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Rings over which every module has a flat δ-cover

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Abstract: Let M be a module. A *δ-cover* of M is an epimorphism from a module F onto M with a δ-small kernel. A δ-cover is said to be a *flat δ-cover* in case F is a flat module. In the present paper, we investigate some properties of (flat) δ-covers and flat modules having a projective δ-cover. Moreover, we study rings over which every module has a flat δ-cover and call them *right generalized δ-perfect* rings. We also give some characterizations of δ-semiperfect and δ-perfect rings in terms of locally (finitely, quasi-, direct-) projective δ-covers and flat δ-covers.

Key words: δ-covers, δ-perfect rings, δ-semiperfect rings, flat modules

1. Preliminaries and Notation

Let R be a ring and $\mathcal{F}$ be a class of R-modules. Due to Enochs and Jenda [9], for an R-module M, a morphism $\varphi : C \to M$, where $C \in \mathcal{F}$, is called an $\mathcal{F}$-cover of M if the following properties are satisfied:

1) For any morphism $\psi : C' \to M$, where $C' \in \mathcal{F}$, there is a morphism $\lambda : C' \to C$ such that $\varphi o \lambda = \psi$; and

2) if μ is an endomorphism of C such that $\varphi o \mu = \varphi$, then μ is an automorphism of C.

If $\mathcal{F}$ is the class of projective modules, then an $\mathcal{F}$-cover is called a *projective cover*. This definition is in agreement with the usual definition of a projective cover. If $\mathcal{F}$ is the class of flat modules, then an $\mathcal{F}$-cover is called a *flat cover*. On the other hand, some authors deal with flat covers in the following sense.

Let M be an R-module. A *flat cover* of M is an epimorphism $f : F \to M$ with a small kernel, where F is a flat module.

In this paper, we will consider the second definition. In fact, the notion of a flat cover in this sense is a natural generalization of a projective cover. But these two notions of flat covers do not coincide. There are examples of modules which do not have flat covers (see [2]) whereas all modules have flat covers in Enochs' sense (see [6]).

A. Amini, B. Amini, Ershad and Sharif investigate in [2] those rings R whose right R-modules have flat covers, and call them *right generalized perfect* (*right G-perfect*, for short) rings.

It is well-known that projective covers play an important role in characterizing perfect and semiperfect rings. Some authors have also characterized these rings in terms of flat covers. Ding and Chen show in [8] that a ring R is right perfect if and only if R is semilocal and every semisimple right R-module has a flat cover. In

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[14], Lomp proves that a ring R is semiperfect if and only if R is semilocal and every simple right R-module has a flat cover.

Recall from [18] that an epimorphism $f : P \to M$ with a δ-small kernel is called a *projective* δ*-cover* of the module M in case P is projective. As a proper generalization of perfect (resp., semiperfect) rings, δ-perfect (resp., δ-semiperfect) rings are defined in [18] as follows: A ring R is said to be δ*-perfect* (resp., δ*-semiperfect*) if every R-module (resp., simple R-module) has a projective δ-cover.

These results motivated us to define the notion of flat δ-covers. In this paper, we deal with rings over which (certain) right modules have flat δ-covers. Firstly, in Section 2, we investigate some basic properties of δ-covers. We prove that if a module has a flat δ-cover, then a generalized projective δ-cover of the module is a projective δ-cover. It is a well-known fact that if a flat module has a projective cover, then it is projective. As Example 2.17 shows, a flat module need not be projective whenever it has a projective δ-cover. However, over a ring with a finitely generated right socle, a finitely generated flat module is projective if it has a projective δ-cover. Section 3 is concerned with those rings R whose right R-modules have flat δ-covers. We call them *'right generalized* δ*-perfect'* (*right* G*-*δ*-perfect*, for short) rings and show that this notion is a proper generalization of δ-perfect rings. As Example 3.8 shows, this notion is not left-right symmetric. We prove that if R is a right G-δ-perfect ring, then $J(R/S_r)$ is right T-nilpotent. This result leads us to generalize some important results proved in [2]. For instance, we are able to show that if R is a right G-δ-perfect ring, then R is right Artinian if and only if R is right Noetherian. In the last section, we give some characterizations of δ-perfect and δ-semiperfect rings in terms of flat δ-covers. We also consider locally projective, finitely projective, quasi-projective and direct-projective δ-covers in order to give some necessary and sufficient conditions for a ring to be δ-perfect or δ-semiperfect.

Throughout this paper, R denotes an associative ring with identity and modules are unitary right R-modules. For a module M, $\mathrm{Soc}(M)$ is the socle and $\mathrm{Rad}(M)$ is the Jacobson radical of M. S_r and $J(R)$ will stand for the right socle and the Jacobson radical of a ring R, respectively. We will denote a direct summand (resp., small submodule) of a module M by $K \leq^{\oplus} M$ (resp., $K \ll M$).

As a generalization of small submodules, in [18], Zhou introduce δ-small submodules as follows:

A submodule N of a module M is said to be δ*-small* if $N + K \neq M$ for any proper submodule K of M with M/K singular, and it is denoted by $N \ll_\delta M$. By this definition, every small or nonsingular semisimple submodule of M is δ-small in M.

The following lemma, which appeared in [18], gives a necessary and sufficient condition for a submodule N of M to be δ-small in M and we will use it throughout the paper.

Lemma 1.1 *[18, Lemma 1.2] The following are equivalent:*

(1) $N \ll_\delta M$

(2) *If* $X + N = M$, *then* $M = X \oplus Y$ *for a projective semisimple submodule* Y *with* $Y \subseteq N$.

According to [18, Lemma 1.5], the submodule $\delta(M) = \sum\{L \subseteq M | L \ll_\delta M\}$ which is also equal to the intersection of all essential maximal submodules of M whenever M is projective (see [18, Lemma 1.9]). We will use the notation δ_r to indicate the intersection of all essential maximal right ideals of R. Note from [18, Corollary 1.7] that $J(R/S_r) = \delta_r/S_r$.

