl's_1g's_1s_0p^{-1}ls_1gs_1s_0ps_1(g')^{-1}s_1s_0p^{-1}s_1g^{-1}, pp), (gs_0pg's_0p^{-1}, pp))$$

and

$$\begin{aligned}
&\tau_{a',b'} * r(x,y) \\
&\quad = ((s_0d_2(ls_1g)s_1d_2(s_1s_0pl's_1g's_1s_0p^{-1})s_0d_2(ls_1g)^{-1} \\
&\qquad s_1d_2(ls_1g)s_1d_2(s_1s_0pl's_1g's_1s_0p^{-1})^{-1}s_1d_2(ls_1g)^{-1} \\
&\qquad\quad (ls_1g)(s_1s_0pl's_1g's_1s_0p^{-1})s_1s_0ps_1(g')^{-1}s_1s_0p^{-1}s_1g^{-1}, pp), (gs_0pg's_0p^{-1}, pp)).
\end{aligned}$$

To obtain the required equality, we must have

$$\begin{aligned}
&s_0d_2(ls_1g)s_1d_2(s_1s_0pl's_1g's_1s_0p^{-1})s_0d_2(ls_1g)^{-1} \\
&\qquad s_1d_2(ls_1g)s_1d_2(s_1s_0pl's_1g's_1s_0p^{-1})^{-1}s_1d_2(ls_1g)^{-1} \\
&\qquad\quad (ls_1g)(s_1s_0pl's_1g's_1s_0p^{-1})s_1s_0ps_1(g')^{-1}s_1s_0p^{-1}s_1g^{-1} \\
&= s_1s_0pl's_1g's_1s_0p^{-1}ls_1gs_1s_0ps_1(g')^{-1}s_1s_0p^{-1}s_1g^{-1}.
\end{aligned}$$

To obtain this equality, we will use the functions $F_{\alpha,\beta}$ from [24].

For any $x_2, y_2 \in NG_2$, from [24], we have

$$\partial_3(F_{(0)(1)}(x_2,y_2)) = s_0d_2x_2s_1d_2y_2s_0d_2x_2^{-1}s_1d_2x_2s_1d_2y_2^{-1}s_1d_2x_2^{-1}x_2y_2x_2^{-1}y_2^{-1} \in \partial_3(NG_3 \cap D_3).$$

Now, we take $x_2 = ls_1g$ and $y_2 = s_1s_0pl's_1g's_1s_0p^{-1}$. Then we have

$$\begin{aligned}
&s_0d_2(ls_1g)s_1d_2(s_1s_0pl's_1g's_1s_0p^{-1})s_0d_2(ls_1g)^{-1} \\
&\qquad s_1d_2(ls_1g)s_1d_2(s_1s_0pl's_1g's_1s_0p^{-1})^{-1}s_1d_2(ls_1g)^{-1} \\
&\qquad\quad (ls_1g)s_1s_0pl's_1g's_1s_0p^{-1}s_1s_0ps_1(g')^{-1}s_1s_0p^{-1}s_1g^{-1} \\
&\quad = s_0d_2x_2s_1d_2y_2s_0d_2x_2^{-1}s_1d_2x_2s_1d_2y_2^{-1}s_1d_2x_2^{-1}x_2y_2(s_1s_0ps_1(g')^{-1}s_1s_0p^{-1}s_1g^{-1}) \\
&\quad \equiv y_2x_2(s_1s_0ps_1(g')^{-1}s_1s_0p^{-1}s_1g^{-1}) \mod (\partial_3(NG_3 \cap D_3)) \\
&\quad = s_1s_0pl's_1g's_1s_0p^{-1}ls_1gs_1s_0ps_1(g')^{-1}s_1s_0p^{-1}s_1g^{-1}.
\end{aligned}$$

Thus, we obtain $\tau_{a',b'} * r(x,y) \equiv l(x,y) * \tau_{a,b} \mod \partial_3(NG_3 \cap D_3)$.

