Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2013) 37: 165 – 181
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doi:10.3906/mat-1106-4

Generalized class invariants with 'Thetanullwerte'

Osmanbey UZUNKOL[*]
Institut für Mathematik, Carl von Ossietzky Universität Oldenburg
D-26111, Oldenburg, Germany

Received: 03.06.2011 • **Accepted:** 02.10.2011 • **Published Online:** 17.12.2012 • **Printed:** 14.01.2013

Abstract: We introduce generalized class invariants as quotients of Thetanullwerte, which realize the computation of class polynomials more efficiently than as quotients of values of the Dedekind η-function. Furthermore, we prove that these invariants are units in the corresponding class field as most of their classical counterparts.

Key words: Class polynomials, theta functions, complex multiplication

1. Introduction

Construction of class fields using special values of analytical functions goes back to Hilbert, who gave the first complete proof of Kronecker-Weber theorem [9]. This theorem states that every finite abelian extension of $\mathbb{Q}$ is a subfield of a suitable cyclotomic field, and that each such field lies in a field generated by the special values of the exponential function $z \mapsto e^{2\pi i z}$.

Kronecker-Weber Theorem is the raison d'être of Kronecker's "Jugendtraum"[*] . He asks to generate all abelian extensions of a given imaginary quadratic number field by special (the so-called **singular**) values of elliptic modular functions. Kronecker's Jugendtraum can be seen, in that sense, as a special fact of the Hilbert's famous 12th problem, which asks to generate all abelian extensions of a given number field by values of suitable analytical functions.

The theory of complex multiplication realizes Kronecker's Jugendtraum. At the first step, one needs to construct the maximal abelian unramified extension (the so-called Hilbert class field H_K) of the imaginary quadratic number field K as an intermediate field in order to obtain the other abelian extensions of K by Weber functions. These functions can be seen as the generalization of the exponential function; see [16] for more details. This intermediate step is not necessary in the case of $\mathbb{Q}$, as its class group is trivial.

By the main theorem of complex multiplication, the value $j(\tau)$ of the modular function j generates the Hilbert class field H_K if τ is an element of the maximal order $\mathcal{O}_K$ of K with discriminant D. The minimal polynomial $H_D(x)$ of $j(\tau)$ has integer coefficients, as this value is an algebraic integer by the theory of elliptic modular functions. The Galois group of H_K/K is isomorphic to the class group Cl_K of K by class field theory. Hence, the degree of $H_D(x)$ is just the class number h_K of K. By the theory of complex multiplication, the conjugates of $j(\tau)$ are $j(\tau_i)$, $1 \leq i \leq h_K$, where $[\tau_i, 1]$ are the representatives of the ideal classes of K. Hilbert's 12th problem remains open for all other types of number fields.

[*]Correspondence: osmanbey.uzunkol@uni-oldenburg.de

[*] Kronecker's youthful dream

Coefficients of the Hilbert class polynomial $H_D(x)$ grow exponentially in the size of the discriminant of the field. Moreover, the polynomial has very large coefficients even for comparably small discriminants. For example, for $D = -260$ we have

$$\begin{aligned} H_{-260}(x) = x^8 - 9997874035270492198400 \cdot x^7 - \\ 99989616189584210186369021747200 0\cdot x^6 - \\ 21507054600723946274941348498171494400000 \cdot x^5 + \\ 46323890873234776715342057877550577588633 6000000 \cdot x^4 + \\ 148655578046498651131500340770766641673797632 00000000 \cdot x^3 + \\ 85980083235988029405783249092189509918128078848 000000000 \cdot x^2 + \\ 30548608836792995170796076852647786030663655751680000000000 0 \cdot x + \\ 33029475056757150289467742566614726794263595581440000000000000. \end{aligned} \tag{1}$$

Weber introduced in his famous book ‘Lehrbuch der Algebra’ [19] the use of other modular functions of higher level whose values, the so-called **class invariants**, generate H_K. The minimal polynomials of these values have significantly smaller coefficients than $H_D(x)$. Most statements are given without rigorous proofs in [19]. Schertz proved later in [14] that these values are indeed class invariants. For example, the polynomial

$$W_{-260}(x) = x^8 - 8 \cdot x^7 + 12 \cdot x^6 + 8 \cdot x^5 - 27 \cdot x^4 + 8 \cdot x^3 + 12 \cdot x^2 - 8 \cdot x + 1, \tag{2}$$

which is the minimal polynomial of a suitable class invariant, generates the same field as the polynomial (1) over K.

These modular functions are defined classically as quotients of the Dedekind η-function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{k=1}^{\infty} (1 - q^k), \quad q = \exp(2\pi i \tau) \text{ with } \Im(\tau) > 0. \tag{3}$$

In [11] and [18] we introduced the possibility to represent these invariants as quotients of ‘Thetanullwerte’. Using the AGM-algorithm of Dupont [4], [5], we showed that the class polynomials can be computed more efficiently using this new representation than using the classical representation of class invariants as quotients of values of the η-function. Furthermore, it is proven in [11] that most class invariants are units in the corresponding ring class fields.

In this paper, we improve and generalize the results of [11].

In Section 2, we recall some basic facts about modular functions, theta functions and class field theory, which is necessary to explain the results in the subsequent sections.

In Section 3, we summarize the generalization of class invariants, which were introduced in the earlier articles.

We prove an identity between values of the η-quotients and the quotients of suitable Thetanullwerte in Section 4, which improves the result of [11] and enables to compute class polynomials, and hence class fields, more efficiently using the corresponding quotient of Thetanullwerte. Moreover, we introduce the identities between generalized Weber class invariants and suitable quotients of Thetanullwerte, which leads to use the faster techniques of [5] to compute the generalized class polynomials. Furthermore, we analyze and compare our results with the existing algorithms.

Using a theorem of Deuring, we prove in Section 5 that the generalized class invariants are units in the corresponding class fields, which can also be seen as a generalization of the results of [11], loc. cit. Theorem 20, 22.

In the last section, we give examples and a comparison table of the results we obtained.

2. Basics

For the basic properties of modular functions and the theory of complex multiplication we refer to [10] and [15]. Moreover, if not stated explicitly, the results of this section can be found in these references.

Let $\mathcal{O}_t$ be the order of an imaginary quadratic number field K of conductor $t \in \mathbb{N}$. The Galois group G of the ring class field Ω_t of K modulo t is isomorphic to the ring class group Cl_t of $\mathcal{O}_t$ by class field theory. Hence, the degree of Ω_t over K is equal to the class number h_t of $\mathcal{O}_t$.

Let $\tau \in \mathcal{O}_t$ be an element of the upper half plane

$$\tau \in \mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\} \tag{4}$$

with discriminant $D := t^2 d$, where d is the field discriminant. Using Ramanujan's modular discriminant function

$$\Delta(\tau) := g_2(\tau)^3 - 27g_3(\tau)^2, \tag{5}$$

the value of modular j-function at $\tau \in \mathbb{H}$ is defined as

$$j(\tau) := 2^6 3^3 g_2(\tau)^3 \Delta(\tau)^{-1}, \tag{6}$$

where the functions g_2 are g_3 are the classical Eisenstein series; see for example [3, p. 3]. The value $j(\tau)$ generates the ring class field Ω_t over K by the main theorem of complex multiplication.