2. Flat δ-covers

Definition 2.1 An epimorphism $f : P \to M$ is called a *δ-cover of M* in the case $\mathrm{Ker}(f) \ll_\delta P$.

We start with some basic properties of δ-covers. The proofs of the following three lemmas are straightforward, so we omit them.

Lemma 2.2 *If $f : P \to M$ and $g : M \to N$ are δ-covers, then $gf : P \to N$ is a δ-cover.*

Lemma 2.3 *If each $f_i : P_i \to M_i$ is a δ-cover for $i = 1, \ldots, n$, then $\oplus_{i=1}^n f_i : \oplus_{i=1}^n P_i \to \oplus_{i=1}^n M_i$ is a δ-cover.*

Lemma 2.4 *If $N \leq^\oplus M$ and $A \ll_\delta M$, then $A \cap N \ll_\delta N$.*

Lemma 2.5 *Let K be a submodule of a projective module F. If F/K has a δ-cover, then it has a δ-cover of the form $f : F/L \to F/K$ with $Ker(f) = K/L$, where $L \subseteq K$.*

Proof Let $f : P \to F/K$ be a δ-cover of F/K and $\pi : F \to F/K$ be the natural epimorphism. Since F is projective, there exists a homomorphism $\lambda : F \to P$ such that $f\lambda = \pi$. Then $P = \mathrm{Ker}(f) + \mathrm{Im}(\lambda)$. It follows from Lemma 1.1 that $P = Y \oplus \mathrm{Im}(\lambda)$ for a semisimple submodule Y with $Y \subseteq \mathrm{Ker}(f)$. Also, by Lemma 2.4, $\mathrm{Ker}(f|_{\mathrm{Im}(\lambda)}) \ll_\delta \mathrm{Im}(\lambda)$. So $f|_{\mathrm{Im}(\lambda)}$ is also a δ-cover of F/K. But $F/\mathrm{Ker}(\lambda) \cong \mathrm{Im}(\lambda)$ and since $f\lambda = \pi$, $\mathrm{Ker}(\lambda) \subseteq K$. If we consider the isomorphism $\lambda' : F/\mathrm{Ker}(\lambda) \to \mathrm{Im}(\lambda)$, then we obtain $\mathrm{Ker}(f|_{\mathrm{Im}(\lambda)}\lambda') \ll_\delta F/\mathrm{Ker}(\lambda)$ by Lemma 2.2. □

Since any finitely generated (resp., cyclic) module is an epimorphic image of a finitely generated (resp., cyclic) free module, we obtain the following result by the proof of Lemma 2.5.

Lemma 2.6 *If $f : P \to M$ is a δ-cover of a finitely generated (cyclic) module M, then there exists a finitely generated (cyclic) direct summand P' of P such that $f|_{P'}$ is a δ-cover of M.*

Definition 2.7 A δ-cover $f : P \to M$ is called a *flat δ-cover* of M in case P is a flat module.

It is clear that if a module has a projective δ-cover, then it also has a flat δ-cover. By Example 3.8 below, the converse does not hold in general. Now we will investigate under which condition a module M has a projective δ-cover whenever it has a flat δ-cover. But we need some results in order to prove one of the main result of this section.

Locally projective modules are introduced by Zimmermann-Huisgen ([19]) and we know from [5, Proposition 6] that an R-module M is *locally projective* if and only if for any $x \in M$ there exist a finite number of homomorphisms $f_i : M \to R$ $(i = 1, \ldots, n)$ and elements $y_i \in M$ $(i = 1, \ldots, n)$ such that $y_1 f_1(x) + \cdots + y_n f_n(x) = x$. It is well-known that the following implications hold for a module:

projective $\Rightarrow$ locally projective $\Rightarrow$ flat.

Proposition 2.8 *If M is a locally projective module, then $M\delta_r = \delta(M)$.*

Proof By [18, Lemma 1.5(2)], the inclusion $M\delta_r \subseteq \delta(M)$ always holds. For the reverse inclusion let $x \in \delta(M)$. Then by hypothesis, there exist a finite number of homomorphisms $f_i : M \to R$ and elements $y_i \in M$ $(i = 1, \ldots, n)$ such that $y_1 f_1(x) + \cdots + y_n f_n(x) = x$. It follows from [18, Lemma 1.5(2)] that $f_i(\delta(M)) \subseteq \delta_r$ for each i and so $f_i(x) \in \delta_r$ for each i. Hence, we obtain that $x \in M\delta_r$. □

Definition 2.9 An epimorphism $f : P \to M$ is called a *generalized (locally) projective δ-cover of M* in case $\mathrm{Ker}(f) \subseteq \delta(P)$ and P is (locally) projective.

For a homomorphism $f : P \to M$, the inclusion $f(\delta(P)) \subseteq \delta(M)$ always holds by [18, Lemma 1.5(2)]. It can be observed that the equality holds whenever $f : P \to M$ is an epimorphism and $\mathrm{Ker}(f) \subseteq \delta(P)$. By this fact, we obtain the following result.

Corollary 2.10 *If a module M has a generalized locally projective δ-cover, then $M\delta_r = \delta(M)$.*

Proof Let $f : P \to M$ be a generalized locally projective δ-cover of M. Then $\delta(M) = f(\delta(P)) = f(P\delta_r) = f(P)\delta_r = M\delta_r$. □

Proposition 2.11 *If M is a locally projective module, then $MS_r = Soc(M)$.*

Proof It follows from a proof similar to that of Proposition 2.8. □

Remark 2.12 1) Note that $[\delta(M) + \mathrm{Soc}(M)]/\mathrm{Soc}(M) \subseteq Rad(M/\mathrm{Soc}(M))$ for any module M: Consider $\overline{m} = m + \mathrm{Soc}(M) \in [\delta(M) + \mathrm{Soc}(M)]/\mathrm{Soc}(M)$, where $m \in \delta(M)$. Suppose that $\overline{m} \notin Rad(M/\mathrm{Soc}(M))$. Then there exists a maximal submodule of M with $\mathrm{Soc}(M) \subseteq L$ and $m \notin L$. So $M = L + mR$. Since $mR \ll_\delta M$, $M = L \oplus Y$ for a projective semisimple submodule Y of mR. But $\mathrm{Soc}(M) \subseteq L$ implies that $Y = 0$. It follows that $M = L$, a contradiction.