Since the Moore complex is of length 2, we have $NG_3 = \{1\}$, and $\partial_3(NG_3 \cap D_3) = \{1\}$, and thus we obtain the required equality. □

Recall from [7] and [22] that a strict 2-category is a category enriched over Cat, where Cat is treated as the 1-category of strict categories. That is, a strict 2-category consists of objects, 1-morphisms between objects, and 2-morphisms between 1-morphisms. The 1-morphisms can be composed along the objects, while the 2-morphisms can be composed in two different directions: along the objects and along the 1-morphisms. The composition of morphisms between objects is called the vertical composition, and the composition of morphisms between 1-morphisms is called the horizontal composition. Thus it has a collection of objects and for each pair of objects x, y a category $hom(x,y)$, and the objects of these hom-categories are the 1-morphisms, and the morphisms of these hom-categories are the 2-morphisms. We also have the interchange law, because the horizontal composition is a functor it commutes with composition in the hom-categories.

Similarly, a strict 2-groupoid is a groupoid enriched over groupoids. In more detail, a strict 2-groupoid $\mathbf{X}$ consists of

(a) a set X_0 of objects;

(b) for each $x, y \in X_0$, a set $X_1(x, y)$ of 1-morphisms from x to y, and a composition of 1-morphisms denoted by $\circ$;

(c) for the 1-morphisms $f, g : x \to y$, a set X_2 of 2-morphisms $f \Rightarrow g$ from f to g, a vertical composition and a horizontal composition of 2-morphisms, denoted by $\circ$ and $*$ respectively.

such that (X_i, X_j) are groupoids for $i = 1, 2$, $j = 0, 1$, $j < i$, and for 2-morphisms $\alpha, \beta, \gamma, \delta \in X_2$, the interchange law holds:

$$(\alpha * \beta) \circ (\gamma * \delta) = (\alpha \circ \gamma) * (\beta \circ \delta).$$

Thus, for the category $\mathbf{C}$, according to the above calculations, we can take $X_0 = C_0$, $(X_1, X_0) = (D_1, C_0)$, $(X_2, X_0) = (C_1, C_0)$ and $(X_2, X_1) = (C_1, D_1)$. The only thing remaining is to check the interchange law.

Proposition 4.6 *If the Moore complex of the simplicial group* $\mathbf{G}$ *is of length 2, then the category* $\mathbf{C}$ *has an interchange law between the horizontal and vertical compositions of 2-morphisms.*

Proof By using the image of $F_{\alpha,\beta}$ functions given in [24], we shall show that the interchange law holds for 2-morphisms in $\mathbf{C}$. Let

$$\alpha = ((l, d_1gq), (g, q)) : (g, q) \Rightarrow (d_2lg, q)$$

and

$$\beta = ((l', d_1g'q), (g', q)) : (g', q) \Rightarrow (d_2l'g', q)$$

be 2-morphisms in $\mathbf{C}$ with $(g, q) = ((d_2l')g', q)$ and hence $d_1gq = d_1g'q$.

Similarly, let

$$\gamma = ((l_1, d_1g_1q_1), (g_1, q_1)) : (g_1, q_1) \Rightarrow (d_2l_1g_1, q_1)$$

and

$$\delta = ((l_1', d_1g_1'q_1), (g_1', q_1)) : (g_1', q_1) \Rightarrow (d_2l_1'g_1', q_1)$$

be 2-morphisms in $\mathbf{C}$ with $(g_1, q_1) = ((d_2l_1')g_1', q_1)$, and hence $d_1g_1q_1 = d_1g_1'q_1$

We must show that

$$(\alpha * \beta) \circ (\gamma * \delta) = (\alpha \circ \gamma) * (\beta \circ \delta).$$