More precisely, the field extension Ω_t over K corresponds to the subgroup $\mathcal{U}_t$ of the ideal group of K generated by the ideals of the form (λ), $\lambda \in \mathbb{Z}$ with $\gcd(\lambda, t) = 1$ and $\lambda \equiv r \bmod t$ for a suitable $r \in \mathbb{Z}$ [14, p. 327].

The values $j(\tau_i)$, $i = 1, \cdots, h_t$, coming from the representatives $[\tau_i, 1]$ of the ideals in Cl_t with $\tau := \tau_1$, form a complete system of conjugate numbers not only over K but also over $\mathbb{Q}$, [10, Remark 1, p. 133]. Since the values $j(\tau)$ for all $\tau \in \mathbb{H}$ are algebraic integers, the minimal polynomial of $j(\tau)$ has coefficients in $\mathbb{Z}$.

Hence, the minimal polynomial

$$H_D(x) = \prod_{i=1}^{h_t} (x - j_i) \in \mathbb{Z}[x] \text{ with } j_i = j(\tau_i) \tag{7}$$

can be computed explicitly using the numerical values of the j-function.

In order to compute Hilbert's class polynomial using (7), an upper bound for the precision is needed to recognize the integer coefficients of the polynomial, which is given in [1, p. 285] as follows:

$$\left\lceil \log_2 \left(2.48h_t + \pi\sqrt{|D|} \sum_{(a,b,c)\in\mathcal{H}(D)} \frac{1}{a} \right) \right\rceil + 1, \tag{8}$$

where we abbreviate with $\mathcal{H}(D)$ the form class group of reduced binary quadratic forms of discriminant D. We remind that the form class group is isomorphic to the ring class group Cl_t; see for example [17].

We need to compute approximately $\sqrt{|D|}$ coefficients by the Brauer-Siegel theorem, which implies that h_t grows like $|D|^{1/2+o(1)}$; see [2]. As discussed in the introduction, the coefficients of $H_D(x)$ are very large compared to the discriminant. Worse, the coefficients grow exponentially in $|D|$.

Weber used other modular functions of higher level, whose values generate Ω_t. These values have class polynomials having smaller coefficients than $H_D(x)$. Using reciprocity law of Shimura, it is easy to check whether $g(\tau)$ is a class invariant or not; see [18] and the references therein.

Definition 1 *A value $g(\tau)$ of a modular function g is said to be **class invariant** if $K(g(\tau)) = K(j(\tau))$.*

The following definition and proposition of [14, p. 329, 335] are useful to write down the conjugates of such a class invariant $g(\tau)$ so as to compute the minimal polynomial of $g(\tau)$ numerically. These new class polynomials are called **Weber class polynomials** and they are abbreviated by $W_D(x)$.

Definition & Proposition 2

1. *An imaginary quadratic integer $\tau \in \mathbb{H}\cap K$ is the zero of a quadratic equation of the form $Ax^2+Bx+C=0$, which is uniquely determined by τ if we postulate the following normalization assumption:*

$$A, B, C \in \mathbb{Z}, \quad \gcd(A, B, C) = 1, \ A > 0.$$

*Such an equation is called **primitive**.*

2. *Let $N \in \mathbb{Z}^{>0}$ and $\tau_1, \tau_2, \cdots, \tau_{h_t} \in \mathbb{H}$, so that*

$$[\tau_1, 1], [\tau_2, 1], \cdots, [\tau_{h_t}, 1]$$

is a system of representatives of Cl_t. Furthermore, let $A_ix^2 + B_ix + C_i = 0$ be primitive equations for τ_i which satisfy the properties

$$\gcd(A_i, N) = 1, \quad B_i \equiv B_j \bmod 2N, 1 \le i, j \le h_t.$$

*Then the elements $\tau_1, \tau_2, \cdots, \tau_{h_t}$ are called an N**-system modulo** t.*

3. *There exists an N-system for every natural number.*

3. Generalized class invariants

3.1. Classical class invariants

Let

$$\gamma_2(\tau) := \sqrt[3]{j(\tau)}, \quad \gamma_3(\tau) := \sqrt{j(\tau) - 12^3}. \tag{9}$$

The Schläfli functions $\mathfrak{f}, \mathfrak{f}_1$ and $\mathfrak{f}_2$ of Weber are the following quotients of values of the Dedekind η-function:

$$\mathfrak{f}(\tau) = \exp\left(-\frac{\pi i}{24}\right)\frac{\eta(\frac{\tau+1}{2})}{\eta(\tau)}, \quad \mathfrak{f}_1(\tau) = \frac{\eta(\frac{\tau}{2})}{\eta(\tau)}, \quad \mathfrak{f}_2(\tau) = \sqrt{2}\frac{\eta(2\tau)}{\eta(\tau)}. \tag{10}$$

These functions satisfy the following identities:

Theorem 3 *Let $\tau \in \mathbb{H}$. We have*

1. $\mathfrak{f}(\tau)^8 = \mathfrak{f}_1(\tau)^8 + \mathfrak{f}_2(\tau)^8$,

2. $\mathfrak{f}(\tau)\mathfrak{f}_1(\tau)\mathfrak{f}_2(\tau) = \sqrt{2}$,

3. $\gamma_2 = \frac{\mathfrak{f}^{24}-16}{\mathfrak{f}^8} = \frac{\mathfrak{f}_1^{24}+16}{\mathfrak{f}_1^8} = \frac{\mathfrak{f}_2^{24}+16}{\mathfrak{f}_2^8}$,

4. $\mathfrak{f}_1(2\tau)\mathfrak{f}_2(\tau) = \sqrt{2}$,

5. $\mathfrak{f}(\tau)\mathfrak{f}_2(\frac{\tau+1}{2}) = \sqrt{2}\exp(\frac{\pi i}{24})$.

Proof The first two results follow by [19, p. 114] and the third identity is the result of [14, p. 327].

Assertion (4) follows immediately from definition of $\mathfrak{f}_1$ and $\mathfrak{f}$.

For the last identity, we use the transformation formula $\eta(\tau+1) = \exp(\frac{\pi i}{12})\eta(\tau)$, [14, Proposition 2, p. 335], which implies that

$$\mathfrak{f}_1(\tau+1) = \frac{\eta(\frac{\tau+1}{2})}{\eta(\tau+1)} = \exp\left(\frac{-\pi i}{12}\right)\frac{\eta(\frac{\tau+1}{2})}{\eta(\tau)} = \exp\left(\frac{-\pi i}{24}\right)\mathfrak{f}(\tau). \tag{11}$$

From identity (4), the last assertion follows:

$$\mathfrak{f}(\tau)\mathfrak{f}_2\left(\frac{\tau+1}{2}\right) = \sqrt{2}\frac{\mathfrak{f}(\tau)}{\mathfrak{f}_1(\tau+1)} = \sqrt{2}\exp\left(\frac{\pi i}{24}\right).$$

□

It follows from Theorem 3 and Definitions (9) and (10) that $\mathbb{Q}(j(\tau)) \subseteq \mathbb{Q}(g(\tau))$, if g is one of the functions $\mathfrak{f}$, $\mathfrak{f}_1$, $\mathfrak{f}_2$, γ_2 and γ_3.