2) It is easy to observe that if P is a locally projective R-module, then P/PI is a locally projective R/I-module for any ideal I of R.

3) We know from [19, Proposition 2.2] that a locally projective module with $Rad(M) = M$ is zero.

4) Recall from [5, Proposition 10] that a countably generated locally projective module is projective.

Proposition 2.13 *Let M be a locally projective module with $\delta(M) = M$. Then M is a projective semisimple module.*

Proof Since $M = \delta(M)$, we get that $Rad(M/\mathrm{Soc}(M)) = M/\mathrm{Soc}(M)$ by Remark 2.12(1). Also, Remark 2.12(2) together with Proposition 2.11 implies that $M/\mathrm{Soc}(M)$ is a locally projective R/S_r-module. It follows from Remark 2.12(3) that $M = \mathrm{Soc}(M)$. Moreover, M is projective because a simple locally projective module is projective by Remark 2.12(4). □

Recall from [13] that a short exact sequence of right R-modules $0 \to A \xrightarrow{\varphi} B \to C \to 0$ is *pure* if it remains exact after being tensored with any left R-module. If this is the case, then $\varphi(A)$ is called *a pure submodule* of B. It is known that direct summands are pure submodules. Due to [16, Theorem 4], if N is a finitely generated pure submodule of a projective module P, then it is a direct summand of P. Let $A \subseteq B \subseteq D$ be right R-modules. If A is pure in B and B is pure in D, then A is pure in D (see [13, Examples 4.84(e)]). Also, it follows from [13, Theorem 4.85] that if M/N is a flat R-module, then N is a pure submodule of M, and the converse holds if M is flat by [13, Corollary 4.86(1)]. We know from [13, Corollary 4.92] that if N is a pure submodule of M, then $NI = N \cap MI$ for each left ideal I of the ring R. If M is a projective module, then the converse holds by [13, Exercise 41, pg. 163]. In addition, a pure submodule of a locally projective module is locally projective by [5, Proposition 7].

Now we are ready to prove the following result as promised.

Theorem 2.14 *Suppose that a module M has a flat δ-cover. A generalized projective δ-cover of M is a projective δ-cover of M.*

Proof Let $f : X \to M$ be a flat δ-cover and $g : P \to M$ a generalized projective δ-cover of M. P being projective implies that there exists a homomorphism $h : P \to X$ such that $fh = g$. Then $X = \mathrm{Ker}(f) + \mathrm{Im}(h)$. Since $\mathrm{Ker}(f) \ll_\delta X$, $X = T \oplus \mathrm{Im}(h)$ for a projective semisimple submodule T with $T \subseteq \mathrm{Ker}(f)$ by Lemma 1.1. As $P/\mathrm{Ker}(h)$ is flat, $\mathrm{Ker}(h)$ is a pure submodule of P. Moreover, $\mathrm{Ker}(h)$ is locally projective. But $\mathrm{Ker}(h) \subseteq \mathrm{Ker}(g) \subseteq \delta(P)$. Due to [13, Corollary 4.92] the purity of $\mathrm{Ker}(h)$ implies that $\mathrm{Ker}(h)\delta_r = \mathrm{Ker}(h) \cap \delta(P) = \mathrm{Ker}(h)$. The fact that $\mathrm{Ker}(h)$ is locally projective together with Proposition 2.8 implies that $\delta(\mathrm{Ker}(h)) = \mathrm{Ker}(h)$. Hence, $\mathrm{Ker}(h)$ is projective semisimple by Proposition 2.13, which means that $\mathrm{Ker}(h) \ll_\delta P$. So $h : P \to \mathrm{Im}(h)$ is a projective δ-cover of $\mathrm{Im}(h)$. Note that $f|_{\mathrm{Im}(h)}h = g$, and so by Lemma 2.2, $\mathrm{Ker}(g) \ll_\delta P$. □

Using the idea of the proof of [12, Theorem 10.5.3] we obtain the following theorem. Note that this result can also be used to prove Theorem 2.14. Indeed, by Proposition 2.15, the submodule $\mathrm{Ker}(h)$ in the proof of Theorem 2.14 is projective semisimple.

Proposition 2.15 *Suppose that P is a projective module, $U \subseteq \delta(P)$ and P/U is flat. Then U is projective semisimple. In this case, every finitely generated submodule of U is a direct summand of P.*

Proof Firstly we will consider the theorem in case $P = F$ is a free module. Let $\{x_i | i \in I\}$ be a basis of F. Take $u \in U$ and let $u = \sum_{i=1}^{n} x_i a_i$, where $a_i \in R$. Consider the finitely generated left ideal $A = \sum_{i=1}^{n} Ra_i$ of R. By the techniques used in [12, Theorem 10.5.3], we obtain that $A = \delta_r A$. Now we can observe that $\frac{A+S_r}{S_r} = \frac{\delta_r A+S_r}{S_r} = \frac{\delta_r}{S_r}\frac{A+S_r}{S_r} = J(\frac{R}{S_r})\frac{A+S_r}{S_r}$. By Nakayama's Lemma, $A \subseteq S_r$. Then $u \in U \cap FS_r = U \cap \mathrm{Soc}(F) = \mathrm{Soc}(U)$. It follows that $U = \mathrm{Soc}(U)$. But U is a pure submodule of a projective module, so it is projective, too.