We obtain

$$\alpha * \beta = ((ll', d_1g'q), (g', q))$$

$$\gamma * \delta = ((l_1l_1', d_1g_1'q_1), (g_1', q_1)),$$

and

$$(\alpha * \beta) \circ (\gamma * \delta) = ((ll's_1g'l_1l_1's_1(g')^{-1}, d_1g'q), (g'g_1', q_1))$$

when $q = d_1g_1'q_1$. On the other hand, we obtain

$$\alpha \circ \gamma = ((ls_1gl_1s_1g^{-1}, d_1gq), (gg_1, q_1))$$

$$\beta \circ \delta = ((l's_1g'l_1's_1(g')^{-1}, d_1g'q), (g'g_1', q_1)),$$

when $q = d_1 g_1 q_1 = d_1 g_1' q_1$ and

$$(\alpha \circ \gamma) * (\beta \circ \delta) = ((l s_1 g l_1 s_1 g^{-1} l' s_1 g' l_1' s_1 (g')^{-1}, d_1 g' q), (g' g_1', q_1)).$$

To obtain required equality, we must show the following equality:

$$l s_1 g l_1 s_1 g^{-1} l' s_1 g' l_1' s_1 (g')^{-1} = l l' s_1 g' l_1 l_1' s_1 (g')^{-1}.$$

We know from [24], for $x, y \in NG_2$, that

$$\begin{aligned} F_{(1),(2)}(x,y) &= [s_1 x, s_2 y][s_2 x, s_2 y] \\ &= s_1 x s_2 y (s_1 x)^{-1} s_2 x (s_2 y)^{-1} (s_2 x)^{-1} \in NG_3 \cap D_3 \end{aligned}$$

and

$$\partial_3(F_{(1),(2)}(x,y)) = s_1 d_2(x) y (s_1 d_2 x)^{-1} x y^{-1} x^{-1} \in \partial_3(NG_3 \cap D_3),$$

and

$$s_1 d_2(x) y (s_1 d_2 x)^{-1} \equiv x y x^{-1} \mod \partial_3(NG_3 \cap D_3).$$

Furthermore, since $(g, q) = (d_2(l') g', q)$, we have

$$\begin{aligned} l s_1 g l_1 s_1 g^{-1} l' s_1 g' l_1' s_1 (g')^{-1} &= l(s_1 d_2 l'(s_1 g' l_1 s_1 (g')^{-1}) s_1 d_2 (l')^{-1}) l' (s_1 g' l_1' s_1 (g')^{-1}) \\ &\equiv l l' (s_1 g' l_1 s_1 (g')^{-1})(l')^{-1} l' (s_1 g' l_1' s_1 (g')^{-1}) \qquad \mod \partial_3(NG_3 \cap D_3) \\ &= l l' s_1 g' l_1 l_1' s_1 (g')^{-1} \end{aligned}$$

and thus we obtain

$$(\alpha \circ \gamma) * (\beta \circ \delta) \equiv (\alpha * \beta) \circ (\gamma * \delta) \mod \partial_3(NG_3 \cap D_3).$$

Since the Moore complex of the simplicial group $\mathbf{G}$ is of length 2, we have $NG_3 \cap D_3 = \{1\}$ and $\partial_3(NG_3 \cap D_3) = \{1\}$, and thus we obtain

$$(\alpha * \beta) \circ (\gamma * \delta) = (\alpha \circ \gamma) * (\beta \circ \delta).$$

□

Acknowledgements

The authors would like to thank Professor Ronald Brown for his valuable suggestions.

References

[1] Arvasi, Z., Koçak, M. and Ulualan, E.: Braided crossed modules and reduced simplicial groups. Taiwanese Journal of Mathematics. 9, 3, 477–488, (2005).

[2] Arvasi, Z. and Porter, T.: Higher dimensional Peiffer elements in simplicial commutative algebras. Theory and Applications of Categories. 3, 1, 1–23, (1997).

[3] Arvasi, Z. and Ulualan, E.: On algebraic models for homotopy 3-types. Journal of Homotopy and Related Structures. 1, 1, 1–27, (2006).

[4] Arvasi, Z. and Ulualan, E.: 3-types of simplicial groups and braided regular crossed modules. Homology, Homotopy and Applications. 9, 1, 139–161, (2007).