The following theorem of Schertz gives the complete list of class invariants introduced by Weber, which implies that $\mathbb{Q}(j(\tau)) = \mathbb{Q}(g(\tau))$ and $K(j(\tau)) = K(g(\tau))$, [14, p. 329]:

Theorem 4 *Let $\tau \in \mathbb{H}$ be a zero of a primitive equation*

$$Ax^2 + Bx + C = 0, \quad \gcd(A,2) = 1, B \equiv 0 \bmod 32$$

with the special discriminant $D(\tau) = t^2 d =: -4m$, $m \in \mathbb{N}$. Then the following numbers $g(\tau)$ are class invariants:

- $\left(\left(\frac{2}{A}\right)\frac{1}{\sqrt{2}}\mathfrak{f}(\tau)^2\right)^3$ *if* $m \equiv 1 \bmod 8$,

- $\mathfrak{f}(\tau)^3$ *if* $m \equiv 3 \bmod 8$,

- $\left(\frac{1}{2}\mathfrak{f}(\tau)^4\right)^3$ *if* $m \equiv 5 \bmod 8$,

- $\left(\left(\frac{2}{A}\right)\frac{1}{\sqrt{2}}\mathfrak{f}(\tau)\right)^3$ *if* $m \equiv 7 \bmod 8$,

- $\left(\left(\frac{2}{A}\right)\frac{1}{\sqrt{2}}\mathfrak{f}_1(\tau)^2\right)^3$ *if* $m \equiv 2 \bmod 4$,

- $\left(\left(\frac{2}{A}\right)\frac{1}{2\sqrt{2}}\mathfrak{f}_1(\tau)^4\right)^3$ *if* $m \equiv 4 \bmod 8$,

where the factor $\left(\frac{2}{A}\right)$ *denotes the Legendre symbol.*

If $\tau = \tau_1, \cdots, \tau_{h_t}$ *is a* 16*-system modulo* t*, then the singular values* $g(\tau_i)$ *above form a complete system of conjugates over* $\mathbb{Q}$*. Therefore, the minimal polynomial over* $\mathbb{Q}$ *is*

$$W_{D(\tau)}(x) = \prod_{i=1}^{h_t}(x - g_i), \text{ where } g_i := g(\tau_i).$$

Moreover, this polynomial has integer coefficients.

3.2. Generalization

The generalized Schläfli functions are defined as follows:

$$\mathfrak{m}_l(\tau) = \sqrt{l}\frac{\eta(l\tau)}{\eta(\tau)}, \quad \mathfrak{m}_j(\tau) = \zeta\frac{\eta(\frac{\tau+j}{l})}{\eta(\tau)}, \ 0 \le j \le l-1, \ l > 2 \tag{12}$$

where ζ is a suitable root of unity; see [7, p. 73].

The singular values of suitable powers of these functions yield class invariants by [6] as in Theorem 4.

Level $l = 3$: For this case, we have the functions

$$\mathfrak{g}_0(\tau) = \frac{\eta(\frac{\tau}{3})}{\eta(\tau)}, \ \mathfrak{g}_1(\tau) = \zeta_{24}^{-1}\frac{\eta(\frac{\tau+1}{3})}{\eta(\tau)}, \ \mathfrak{g}_2(\tau) = \frac{\eta(\frac{\tau+2}{3})}{\eta(\tau)}, \ \mathfrak{g}_3(\tau) = \sqrt{3}\frac{\eta(3\tau)}{\eta(\tau)}. \tag{13}$$

We have now the following theorem.

Theorem 5 *Let* $\tau \in \mathbb{H}$*. Then the following identities hold:*

1. $\mathfrak{g}_0(\tau)\mathfrak{g}_1(\tau)\mathfrak{g}_2(\tau)\mathfrak{g}_3(\tau) = \sqrt{3}$,

2. $\prod_{i=0}^{3}(x - \mathfrak{g}_i(\tau)^{12}) = x^4 + 36x^3 + 270x^2 + (756 - j(\tau))x + 3^6$,

3. $\mathfrak{g}_0(3\tau)\mathfrak{g}_3(\tau) = \sqrt{3}$,

4. $\mathfrak{g}_3(\frac{\tau+1}{3})\mathfrak{g}_1(\tau) = \sqrt{3}$,

5. $\mathfrak{g}_3(\frac{\tau+2}{3})\mathfrak{g}_2(\tau) = \zeta_{12}\sqrt{3}$.

Proof We refer to [19, p. 255] for the proof of the first two identities. The third one follows from

$$\mathfrak{g}_0(3\tau)\mathfrak{g}_3(\tau) = \sqrt{3}\frac{\eta(\tau)}{\eta(3\tau)}\frac{\eta(3\tau)}{\eta(\tau)} = \sqrt{3}.$$

Using the transformation formula for the Dedekind η-function $\eta(\tau + 1) = \zeta_{24}\eta(\tau)$, $\tau \in \mathbb{H}$, we obtain

$$\mathfrak{g}_3\left(\frac{\tau+1}{3}\right)\mathfrak{g}_1(\tau) = \sqrt{3}\zeta_{24}^{-1}\frac{\eta(\tau+1)}{\eta(\frac{\tau+1}{3})}\frac{\eta(\frac{\tau+1}{3})}{\eta(\tau)} = \sqrt{3}\zeta_{24}^{-1}\zeta_{24} = \sqrt{3}.$$

The last assertion follows similarly. □

The following theorem generalizes Theorem 4 to the case $l = 3$ for $\mathfrak{g}_i$, $0 \le i \le 2$, see [7, p. 73]:

Theorem 6 *Let $\mathcal{O}_K = [\tau, 1]$ be the maximal order of an imaginary quadratic number field K of discriminant D with $Tr_{K/\mathbb{Q}}(\tau) \in \{-1, 0\}$.*

Then the following values are class invariants, whose class polynomials have integer coefficients:

	$D \equiv 1 \bmod 9$	$D \equiv 4 \bmod 9$	$D \equiv 7 \bmod 9$	$D \equiv 3 \bmod 9$	$D \equiv 6 \bmod 9$
$D \equiv 1(4)$	$\zeta_3\mathfrak{g}_0^2, \zeta_3^2\mathfrak{g}_1^2$	$\mathfrak{g}_0^2, \mathfrak{g}_1^2$	$\zeta_3^2\mathfrak{g}_0^2, \zeta_3\mathfrak{g}_1^2$	$\frac{1}{3\sqrt{-3}}\mathfrak{g}_2^6$	$\frac{1}{\sqrt{-3}}\mathfrak{g}_2^2$
$D \equiv 0(8)$	$\zeta_3^2\zeta_4\mathfrak{g}_1^2, \zeta_3\zeta_4\mathfrak{g}_2^2$	$\zeta_3\zeta_4\mathfrak{g}_1^2, \zeta_3^2\mathfrak{g}_2^2$	$\zeta_4\mathfrak{g}_1^2, \zeta_4\mathfrak{g}_2^2$	$\frac{1}{3\sqrt{3}}\mathfrak{g}_0^6$	$\frac{1}{\sqrt{3}}\mathfrak{g}_0^2$
$D \equiv 4(8)$	$\zeta_3\mathfrak{g}_1^4, \zeta_3^2\mathfrak{g}_2^4$	$\zeta_3^2\mathfrak{g}_1^4, \zeta_3\mathfrak{g}_2^4$	$\mathfrak{g}_1^4, \mathfrak{g}_2^4$	$\frac{1}{3^3}\mathfrak{g}_0^{12}$	$\frac{1}{3}\mathfrak{g}_0^4$