Now let P be a projective module and U be a submodule of P such that $U \subseteq \delta(P)$ and P/U is flat. Then P is a direct summand of a free module F. Again using the techniques in [12, Theorem 10.5.3], we get $U \subseteq \delta(F)$. From the proof above it follows that U is projective semisimple.

In this case, every submodule of U is a pure submodule of P. Because U is a pure submodule of P, every finitely generated submodule of U is a direct summand of P by [16, Theorem 4]. □

By Proposition 2.15, we obtain the following result which will turn out to be a useful tool in characterizing δ-semiperfect rings in Section 4.

Proposition 2.16 *If a flat module F has a projective δ-cover, then so does every finitely generated pure submodule of F.*

Proof Let $f : P \to F$ be a projective δ-cover of F. Then $\mathrm{Ker}(f)$ is projective semisimple by Proposition 2.15. Consider a finitely generated pure submodule $L = \sum_{i=1}^{n} x_i R$ of F. Since f is epic, there exists $p_i \in P$ such that $f(p_i) = x_i$ for each $i = 1, \ldots, n$. So $\sum_{i=1}^{n} p_i R \subseteq T = f^{-1}(L)$. To show that $T = \mathrm{Ker}(f) + \sum_{i=1}^{n} p_i R$, let $t \in T$. Then $f(t) = \sum_{i=1}^{n} x_i r_i = f(\sum_{i=1}^{n} p_i r_i)$ $(r_i \in R)$ which gives that $t - \sum_{i=1}^{n} p_i r_i \in \mathrm{Ker}(f)$. Hence, we get the desired equality. As $\mathrm{Ker}(f)$ is projective semisimple, $\mathrm{Ker}(f) \ll_\delta T$ so that $T = \sum_{i=1}^{n} p_i R \oplus Y$, where Y is a projective semisimple submodule of $\mathrm{Ker}(f)$. Because F is flat and L is pure, $P/T \cong F/L$ is flat which means that T is a pure submodule of P. It follows that $\sum_{i=1}^{n} p_i R$ is also a pure submodule of P

and so it is projective by [16, Theorem 4]. So, T is projective. Thus, $f|_T : T \to L$ is a projective δ-cover of L. □

It is known that if a flat module has a projective cover, then it is projective. But, as the following example shows, this is not the case for a flat module which has a projective δ-cover even if the flat module is cyclic.

Example 2.17 [18, Example 4.1] Let $Q = \prod_{i=1}^{\infty} F_i$, where each $F_i = \mathbb{Z}_2$. Let R be the subring of Q generated by $S = \oplus_{i=1}^{\infty} F_i$ and 1_Q. Consider the singular simple R-module R/S. Since R is a (von Neumann) regular ring, R/S is a flat R-module. Zhou shows that R is a δ-semiperfect ring so that R/S has a projective δ-cover. But if R/S was projective, then R would be semisimple. This leads to a contradiction.

On the other hand, we obtain the following result.

Proposition 2.18 *Let R be a ring with a finitely generated right socle S_r. If F is a finitely generated flat module with a projective δ-cover, then it is projective.*

Proof Let $f : P \to F$ be a projective δ-cover of a finitely generated flat module F. Then, by Lemma 2.6, we can assume that P is also finitely generated. By Theorem 2.15, $\text{Ker}(f)$ is projective semisimple so that $\text{Ker}(f) \subseteq \text{Soc}(P) = PS_r$. $\text{Soc}(P)$ being finitely generated implies that $\text{Ker}(f)$ is finitely generated. But $\text{Ker}(f)$ is a pure submodule of P. Hence, $\text{Ker}(f) \leq^{\oplus} P$ by [16, Theorem 4]. Thus, F is projective. □

As Example 2.17 shows, the condition that 'S_r is finitely generated' is not superfluous in Proposition 2.18.

3. Generalized δ-perfect rings

Definition 3.1 A ring R is said to be *right generalized δ-perfect* (*right G-δ-perfect*, for short) if every right R-module has a flat δ-cover. Left G-δ-perfect rings are defined similarly. We call R a *G-δ-perfect* ring in case it is both right and left G-δ-perfect.

We start this section with some examples.

Example 3.2 Trivially, every flat module has a flat δ-cover. Hence, every regular ring is G-δ-perfect.

Example 3.3 A right δ-perfect ring is a right G-δ-perfect ring. The converse need not be true as Example 3.8 shows.

Example 3.4 $\mathbb{Z}$ is not a G-δ-perfect ring.

Proof Let $n \geq 2$ and consider the $\mathbb{Z}$-module $M = \mathbb{Z}/n\mathbb{Z}$. Assume that $f : F \to M$ is a flat δ-cover of M. It follows from the proof of Lemma 2.5 that M has a flat δ-cover of the form $\mathbb{Z}/K$ which is isomorphic to F because projective semisimple $\mathbb{Z}$-modules are zero. So $\mathbb{Z}/K$ is a cyclic flat $\mathbb{Z}$-module. But it is projective since $\mathbb{Z}$ is Noetherian. Then $K \leq^{\oplus} \mathbb{Z}$. As $K \neq \mathbb{Z}$ we obtain that $K = 0$. So $F \cong \mathbb{Z}$. Let $g : F \to \mathbb{Z}$ be the isomorphism. Since $\text{Ker}(f) \ll_\delta F$, $g(\text{Ker}(f)) \ll_\delta \mathbb{Z}$ by [18, Lemma 1.3(2)]. Since $\delta(\mathbb{Z}) = 0$ and g is an isomorphism, we have that $\text{Ker}(f) = 0$. So f is an isomorphism which means that $M \cong \mathbb{Z}$. But this is a contradiction. Thus, M does not have a flat δ-cover. □

Example 3.5 Let $\mathbb{Q}$ be the set of rational numbers. Since $\mathbb{Q}$ is a flat $\mathbb{Z}$-module and $\mathbb{Z} \ll_\delta \mathbb{Q}$, the natural epimorphism $\pi : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is a flat δ-cover of $\mathbb{Q}/\mathbb{Z}$. But it can be shown by a proof similar to that of [2, Example 2.1(d)] that its direct summand $\mathbb{Z}_{p^\infty}$ (the Prufer p-group) does not have a flat δ-cover.