[5] Baez, J.C. and Crans, A.S.: Higher dimensional algebra VI: Lie 2-algebras. Theory and Applications of Categories. 12, 15, 492–538, (2004).

[6] Baez, J.C. and Lauda, A.D.: Higher dimensional algebra V:2-groups. Theory and Applications of Categories. 12, 423–492, (2004).

[7] Baez, J.C. and Neuchl, M.: Higher-dimensional algebra I: Braided monoidal 2-categories. Adv. Math. 121, 196–244, (1996).

[8] Berger, C.: Double loop spaces, braided monoidal categories and algebraic 3-types of spaces. Contemporary Mathematics. 227, 46–66, (1999).

[9] Bullejos, M., Carrasco, P. and Cegarra, A.M.: Cohomology with coefficients in symmetric cat-groups. An extension of Eilenberg-MacLane's classification theorem. Math. Proc. Camb. Phil. Soc. 114, 163–189, (1993).

[10] Breen, L.: Théorie de Schreier supérieure. Ann. Sci. Ecol Norm. Sup. 25, 465–514, (1992).

[11] Brown, R.: Possible connections between whiskered categories and groupoids, Leibniz algebras, automorphism structures and local-to-global questions. Journal of Homotopy and Related Structures. 5, 1, 305–318, (2010).

[12] Brown, R. and Gilbert, N.D.: Algebraic models of 3-types and automorphism structures for crossed modules. Proc. London Math. Soc., 3, 59, 51-73, (1989).

[13] Brown, R. and İçen, İ.: Homotopies and automorphisms of crossed modules over groupoids. Appl. Categorical Structure, 11, 185–206, (2003).

[14] Brown, R. and Spencer, C.B.: $\mathcal{G}$-Groupoids, crossed modules and the fundamental groupoid of a topological group. Proc. Konink. Neder. Akad. van Wetenschappen Amsterdam, 79, 4, (1976).

[15] Carrasco, P. and Cegarra, A.M.: Group-theoretic algebraic models for homotopy types. Journal Pure Appl. Algebra, 75, 195–235, (1991).

[16] Castiglioni, J.L. and Ladra, M.: Peiffer elements in simplicial groups and algebras, Jour. Pure and Applied Algebra 212, 9, 2115–2128, (2008).

[17] Conduché, D.: Modules croisés généralisés de longueur 2. Jour. Pure and Applied Algebra, 34, 155–178, (1984).

[18] Duskin, J.: Simplicial methods and the interpretation of triple cohomology. Memoirs A.M.S. 3, 163, (1975).

[19] Garzon, A.R. and Miranda, J.G.: Homotopy theory for (braided) cat-groups. Chaiers de Topologie Geometrie Differentielle Categorique, XXXVIII-2, (1997).

[20] Joyal, A. and Street, R.: Braided monoidal categories. Macquarie Mathematics Report, 860081, Macquarie University, (1986).

[21] Joyal, A. and Street, R.: Braided tensor categories. Advances in Math, 1, 82, 20–78, (1993).

[22] Kapranov, M. and Voevodsky, V.: Braided monoidal 2-categories and Manin-Schechtman higher braid groups. Jour. Pure Appl. Algebra, 92 241–267, (1994).

[23] May, J.P.: Simplicial objects in algebraic topology. Math. Studies, 11, Van Nostrand, (1967).

[24] Mutlu, A. and Porter, T.: Applications of Peiffer pairings in the Moore complexes of a simplicial group. Theory and Applications of Categories, 4, 7, 148–173, (1998).

[25] Mutlu, A. and Porter, T.: Freeness conditions for 2-crossed modules and complexes. Theory and Applications of Categories, 4, 8, 174–194, (1998).

[26] Ulualan, E.: Relations among algebraic models of 1-connected homotopy 3-types. Turk. J Math., 31, 23–41, (2007).

[27] Whitehead, J.H.C.: Combinatorial homotopy II. Bull. Amer. Math. Soc., 55, 453–496, (1949).