Level $l = 5$**:** In this case, the functions are:

$$\mathfrak{h}_0(\tau) = \frac{\eta(\frac{\tau}{5})}{\eta(\tau)},\ \mathfrak{h}_1(\tau) = \zeta_8\frac{\eta(\frac{\tau+1}{5})}{\eta(\tau)},\ \mathfrak{h}_2(\tau) = \zeta_{12}\frac{\eta(\frac{\tau+2}{5})}{\eta(\tau)}, \tag{14}$$

$$\mathfrak{h}_3(\tau) = \zeta_{24}\frac{\eta(\frac{\tau+3}{5})}{\eta(\tau)},\ \mathfrak{h}_4(\tau) = \zeta_3^{-1}\frac{\eta(\frac{\tau+4}{5})}{\eta(\tau)},\ \mathfrak{h}_5(\tau) = \sqrt{5}\frac{\eta(5\tau)}{\eta(\tau)}.$$

The following theorem holds for these functions:

Theorem 7 *Let $\tau \in \mathbb{H}$. Then, we have the following identities:*

1. $\mathfrak{h}_0(\tau)\mathfrak{h}_1(\tau)\mathfrak{h}_2(\tau)\mathfrak{h}_3(\tau)\mathfrak{h}_4(\tau)\mathfrak{h}_5(\tau) = \sqrt{5}$,
2. $\mathfrak{h}_0(\tau)^6 + \mathfrak{h}_1(\tau)^6 + \mathfrak{h}_2(\tau)^6 + \mathfrak{h}_3(\tau)^6 + \mathfrak{h}_4(\tau)^6 + \mathfrak{h}_5(\tau)^6 = -30$,
3. $\mathfrak{h}_0(5\tau)\mathfrak{h}_5(\tau) = \sqrt{5}$,
4. $\mathfrak{h}_5(\frac{\tau+1}{5})\mathfrak{h}_1(\tau) = \zeta_6\sqrt{5}$,
5. $\mathfrak{h}_5(\frac{\tau+2}{5})\mathfrak{h}_2(\tau) = \zeta_8\sqrt{5}$,
6. $\mathfrak{h}_5(\frac{\tau+3}{5})\mathfrak{h}_3(\tau) = \zeta_{12}\sqrt{5}$,
7. $\mathfrak{h}_5(\frac{\tau+4}{5})\mathfrak{h}_4(\tau) = \zeta_3^{-1}\zeta_{24}\sqrt{5}$.

Proof For the proof of the first two identities we refer to [8, p. 439, 440]. The third identity follows from

$$\mathfrak{h}_0(5\tau)\mathfrak{h}_5(\tau) = \sqrt{5}\frac{\eta(\tau)}{\eta(5\tau)}\frac{\eta(5\tau)}{\eta(\tau)} = \sqrt{5}.$$

By the transformation $\eta(\tau+1) = \zeta_{24}\eta(\tau)$, we obtain

$$\mathfrak{h}_5(\frac{\tau+1}{5})\mathfrak{h}_1(\tau) = \sqrt{5}\zeta_8 \frac{\eta(\tau+1)}{\eta(\frac{\tau+1}{5})}\frac{\eta(\frac{\tau+1}{5})}{\eta(\tau)} = \sqrt{5}\zeta_8\zeta_{24} = \zeta_6\sqrt{5}.$$

The other identities follow similarly. □

The following theorem generalizes 4 for the function $\mathfrak{h}_5^2$, see [6, p. 17]:

Theorem 8 *Let $\mathcal{O}_t = [\tau, 1]$ be on order of an imaginary quadratic number field K of discriminant $D = t^2 d_K$, where d_K is the discriminant of K. Then $\mathfrak{h}_5^2(\tau)$ is a class invariant if $3 \not| D$.*

In contrast to other class invariants, the coefficients of the class polynomial of $\mathfrak{h}_5^2$ in Theorem 8 are not integers. They lie in $\mathcal{O}_K$, see [6].

Remark 9 *We refer to [6] for other generalized class invariants of level l. The sufficient conditions of having class polynomials with integer coefficients are also given in [6, Theorem 10, 11].*

4. Theta representation

4.1. Classical case

We improve in this section our results in [11] for classical class invariants by introducing other identities between some quotients of Thetanullwerte and third powers of Schläfli functions. The reason for using Thetanullwerte instead of values of the Dedekind η-function is that the asymptotically fastest algorithm to compute the n most significant digits of values of Schläfli functions is based on a computation using an identity between Thetanullwerte and the values of the Dedekind η-function. This algorithm uses the AGM method to compute the Thetanullwerte, see [4] and [5]. Hence, we can directly compute the invariants by using the identities which we show in this section. We refer to [11] for a more detailed analysis. The comparison will be given in the last section.

The following functions are the even Thetanullwerte, also known as Jacobi theta functions.

Definition 10 *Let $\tau \in \mathbb{H}$. We define*

1. $\theta_{00}(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2/2}$,
2. $\theta_{10}(\tau) := \sum_{n\in\mathbb{Z}} q^{(n+\frac{1}{2})^2/2}$,
3. $\theta_{01}(\tau) := \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/2}$.

By [19, p. 112, 114] we have the following identities:

Theorem 11 *The following assertions hold for $\tau \in \mathbb{H}$:*

1. $\theta_{00}(\tau) = \eta(\tau)\mathfrak{f}(\tau)^2$,
2. $\theta_{01}(\tau) = \eta(\tau)\mathfrak{f}_1(\tau)^2$,
3. $\theta_{10}(\tau) = \eta(\tau)\mathfrak{f}_2(\tau)^2$.

Definition 12 *Let $\tau \in \mathbb{H}$. We define the modified Schläfli functions as follows:*

1. $\mathfrak{F}(\tau) = 2\exp(\frac{\pi i}{8})\frac{\theta_{00}(\tau)}{\theta_{10}(\frac{\tau+1}{2})}$,

2. $\mathfrak{F}_1(\tau) = \frac{2\theta_{01}(\tau)}{\theta_{10}(\tau/2)}$,

3. $\mathfrak{F}_2(\tau) = \frac{\sqrt{2}\theta_{10}(\tau)}{\theta_{01}(2\tau)}$.

The following theorem gives identities between classical Schläfli functions and modified Schläfli functions.

Theorem 13 *The following identities hold for all $\tau \in \mathbb{H}$:*

1. $\mathfrak{F}(\tau) = \mathfrak{f}(\tau)^3$,

2. $\mathfrak{F}_1(\tau) = \mathfrak{f}_1(\tau)^3$,

3. $\mathfrak{F}_2(\tau) = \mathfrak{f}_2(\tau)^3$.