Example 3.5 shows that a submodule of a module which has a flat δ-cover need not have a flat δ-cover. However, we have the following result which can be obtained by a proof similar to that of [2, Proposition 3.11].

Proposition 3.6 *Let R be a ring such that $\delta(M) = M\delta_r \ll_\delta M$ for any flat module M. Assume that L/K is a flat module, where $K \subseteq L$. If L has a flat δ-cover, then so does K.*

Now we consider some basic properties of right G-δ-perfect rings.

Proposition 3.7 (1) *Being a right G-δ-perfect ring is a Morita invariant.*

(2) *The class of right G-δ-perfect rings is closed under taking quotient rings.*

(3) *The class of right G-δ-perfect rings is closed under finite direct product of rings.*

Proof (1) Similar to [3, Proposition 5.14] we can easily observe that $K \ll_\delta M$ if and only if for every module N and for every homomorphism $h : N \to M$ $\text{Im}(h)+K = M$ with $M/\text{Im}(h)$ singular implies that $\text{Im}(h) = M$. As a consequence of this result (similar to [3, Corollary 5.15]) we get that an epimorphism $g : M \to N$ has a δ-small kernel if and only if for all homomorphisms h with $M/\text{Im}(h)$ singular if gh is epic, then h is epic. Combining this fact with [13, Exercise 18.2, pg. 501] and with [3, Lemma 21.3] we obtain that the property that 'having a δ-cover' is preserved under a category equivalence. Hence, by [3, Exercise 22.12, pg. 268], we get the desired result. (2) and (3) follow from a proof similar to that of [2, Proposition 2.6]. □

Example 3.8 *There exists a right G-δ-perfect ring that is not right δ-perfect.*

Proof Consider a non-semisimple regular ring R with $\delta_r = 0$ and a right δ-perfect ring S that is not regular. (For examples of such rings see [18, Examples 4.2 and 4.3].) Then the ring $R \times S$ is right G-δ-perfect by Proposition 3.7(3), but it is not right δ-perfect since $(R \times S)/\delta(R \times S) \cong R \times S/\delta(S)$ is not semisimple. Note also that $R \times S$ is not regular. □

Recall that a subset S of a ring R is said to be *right T-nilpotent* in case for every sequence $a_1, a_2, \ldots$ in S there is an integer $n \geq 1$ such that $a_n \ldots a_2 a_1 = 0$. The following theorem describes the right T-nilpotency of $J(R/S_r)$.

Theorem 3.9 *The following statements are equivalent:*

(1) *$J(R/S_r)$ is right T-nilpotent.*

(2) *$\delta(M) \ll_\delta M$ for every (non-semisimple) projective module M.*

(3) *$\delta(F) \ll_\delta F$ for every countably generated (non-semisimple) free module F.*

Proof (1) ⇒ (2) Let $M = \delta(M) + K$ with M/K singular for a proper submodule K of a projective module M. Since M is projective, $K \leq_e M$ which implies that $\text{Soc}(M) = MS_r \subseteq K$. So $\frac{M}{K}$ is a nonzero right $\frac{R}{S_r}$-module. But by [3, Lemma 28.3(b)], $\frac{M}{K}J(\frac{R}{S_r}) \neq \frac{M}{K}$ which means that $\frac{M}{K}\frac{\delta_r}{S_r} \neq \frac{M}{K}$. On the other hand, $\frac{M}{K}\frac{\delta_r}{S_r} = \frac{(M\delta_r+K)}{K}\frac{R}{S_r} = \frac{M}{K}$, a contradiction. Consequently, $M = K$.

(2) ⇒ (3) It is obvious.

(3) ⇒ (1) It follows from a proof similar to that of (4) ⇒ (1) of Theorem 3.7 in [18]. □

In [2], it is shown that if R is a right G-perfect ring, then $J(R)$ is right T-nilpotent. But it is evident from [18, Example 4.3] that δ_r need not be right T-nilpotent whenever R is a right G-δ-perfect ring. However, considering the characterization of δ-perfect rings (see [18, Theorem 3.8]), it is natural to expect the following result.

Theorem 3.10 *If R is a right G-δ-perfect ring, then $J(R/S_r)$ is right T-nilpotent. In particular, idempotents lift modulo δ_r.*

Proof By Theorem 3.9, it is enough to show that $\delta(F) \ll_\delta F$ for a countably generated free R-module F. By assumption, $F/\delta(F)$ has a flat δ-cover. Also, the natural epimorphism $\pi : F \to F/\delta(F)$ is a generalized projective δ-cover of $F/\delta(F)$. It follows from Theorem 2.14 that π is a projective δ-cover. Hence, $\delta(F) \ll_\delta F$. In particular, idempotents of the ring R/S_r lift modulo $J(R/S_r)$. By [17, Lemma 1.3], idempotents of R lift modulo δ_r. □

Remark 3.11 Note that alternatively Theorem 3.10 can also be proved with the help of Proposition 2.15. We can consider $(R/\delta_r)^{(\mathbb{N})}$ and its flat δ-cover. Then apply the proof of Lemma 2.5 considering $R^{(\mathbb{N})}$. The rest of the proof follows from Proposition 2.15 and Lemma 2.2.

The next example shows that the notion of G-δ-perfect rings is not left-right symmetric.