Proof

By multiplying the three functions $\theta_{00},\ \theta_{01},\ \theta_{10}$ we obtain by using Theorems 3.(2) and 11 :

$$\theta_{00}(\tau)\theta_{01}(\tau)\theta_{10}(\tau) = \eta(\tau)^3(\mathfrak{f}(\tau)\mathfrak{f}_1(\tau)\mathfrak{f}_2(\tau))^2 = (\sqrt{2})^2\eta(\tau)^3 = 2\eta(\tau)^3.$$

Hence, we have

$$\eta(\tau)^3 = \frac{\theta_{00}(\tau)\theta_{01}(\tau)\theta_{10}(\tau)}{2}.$$

Now, by Theorem 11.(1), the third power of $\theta_{00}(\tau)$ can be written as follows: $\theta_{00}(\tau)^3 = \eta(\tau)^3\mathfrak{f}(\tau)^6$. It means that

$$\mathfrak{f}(\tau)^6 = \frac{\theta_{00}(\tau)^3}{\eta(\tau)^3} = \frac{2\theta_{00}(\tau)^3}{\theta_{00}(\tau)\theta_{01}(\tau)\theta_{10}(\tau)} = \frac{2\theta_{00}(\tau)^2}{\theta_{01}(\tau)\theta_{10}(\tau)}.$$

Using Theorem 11.(3) and (4), we obtain similarly

$$\mathfrak{f}_1(\tau)^6 = \frac{2\theta_{01}(\tau)^2}{\theta_{00}(\tau)\theta_{10}(\tau)}, \quad \mathfrak{f}_2(\tau)^6 = \frac{2\theta_{10}(\tau)^2}{\theta_{00}(\tau)\theta_{01}(\tau)}.$$

On the other hand, we obtain the duplication formula of $\theta_{00}(\tau)$ and $\theta_{01}(\tau)$ using Definition 10 and the identity $2(n^2+m^2) = (n+m)^2 + (n-m)^2$ (see also [13, p. 63]):

$$\theta_{10}(\tau)^2 = 2\theta_{00}(2\tau)\theta_{10}(2\tau). \tag{15}$$

Therefore, we have

$$\mathfrak{f}_1(\tau)^6 = \frac{2\theta_{01}(\tau)^2}{\theta_{00}(\tau)\theta_{10}(\tau)} = \frac{4\theta_{01}(\tau)^2}{\theta_{10}(\tau/2)^2} = \left(\frac{2\theta_{01}(\tau)}{\theta_{10}(\tau/2)}\right)^2 = \mathfrak{F}_1(\tau)^2.$$

It follows by comparing the sign of q-expansions of both side of the identity:

$$\mathfrak{F}_1(\tau) = \mathfrak{f}_1(\tau)^3.$$

Now, Theorem 3.(4) implies that

$$\mathfrak{f}_2(\tau)^3 = \frac{2\sqrt{2}}{\mathfrak{f}_1(2\tau)^3} = \frac{2\sqrt{2}\theta_{10}(\tau)}{2\theta_{01}(2\tau)} = \mathfrak{F}_2(\tau).$$

Considering the third power of $\mathfrak{f}(\tau)$, it follows by Theorem 3.(5) that

$$\mathfrak{f}(\tau)^3 = \frac{e^{\frac{\pi i}{8}}2\sqrt{2}}{\mathfrak{f}_2(\frac{\tau+1}{2})^3} = \frac{e^{\frac{\pi i}{8}}2\sqrt{2}\theta_{01}(\tau+1)}{\sqrt{2}\theta_{10}(\frac{\tau+1}{2})} = \mathfrak{F}(\tau).$$

□

We immediately obtain the following theorem by Theorem 4 and Theorem 13, which enables to compute class polynomials using the new representations of Theorem 13:

Theorem 14 *Let $\tau \in \mathbb{H}$ be a zero of a primitive equation*

$$Ax^2 + Bx + C = 0 \quad \gcd(A,2) = 1, \; B \equiv 0 \bmod 32$$

with the special discriminant $D(\tau) = t^2 d =: -4m$. Then the following numbers $g(\tau)$ are class invariants:

1. $\left(\frac{2}{A}\right)\frac{1}{2\sqrt{2}}\mathfrak{F}(\tau)^2$ *if* $m \equiv 1 \bmod 8$,
2. $\mathfrak{F}(\tau)$ *if* $m \equiv 3 \bmod 8$,
3. $\frac{1}{8}\mathfrak{F}(\tau)^4$ *if* $m \equiv 5 \bmod 8$,
4. $\left(\frac{2}{A}\right)\frac{1}{2\sqrt{2}}\mathfrak{F}(\tau)$ *if* $m \equiv 7 \bmod 8$,
5. $\left(\frac{2}{A}\right)\frac{1}{2\sqrt{2}}\mathfrak{F}_1(\tau)^2$ *if* $m \equiv 2 \bmod 4$,
6. $\left(\frac{2}{A}\right)\frac{1}{16\sqrt{2}}\mathfrak{F}_1(\tau)^4$ *if* $m \equiv 4 \bmod 8$,

where the factor $\left(\frac{2}{A}\right)$ denotes the Legendre symbol.

As before if $\tau = \tau_1, \cdots, \tau_{h_t}$ is a 16-system modulo t, then the singular values $g(\tau_i)$ above form a complete system of conjugates over $\mathbb{Q}$. Therefore, the minimal polynomial over $\mathbb{Q}$ is

$$W_{D(\tau)}(x) = \prod_{i=1}^{h_t}(x - g_i), \text{ where } g_i := g(\tau_i),$$

and has integer coefficients.

4.2. Generalized invariants

We will derive identities to represent the Schläfli functions of level $l = 3$ and $l = 5$ as quotients of Thetanullwerte.

Level $l = 3$: We have the following theorem in this case.

Theorem 15 *For $\tau \in \mathbb{H}$, we have:*

1. $\mathfrak{g}_0(\tau) = \frac{\theta_{10}(\tau/6)}{\theta_{10}(\tau/2)}\frac{\mathfrak{f}_1(\tau/3)}{\mathfrak{f}_1(\tau)}$, $\mathfrak{g}_0(\tau)^3 = \frac{\theta_{10}(\tau/6)^2}{\theta_{10}(\tau/2)^2}\frac{\theta_{01}(\tau/3)}{\theta_{01}(\tau)}$,

2. $\mathfrak{g}_1(\tau) = \zeta_{48}\frac{\theta_{10}(\frac{\tau+1}{6})\mathfrak{f}_1(\frac{\tau+1}{3})}{\theta_{10}(\frac{\tau+1}{2})\mathfrak{f}(\tau)}$, $\mathfrak{g}_1(\tau)^3 = \frac{\theta_{10}(\frac{\tau+1}{6})^2\theta_{01}(\frac{\tau+1}{3})}{\theta_{10}(\frac{\tau+1}{2})^2\theta_{00}(\tau)}$,

3. $\mathfrak{g}_2(\tau) = \frac{\theta_{10}(\frac{\tau+2}{6})\mathfrak{f}_1(\frac{\tau+2}{3})}{\theta_{10}(\frac{\tau}{2})\mathfrak{f}_1(\tau)}$, $\mathfrak{g}_2(\tau)^3 = \frac{\theta_{10}(\frac{\tau+2}{6})^2\theta_{01}(\frac{\tau+2}{3})}{\theta_{10}(\frac{\tau}{2})^2\theta_{01}(\tau)}$,

4. $\mathfrak{g}_3(\tau) = \sqrt{3}\frac{\theta_{10}(\frac{3\tau}{2})\mathfrak{f}_1(3\tau)}{\theta_{10}(\frac{\tau}{2})\mathfrak{f}_1(\tau)}$, $\mathfrak{g}_3(\tau)^3 = 3\sqrt{3}\frac{\theta_{10}(\frac{3\tau}{2})^2\theta_{01}(3\tau)}{\theta_{10}(\frac{\tau}{2})^2\theta_{01}(\tau)}$.

Proof We showed in the proof of Theorem 13 that the identity

$$\eta(\tau)^3 = \frac{\theta_{00}(\tau)\theta_{01}(\tau)\theta_{10}(\tau)}{2}$$

holds. This implies that

$$\mathfrak{g}_0(\tau)^3 = \frac{\theta_{00}(\tau/3)\theta_{01}(\tau/3)\theta_{10}(\tau/3)}{\theta_{00}(\tau)\theta_{01}(\tau)\theta_{10}(\tau)}.$$

By the duplication formula (15) and Theorem 11.(2), we obtain

$$\mathfrak{g}_0(\tau)^3 = \frac{\theta_{10}(\tau/6)^2\theta_{01}(\tau/3)}{\theta_{10}(\tau/2)^2\theta_{01}(\tau)} = \frac{\eta(\tau/3)}{\eta(\tau)}\frac{\theta_{10}(\tau/6)^2\mathfrak{f}_1(\tau/3)^2}{\theta_{10}(\tau/2)^2\mathfrak{f}_1(\tau)^2} = \mathfrak{g}_0(\tau)\frac{\theta_{10}(\tau/6)^2\mathfrak{f}_1(\tau/3)^2}{\theta_{10}(\tau/2)^2\mathfrak{f}_1(\tau)^2}.$$

Hence by comparing the sign of the both sides, we get

$$\mathfrak{g}_0(\tau) = \frac{\theta_{10}(\tau/6)\mathfrak{f}_1(\tau/3)}{\theta_{10}(\tau/2)\mathfrak{f}_1(\tau)}.$$

Moreover, by Theorem 5.(3), we have:

$$\mathfrak{g}_3(\tau) = \frac{\sqrt{3}}{\mathfrak{g}_0(3\tau)} = \sqrt{3}\frac{\theta_{10}(\frac{3\tau}{2})\mathfrak{f}_1(3\tau)}{\theta_{10}(\frac{\tau}{2})\mathfrak{f}_1(\tau)},$$

and

$$\mathfrak{g}_3(\tau)^3 = \frac{3\sqrt{3}}{\mathfrak{g}_0(3\tau)^3} = 3\sqrt{3}\frac{\theta_{10}(\frac{3\tau}{2})^2\theta_{01}(3\tau)}{\theta_{10}(\frac{\tau}{2})^2\theta_{01}(\tau)}.$$

Theorem 5.(4) and the transformation (11) imply the identities

$$\mathfrak{g}_1(\tau) = \frac{\sqrt{3}}{\mathfrak{g}_3(\frac{\tau+1}{3})} = \frac{\theta_{10}(\frac{\tau+1}{6})\mathfrak{f}_1(\frac{\tau+1}{3})}{\theta_{10}(\frac{\tau+1}{2})\mathfrak{f}_1(\tau+1)} = \zeta_{48}\frac{\theta_{10}(\frac{\tau+1}{6})\mathfrak{f}_1(\frac{\tau+1}{3})}{\theta_{10}(\frac{\tau+1}{2})\mathfrak{f}(\tau)},$$

and

$$\mathfrak{g}_1(\tau)^3 = \frac{3\sqrt{3}}{\mathfrak{g}_3(\frac{\tau+1}{3})^3} = \frac{\theta_{10}(\frac{\tau+1}{6})^2\theta_{01}(\frac{\tau+1}{3})}{\theta_{10}(\frac{\tau+1}{2})^2\theta_{01}(\tau+1)}.$$

We obtain now the second part of the second assertion

$$\mathfrak{g}_1(\tau)^3 = \frac{\theta_{10}(\frac{\tau+1}{6})^2\theta_{01}(\frac{\tau+1}{3})}{\theta_{10}(\frac{\tau+1}{2})^2\theta_{00}(\tau)}$$

with the transformation formulas for Thetanullwerte

$$\theta_{01}(\tau+1)=\theta_{00}(\tau), \quad \theta_{00}(\tau+1)=\theta_{01}(\tau), \quad \theta_{10}(\tau+1)=\zeta_8\theta_{10}(\tau). \tag{16}$$

Theorem 5.(5) implies that

$$\mathfrak{g}_2(\tau)=\frac{\zeta_{12}\sqrt{3}}{\mathfrak{g}_3(\frac{\tau+2}{3})}=\zeta_{12}\frac{\theta_{10}(\frac{\tau+2}{6})\mathfrak{f}_1(\frac{\tau+2}{3})}{\theta_{10}(\frac{\tau+2}{2})\mathfrak{f}_1(\tau+2)}.$$

By transformation formulas (11) and (16), the following identities hold:

$$\mathfrak{f}_1(\tau+2)=\zeta_{48}^{-1}\mathfrak{f}(\tau+1)=\zeta_{24}^{-1}\frac{\eta(\frac{\tau+2}{2})}{\eta(\tau+1)}=\zeta_{24}^{-1}\mathfrak{f}_1(\tau), \quad \theta_{10}\left(\frac{\tau+2}{2}\right)=\zeta_8\theta_{10}(\tau/2).$$

Hence, we have the first part of the third assertion:

$$\mathfrak{g}_2(\tau)=\zeta_{12}\frac{\theta_{10}(\frac{\tau+2}{6})\mathfrak{f}_1(\frac{\tau+2}{3})}{\theta_{10}(\frac{\tau+2}{2})\mathfrak{f}_1(\tau+2)}=\zeta_{12}\frac{\theta_{10}(\frac{\tau+2}{6})\mathfrak{f}_1(\frac{\tau+2}{3})}{\zeta_8\theta_{10}(\frac{\tau}{2})\zeta_{24}^{-1}\mathfrak{f}_1(\tau)}=\frac{\theta_{10}(\frac{\tau+2}{6})\mathfrak{f}_1(\frac{\tau+2}{3})}{\theta_{10}(\frac{\tau}{2})\mathfrak{f}_1(\tau)}.$$

Lastly, we get by using Theorem 5.(5)

$$\mathfrak{g}_2(\tau)^3=\frac{\zeta_4 3\sqrt{3}}{\mathfrak{g}_3(\frac{\tau+2}{3})^3}=\zeta_4\frac{\theta_{10}(\frac{\tau+2}{6})^2\theta_{01}(\frac{\tau+2}{3})}{\theta_{10}(\frac{\tau+2}{2})^2\theta_{01}(\tau+2)}$$

holds. By formula (16), the last identity follows:

$$\mathfrak{g}_2(\tau)^3=\zeta_4\frac{\theta_{10}(\frac{\tau+2}{6})^2\theta_{01}(\frac{\tau+2}{3})}{\zeta_4\theta_{10}(\frac{\tau}{2})^2\theta_{01}(\tau)}=\frac{\theta_{10}(\frac{\tau+2}{6})^2\theta_{01}(\frac{\tau+2}{3})}{\theta_{10}(\frac{\tau}{2})^2\theta_{01}(\tau)}.$$

□

Level $l=5$: We can represent also the Schläfli functions of level 5 using some quotients of Thetanullwerte with the help of the following theorem, whose proof follows under the consideration of transformations of $\mathfrak{f}_1$ together with the transformation formulas (16), similar to the proof of Theorem 15.