Example 3.12 *There exists a right G-δ-perfect ring that is not left G-δ-perfect.*

Proof Let R be the ring of all countably infinite square upper triangular matrices over a field F that are constant on the main diagonal and have only finitely many nonzero entries off the main diagonal. It is shown in ([15, Example B.46]) that $J(R)$ is not left T-nilpotent. So $J(R/S_r)$ is not left T-nilpotent. Hence, R is not left G-δ-perfect by Theorem 3.10. On the other hand, R is right G-δ-perfect since R is right perfect. □

According to [18, Theorem 3.5], a ring R is δ-semiregular if and only if R/δ_r is regular and idempotents lift modulo δ_r. Büyükaşık and Lomp prove in [7] that a δ-semiperfect ring with a finitely generated right socle is semiperfect. This fact together with Theorem 3.10 enables us to prove the following result which generalizes Proposition 2.4 in [2].

Proposition 3.13 *Let R be a right G-δ-perfect ring. Then R is right Noetherian if and only if R is right Artinian.*

Proof The necessity is obvious. For the sufficiency, let M be a simple R-module. Then, by Lemma 2.6, M has a flat δ-cover $f : F \to M$ such that F is cyclic. Since finitely generated flat modules are projective over a Noetherian ring, F is projective. Hence, every simple R-module has a projective δ-cover which means that R is δ-semiperfect. By [7, Remark 4.4], R is semiperfect. It follows that R/S_r is semiperfect. Since $J(R/S_r)$ is nil by Theorem 3.10, R/S_r is right Noetherian semiprimary ring. It follows from Hopkin's Theorem that R/S_r is an Artinian ring and so an Artinian R-module. Since S_r is Artinian, R is right Artinian. □

Theorem 3.14 *Let R be a ring such that every cyclic flat right R-module is projective. If R is right G-δ-perfect, then R/δ_r is regular.*

Proof By Proposition 3.7(2), it is enough proof that R is regular whenever $\delta_r = 0$. Assume that R is not regular. Then there exists a cyclic right R-module M that is not flat by [13, Theorem 4.21]. But M has a flat δ-cover $f : F \to M$ and since M is cyclic we can assume that F is cyclic by Lemma 2.6. Then F is projective by hypothesis. Therefore, we get that $\mathrm{Ker}(f) \subseteq \delta(F) = F\delta_r = 0$. Thus, $F \cong M$ is projective, which is a contradiction. □

Recall from [13, pg. 297 and 321] that a ring R is said to be *strongly (π-)regular* if, for any $a \in R$, there exists $x \in R$ (and a positive integer n) such that $a = a^2x$ ($a^n = a^{n+1}x$). Recall also that a ring R is said to be *right* (resp., *left*) *duo* in case every right (resp., left) ideal of R is a two-sided ideal. It is known that a strongly regular ring is right and left duo. By the next theorem, we can conclude that a right duo and a right G-δ-perfect ring R with $J(R) = 0$ is strongly regular. Note also that the next theorem is a generalization of [2, Theorem 2.7] since strongly regular rings are regular.

Theorem 3.15 *If R is right duo and right G-δ-perfect, then $R/J(R)$ is strongly regular.*

Proof By Proposition 3.7(2), we can assume, without loss of generality that $J(R) = 0$. Let x be a nonzero element of R. By Lemma 2.5, R/xR has a flat δ-cover of the form $f : R/I \to R/xR$, where $I \subseteq xR$ and $\mathrm{Ker}(f) = xR/I$. Hence, $xR/I \subseteq \delta(R/I)$. R/I being right G-δ-perfect implies that $\delta(R/I)/\mathrm{Soc}(R/I)$ is nil so that there exists a positive integer n such that $\overline{x}^n = x^n + I \in \mathrm{Soc}(R/I)$. Since $\overline{x}^nR$ is an Artinian R-module, there exists a positive integer $k \geq n$ such that $\overline{x}^kR = \overline{x}^{k+1}R = \ldots$. Then there exists $r \in R$ such that $x^k - x^{k+1}r \in I$. Since R/I is flat, it follows from [13, Theorem 4.23] that there is an element $a \in I$ such that $x^k - x^{k+1}r = a(x^k - x^{k+1}r)$ and hence $x^k - x^{k+1}r = a^{k+1}(x^k - x^{k+1}r)$. Also, we have that $I^{k+1} \subseteq (xR)^{k+1} \subseteq x^{k+1}R$ because R is right duo. But then $x^k \in x^{k+1}R$. So, R is strongly π-regular. By [4, Theorem 3], we may assume that $x^k = x^{k+1}r$ and $xr = rx$ for some $r \in R$. It follows that $(x^{k-1} - x^kr)^2 = 0$. But since $J(R) = 0$, R is semiprime and so $x^{k-1} - x^kr = 0$. If we continue this process, then we get that $x = x^2r$. Thus, R is strongly regular. □

We observe from [18, Example 4.3] that the above theorem need not be true if R is not right duo.

We obtain some conditions under which a right G-δ-perfect ring is δ-semiregular by Theorems 3.14 and 3.15.

Corollary 3.16 *Assume that R is a right duo ring or a ring such that every cyclic flat right R-module is projective. If R is right G-δ-perfect, then R is δ-semiregular.*

Recall that a ring R is said to be *right max* if every nonzero right R-module has a maximal submodule. Due to Hamsher [10], if R is commutative, then R is a max ring if and only if $R/J(R)$ is regular and $J(R)$ is T-nilpotent. By Theorems 3.10 and 3.15 we have the following corollaries as generalizations of [2, Corollaries 2.9 and 2.10].

Corollary 3.17 *If R is a commutative G-δ-perfect ring, then R is a max ring. In particular, every prime ideal of R is maximal.*

Corollary 3.18 *Let R be a commutative G-δ-perfect ring. Then a module M is Noetherian if and only if M is Artinian.*

Using the idea of the proofs of [2, Theorem 3.3] and [2, Corollary 3.4], one can immediately obtain the following two results.