Theorem 16 *We have the following identities for $\tau\in\mathbb{H}$:*

1. $\mathfrak{h}_0(\tau)=\frac{\theta_{10}(\tau/10)}{\theta_{10}(\tau/2)}\frac{\mathfrak{f}_1(\tau/5)}{\mathfrak{f}_1(\tau)}$, $\mathfrak{h}_0(\tau)^3=\frac{\theta_{10}(\tau/10)^2}{\theta_{10}(\tau/2)^2}\frac{\theta_{01}(\tau/5)}{\theta_{01}(\tau)}$,

2. $\mathfrak{h}_1(\tau)=\zeta_6\zeta_{48}\frac{\theta_{10}(\frac{\tau+1}{10})\mathfrak{f}_1(\frac{\tau+1}{5})}{\theta_{10}(\frac{\tau+1}{2})\mathfrak{f}(\tau)}$, $\mathfrak{h}_1(\tau)^3=-\frac{\theta_{10}(\frac{\tau+1}{10})^2\theta_{01}(\frac{\tau+1}{5})}{\theta_{10}(\frac{\tau+1}{2})^2\theta_{00}(\tau)}$,

3. $\mathfrak{h}_2(\tau)=\zeta_{24}\frac{\theta_{10}(\frac{\tau+2}{10})\mathfrak{f}_1(\frac{\tau+2}{5})}{\theta_{10}(\frac{\tau}{2})\mathfrak{f}_1(\tau)}$, $\mathfrak{h}_2(\tau)^3=\zeta_8\frac{\theta_{10}(\frac{\tau+2}{10})^2\theta_{01}(\frac{\tau+2}{5})}{\theta_{10}(\frac{\tau}{2})^2\theta_{01}(\tau)}$,

4. $\mathfrak{h}_3(\tau)=\zeta_{48}\frac{\theta_{10}(\frac{\tau+3}{10})\mathfrak{f}_1(\frac{\tau+3}{5})}{\theta_{10}(\frac{\tau+1}{2})\mathfrak{f}(\tau)}$, $\mathfrak{h}_3(\tau)^3=\frac{\theta_{10}(\frac{\tau+3}{10})^2\theta_{01}(\frac{\tau+3}{5})}{\theta_{10}(\frac{\tau+1}{2})^2\theta_{00}(\tau)}$,

5. $\mathfrak{h}_4(\tau)=\zeta_{24}^{-1}\frac{\theta_{10}(\frac{\tau+4}{10})\mathfrak{f}_1(\frac{\tau+4}{5})}{\theta_{10}(\frac{\tau}{2})\mathfrak{f}_1(\tau)}$, $\mathfrak{h}_4(\tau)^3=\zeta_8\frac{\theta_{10}(\frac{\tau+4}{10})^2\theta_{01}(\frac{\tau+4}{5})}{\theta_{10}(\frac{\tau}{2})^2\theta_{01}(\tau)}$,

6. $\mathfrak{h}_5(\tau)=\sqrt{5}\frac{\theta_{10}(\frac{5\tau}{2})\mathfrak{f}_1(5\tau)}{\theta_{10}(\frac{\tau}{2})\mathfrak{f}_1(\tau)}$, $\mathfrak{h}_5(\tau)^3=5\sqrt{5}\frac{\theta_{10}(\frac{5\tau}{2})^2\theta_{01}(5\tau)}{\theta_{10}(\frac{\tau}{2})^2\theta_{01}(\tau)}$.

Arbitrary level l: We can represent the following generalized Schläfli functions of arbitrary level ℓ as quotients of Thetanullwerte using the same method of the proof of Theorem 15:

$$\mathfrak{m}_\ell(\tau) = \sqrt{\ell}\frac{\eta(\ell\tau)}{\eta(\tau)}, \quad \mathfrak{m}_0(\tau) = \frac{\eta(\frac{\tau}{\ell})}{\eta(\tau)}. \tag{17}$$

Theorem 17 *For $\tau \in \mathbb{H}$, we have:*

1. $\mathfrak{m}_0(\ell\tau)\mathfrak{m}_\ell(\tau) = \sqrt{\ell}$,
2. $\mathfrak{m}_0(\tau) = \frac{\theta_{10}(\tau/2\ell)\mathfrak{f}_1(\tau/\ell)}{\theta_{01}(\tau/2)\mathfrak{f}_1(\tau)}$,
3. $\mathfrak{m}_0(\tau)^3 = \frac{\theta_{10}(\tau/2\ell)^2}{\theta_{10}(\tau/2)^2}\frac{\theta_{01}(\tau/\ell)}{\theta_{01}(\tau)}$,
4. $\mathfrak{m}_\ell(\tau) = \sqrt{\ell}\frac{\theta_{10}(\ell\tau/2)\mathfrak{f}_1(\ell\tau)}{\theta_{10}(\tau/2)\mathfrak{f}_1(\tau)}$,
5. $\mathfrak{m}_\ell(\tau)^3 = \ell\sqrt{\ell}\frac{\theta_{10}(\ell\tau/2)^2\theta_{01}(\ell\tau)}{\theta_{10}(\tau/2)^2\theta_{01}(\tau)}$.

Remark 18 *The function*

$$\zeta\frac{\eta(\frac{\tau+k}{\ell})}{\eta(\tau)},$$

of arbitrary level ℓ can be represented by Theorem 17 by using suitable roots of unity ζ and the transformation formula (16) as quotients of some Thetanullwerte.

5. Unit property

In this section we prove that the invariants of Theorem 6 are units in the corresponding Hilbert class fields. This result generalizes the results of [11, Theorem 20, 21], in which we proved that most of the invariants introduced in Theorem 14 are units in the corresponding ring class fields.

For the modular discriminant function and the Dedekind η-function, we have the following identity, which we will need later:

$$\Delta(\tau) = (2\pi)^{12}\eta(\tau)^{24}. \tag{18}$$

Furthermore, let P be a primitive matrix of determinant p, where p is a prime number, that is $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2\times 2}$ with $\det(P) = p$ and $\gcd(a,b,c,d) = 1$.

For the quotient (see [3, p. 11])

$$\varphi_P(\tau) := p^{12}\frac{\Delta\left(P\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}\right)}{\Delta\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}} \text{ where } \Delta\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = \omega_2^{-12}\Delta(\tau), \tag{19}$$

we have the following two cases of theorem of Deuring [3, p. 43], which we need in order to prove our claim stated at the beginning. (For the other cases we refer to [3, p. 43].):

Theorem 19 *Let $t > 0$ be an integer, p be a prime number and $l \geq 0$ be the greatest power of p with $p^l | t$. Let further a, b, c and d be integers such that the matrix $P := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has determinant p. Assume that $\{\omega_1, \omega_2\}$ is a basis of a fractional $\mathcal{O}_t$-ideal I with $\tau := \frac{\omega_1}{\omega_2} \in \mathbb{H}$.*

1. *Let $l = 0$. If p splits completely in K, $(p) = \mathfrak{p}\overline{\mathfrak{p}}$, then we have:*

   if $P \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$ is a basis of the ideal $I_{\mathcal{O}_t}\mathfrak{p}_{\mathcal{O}_t}$ (or $I_{\mathcal{O}_t}\overline{\mathfrak{p}}_{\mathcal{O}_t}$), then $\frac{\varphi_P(\tau)}{\overline{\mathfrak{p}}^{12}}$ (resp. $\frac{\varphi_P(\tau)}{\mathfrak{p}^{12}}$) is a unit.