Proposition 3.19 *Let R be a ring. If R/δ_r is regular and $J(R/S_r)$ is right T-nilpotent, then every module of the form F/K, where F is a free module and K is a countably generated submodule of F, has a flat δ-cover.*

Corollary 3.20 *Let R be a right max ring with R/δ_r regular. Let F be a free module and $K \subseteq F$. Suppose that $\Omega = \{T \subseteq F|\ T$ is an essential maximal submodule of F not containing $K\}$ is countable. Then F/K has a flat δ-cover.*

Remark 3.21 It is easy to observe that if $J(R/S_r)$ is right T-nilpotent, then $J(R)$ is right T-nilpotent, too. Therefore, it follows from Theorem 3.10 that if R is a semilocal ring, then R is right G-δ-perfect if and only if R is right G-perfect, if and only if R is right perfect, if and only if R is right δ-perfect. We do not know an example of a G-δ-perfect ring that is not G-perfect.

4. Some characterizations of δ-semiperfect and δ-perfect rings

We start this section with some characterizations of δ-semiperfect rings. Firstly, we consider generalized (locally) projective δ-covers.

Theorem 4.1 *Let R be a ring. Suppose that idempotents lift modulo δ_r. Then the following statements are equivalent:*

(1) *R is δ-semiperfect.*

(2) *Every simple right R-module has a generalized locally projective δ-cover.*

(3) *Every simple right R-module has a generalized projective δ-cover.*

Proof (1) $\Leftrightarrow$ (3): It follows from [1, Lemma 4.3] and [18, Theorem 3.6].

(1) $\Rightarrow$ (2) It is obvious. (2) $\Rightarrow$ (1): To show that $\overline{R} = R/\delta_r$ is semisimple we need to prove each simple right $\overline{R}$-module S is projective. If we regard S as a simple R-module, then S has a generalized locally projective δ-cover $f : P \to S$. Since P is locally projective, $\mathrm{Ker}(f) \subseteq P\delta_r$ by Proposition 2.8.

If $P\delta_r = P$, then $S = f(P) = f(P\delta_r) = f(P)\delta_r = S\delta_r = 0$, which is impossible. Then $P\delta_r \neq P$. Since $\mathrm{Ker}(f)$ is maximal in P, we have that $\mathrm{Ker}(f) = P\delta_r$ and so $P/P\delta_r \cong S$. Since P is a locally projective R-module, $P/P\delta_r$ is a locally projective $\overline{R}$-module, so S is a locally projective $\overline{R}$-module. But S is simple so that it is projective. Thus, $\overline{R}$ is semisimple. □

Corollary 4.2 *Let R be a ring. Suppose that idempotents lift modulo δ_r. Then the following statements are equivalent:*

(1) *R is δ-semiperfect.*

(2) *Every finitely generated (cyclic) right R-module has a generalized locally projective δ-cover.*

(3) *Every finitely generated (cyclic) right R-module has a generalized projective δ-cover.*

Recall from [8] that an R-module M is called *finitely projective* if, for any finitely generated submodule M_0 of M, there exist a finitely generated free module F and homomorphisms $f : M_0 \to F$ and $g : F \to M$ such that $g(f(x)) = x$ for all $x \in M_0$. Note that a finitely generated finitely projective module is projective. Also, it is well-known that the following implications hold for a module:

$$\text{locally projective} \Rightarrow \text{finitely projective} \Rightarrow \text{flat.}$$

Note that we will call a δ- cover $f : P \to M$ of a module M a *locally (finitely) projective δ-cover* in case P is a locally (finitely) projective module.

Theorem 4.3 *The following statements are equivalent for a ring R:*

(1) *R is δ-semiperfect.*

(2) *Every simple right R-module has a locally projective δ-cover.*

(3) *Every simple right R-module has a finitely projective δ-cover.*

(4) *R/δ_r is semisimple and every simple right R-module has a flat δ-cover.*

Proof The implications $(1) \Rightarrow (2) \Rightarrow (3)$ and $(1) \Rightarrow (4)$ are obvious.

$(3) \Rightarrow (1)$ Let S be a simple right R-module and $f : P \to S$ be a finitely projective δ-cover. Since S is cyclic, by Lemma 2.6, there exists a cyclic direct summand P' of P such that $f|_{P'}$ is a finitely projective δ-cover of S. Then P' is projective. Hence, S has a projective δ-cover. Thus, R is δ-semiperfect.

$(4) \Rightarrow (1)$ By [18, Theorem 1.8], we can consider $R/\delta_r = \oplus_{i=1}^n S_i$, where S_i is simple singular R-module for each $i = 1, \ldots, n$. It is enough to show that each simple singular R-module has a projective δ-cover in order to prove that R is δ-semiperfect. Let M be a simple singular R-module. Then M is isomorphic to one of S_i's. By hypothesis, each S_i has a flat δ-cover. Let $f_i : F_i \to S_i$ be the flat δ-cover of S_i. Then $f = \oplus_{i=1}^n f_i : F = \oplus_{i=1}^n F_i \to R/\delta_r$ is a flat δ-cover of R/δ_r by Lemma 2.3. Since R is projective, there exists $g : R \to F$ such that $fg = \pi$, where $\pi : R \to R/\delta_r$ is the natural epimorphism. Then $F = \mathrm{Ker}(f) + \mathrm{Im}(g)$. But $\mathrm{Ker}(f) \ll_\delta F$ so that $F = Y \oplus \mathrm{Im}(g)$, where Y is a projective semisimple submodule of $\mathrm{Ker}(f)$. Since $\mathrm{Ker}(g) \subseteq \mathrm{Ker}(\pi) = \delta_r$, $g : R \to \mathrm{Im}(g)$ is a projective δ-cover of $\mathrm{Im}(g)$. Hence, $g \oplus id_Y : R \oplus Y \to F$ is a projective δ-cover of F. By Lemma 2.6, we can assume that each F_i is cyclic. It follows from Proposition 2.16 that each F_i has a projective δ-cover. Thus, $S_i \cong M$ has a projective δ-cover. □

Recall that an M-projective module M is *quasi-projective* (see [3]) and that a module M is called *direct-projective* if, for every direct summand X of M, every epimorphism $M \to X$ splits (see [11]). Note that a quasi-projective module is direct-projective.