2. *If p ramifies in K, $(p) = \mathfrak{p}^2$, then we have:*

   $\frac{\varphi_P(\tau)}{p^6}$ is a unit if $P \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$ is a basis of the ideal $I_{\mathcal{O}_t}\mathfrak{p}_{\mathcal{O}_t}$.

We have now the following theorem:

Theorem 20 *The class invariants introduced in Theorem 6 are units in the corresponding Hilbert class fields.*

Proof Firstly, we consider the cases $D \equiv 1 \bmod 3$ and $D \equiv 1 \bmod 4$. It follows that $D \equiv 1 \bmod 12$.

Hence, we get $\mathcal{O}_K = \mathbb{Z}[\frac{1+\sqrt{D}}{2}]$ and that 3 splits in $\mathcal{O}_K$.

For the number l in Theorem 19, the equality $l = 0$ holds, since the conductor $t = 1$ by Theorem 6.

By Theorem 19.(1) the quotients $\frac{\varphi_P(\tau)}{3^{12}}$ and $\frac{\varphi_Q(\tau)}{3^{12}}$ are units with the matrices $P = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$, respectively.

We obtain now by identity (19) the units

$$\frac{\varphi_P(\tau)}{3^{12}} = 3^{-12}\frac{\Delta(\tau/3)}{3^{-12}\Delta(\tau)} = \frac{\Delta(\tau/3)}{\Delta(\tau)} \text{ and } \frac{\varphi_Q(\tau)}{3^{12}} = \frac{\Delta(\frac{\tau+1}{3})}{\Delta(\tau)}.$$

Due to the identity (18), the 12th roots of these units are class invariants for the cases $D \equiv 1 \bmod 4$ and $D \equiv 1, 4, 7 \bmod 9$.

Similarly, one can obtain for $D \equiv 0, 4 \bmod 8$ and $D \equiv 1, 4, 7 \bmod 9$ that the corresponding invariants are units, because we have $D \equiv 4 \bmod 12$. This is due to the fact that 3 splits in $\mathcal{O}_k$ and we can consider the matrices Q and $R = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$.

By 19.(1) it follows that $\mathfrak{g}_2(\tau)^2$ is a unit.

It follows $D \equiv 0 \bmod 12$ for the cases $D \equiv 0 \bmod 3$ and $D \equiv 0 \bmod 4$.

Hence, 3 is ramified in $\mathcal{O}_k$. We obtain the following unit by Theorem 19.(2) and the identity (19) with using the matrix $P = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

$$\frac{\varphi_P(\tau)}{3^6} = \frac{1}{3^6}\frac{\Delta(\frac{\tau}{3})}{\Delta(\tau)}.$$

By the identity (19), the following holds:

$$\left(\frac{\mathfrak{g}_0(\tau)^2}{\sqrt{3}}\right)^{12} = \frac{1}{3^6}\frac{\Delta(\frac{\tau}{3})}{\Delta(\tau)}.$$

Altogether the class invariants

$$\frac{\mathfrak{g}_0(\tau)^2}{\sqrt{3}},\ \frac{\mathfrak{g}_0(\tau)^4}{3},\ \frac{\mathfrak{g}_0(\tau)^6}{3\sqrt{3}} \text{ and } \frac{\mathfrak{g}_0(\tau)^{12}}{3^3}$$

are units for $D \equiv 24 \bmod 72$, $D \equiv 60 \bmod 72$, $D \equiv 48 \bmod 72$, $D \equiv 12 \bmod 72$, respectively.

From the last two cases, $D \equiv 1 \bmod 4$ and $D \equiv 0 \bmod 3$, we have the property $D \equiv 9 \bmod 12$, and hence $3|D$. We have the following again by Theorem 19.(2) and the identity (19) with the matrix $R = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$:

$$\frac{\varphi_R(\tau)}{3^6} = \frac{1}{3^6}\frac{\Delta(\frac{\tau+2}{3})}{\Delta(\tau)},$$

and by the identity (19)

$$\left(\frac{\mathfrak{g}_2(\tau)^2}{\sqrt{3}}\right)^{12} = \frac{1}{3^6}\frac{\Delta(\frac{\tau+2}{3})}{\Delta(\tau)}.$$

Therefore, the class invariants

$$\frac{\mathfrak{g}_2(\tau)^2}{\sqrt{-3}} \text{ and } \frac{\mathfrak{g}_2(\tau)^6}{3\sqrt{-3}}$$

are units when $D \equiv 33 \bmod 72$ and $D \equiv 57 \bmod 72$, respectively. □

6. Examples

We compare in this section the time needed to compute the class polynomials as quotients of values of the Dedekind $\eta-$function by Theorem 4 with the time needed when using the different representations of class invariants using Thetanullwerte (the representations in [11] and in Theorem 14).

We introduced the following representation of class invariants using Thetanullwerte, [11, Theorem 8, p. 4]:

Theorem 21 *For $\tau \in \mathbb{H}$, we have:*

$$\mathfrak{G}(\tau) := \frac{2\theta_{00}(\tau)^2}{\theta_{01}(\tau)\theta_{10}(\tau)} = \mathfrak{f}(\tau)^6, \quad \mathfrak{G}_1(\tau) := \frac{2\theta_{01}(\tau)^2}{\theta_{00}(\tau)\theta_{10}(\tau)} = \mathfrak{f}_1(\tau)^6,$$

and

$$\mathfrak{G}_2(\tau) := \frac{2\theta_{10}(\tau)^2}{\theta_{00}(\tau)\theta_{01}(\tau)} = \mathfrak{f}_2(\tau)^6.$$

We compute the class polynomials using the invariants of Theorems 4, 21 and 14 with a fixed precision for several discriminants. We obtained the following table using MAGMA, see [12]. Moreover, the values of η-functions are computed by [11, Theorem 18, p. 9].

Let D denote the discriminant, h_D the class number, Prec the fixed precision, η, old$-\theta$, new$-\theta$ the time in seconds needed to compute the class polynomial using Theorems 4, 21 and 14, respectively. Then, we computed the following table of examples:

D	h_D	Prec	η	old-θ	new-θ
-740	16	40	0.11	0.03	0.02
-1040	21	45	0.12	0.04	0.03
-3188	30	90	0.23	0.06	0.04
-7196	50	120	0.48	0.11	0.09
-7796	70	149	0.69	0.16	0.13
-12344	84	94	0.59	0.14	0.12
-42800	108	201	1.44	0.32	0.26
-43316	128	355	3.72	0.93	0.70
-66404	168	403	4.74	1.05	0.87
-204716	264	679	16.34	3.95	3.69
-345236	340	1073	55.15	13.12	9.70
-825020	504	1329	147.18	34.87	27.80
-1057124	1032	1756	423.55	96.76	73.19
-14123480	1752	5179	16715.51	1835.23	1386.31

These examples show also experimentally that the computations can be performed more efficiently using Theorem 14.

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