We need the following lemma in order to prove the next result.

Lemma 4.4 *[11] Let P be projective and $P \oplus M$ direct projective. If there is an epimorphism $f : P \to M$, then M is projective.*

A δ-cover $f : P \to M$ of a module M is said to be a *quasi-projective (direct-projective) δ-cover* in case P is a quasi-projective (direct-projective) module.

Theorem 4.5 *If every right R-module has a direct-projective δ-cover, then every right R-module has a projective δ-cover.*

Proof Let M be a module and consider the epimorphism $f : F \to M$, where F is free. By assumption, $F \oplus M$ has a direct-projective δ-cover. Let $g : P \to F \oplus M$ be the direct-projective δ-cover of $F \oplus M$. Consider the canonical projection $\pi : F \oplus M \to F$. Since F is projective, we have a monomorphism $h : F \to P$ such that $\pi gh = id_F$. So $P = F \oplus T$, where $T = \mathrm{Ker}(\pi g)$. Now we claim that $\overline{g} = g|_T : T \to M$ is a projective δ-cover of M. To show that $\mathrm{Ker}(\overline{g}) \ll_\delta T$, let $T = \mathrm{Ker}(\overline{g}) + N$, where N is a submodule of T. Since $P = F \oplus T = F + \mathrm{Ker}(\overline{g}) + N$ and $\mathrm{Ker}(\overline{g}) \subseteq \mathrm{Ker}(g) \ll_\delta P$, $P = F \oplus N \oplus Y$ for a projective semisimple submodule Y of $\mathrm{Ker}(\overline{g})$. The equality $P = F \oplus T = F \oplus N \oplus Y$ gives that $T = N \oplus Y$ and so by Lemma 1.1, $\mathrm{Ker}(\overline{g}) \ll_\delta T$.

Now we will show that T is projective. Again by projectivity of F there exists $f' : F \to T$ such that $\overline{g} f' = f$. Hence, $T = \mathrm{Ker}(\overline{g}) + \mathrm{Im}(f')$. Since $\mathrm{Ker}(\overline{g}) \ll_\delta T$, there exists a projective semisimple submodule Y

of $\mathrm{Ker}(\overline{g})$ such that $T = Y \oplus \mathrm{Im}(f')$. Since $F \oplus \mathrm{Im}(f') \leq^{\oplus} P$, it is direct-projective. Then $\mathrm{Im}(f')$ is projective by Lemma 4.4. It follows that T is projective. □

By a proof similar to that of Theorem 4.5, we can observe that if every finitely generated right R-module has a direct projective δ-cover, then every finitely generated right R-module has a projective δ-cover. The next result is an immediate consequence of this fact and Theorem 4.3.

Corollary 4.6 *The following statements are equivalent for a ring R:*

(1) *R is δ-semiperfect.*

(2) *Every finitely generated right R-module has a quasi-projective δ-cover.*

(3) *Every finitely generated right R-module has a direct-projective δ-cover.*

(4) *Every finitely generated (cyclic) right R-module has a locally projective δ-cover.*

(5) *Every finitely generated (cyclic) right R-module has a finitely projective δ-cover.*

(6) *R/δ_r is semisimple and every finitely generated (cyclic) right R-module has a flat δ-cover.*

Next, we will deal with δ-perfect rings.

Theorem 4.7 *Let R be a ring such that $J(R/S_r)$ is right T-nilpotent. Then the following statements are equivalent:*

(1) *R is right δ-perfect.*

(2) *Every semisimple right R-module has a generalized locally projective δ-cover.*

(3) *Every semisimple right R-module has a generalized projective δ-cover.*

Proof The equivalency $(1) \Leftrightarrow (3)$ follows from [1, Lemma 4.3], [18, Theorem 3.6] and [17, Lemma 1.3], and the proof of $(1) \Leftrightarrow (2)$ is similar to that of $(1) \Leftrightarrow (2)$ in Theorem 4.1. □

We conclude this section with the following theorem which states some equivalent conditions for a ring to be δ-perfect.

Theorem 4.8 *The following statements are equivalent for a ring R:*

(1) *R is right δ-perfect.*

(2) *Every right R-module has a quasi-projective δ-cover.*

(3) *Every right R-module has a direct-projective δ-cover.*

(4) *Every semisimple right R-module has a locally projective δ-cover.*

(5) *Every semisimple right R-module has a finitely projective δ-cover.*

(6) *R/δ_r is semisimple and every semisimple right R-module has a flat δ-cover.*

Proof The implications $(1) \Rightarrow (2) \Rightarrow (3)$ and $(1) \Rightarrow (4) \Rightarrow (5)$ are obvious.

$(3) \Rightarrow (1)$ It follows from Theorem 4.5.

$(5) \Rightarrow (6)$ By Theorem 4.3, R is δ-semiperfect. Every semisimple right R-module has a flat δ-cover since finitely projective modules are flat.

$(6) \Rightarrow (1)$ By [18, Theorem 3.8], we only need to prove that $J(R/S_r)$ is right T-nilpotent. Since R/δ_r is semisimple, $F/\delta(F)$ is a semisimple R-module for a countably generated free module F and so $F/\delta(F)$ has a flat δ-cover by assumption. Hence, the rest of the proof is similar to that of Theorem 3.10. □

The condition that 'R/δ_r is semisimple' in Theorems 4.3 and 4.8 is not superfluous because of Example 3.8.

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