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C.P. modules and their applications

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Abstract: Let R be a ring. A left R-module M is called a c.p. module if every cyclic submodule of M is projective. This notion is a generalization of left p.p. rings in the general module theoretic setting. The aim of this article is to investigate these modules. Some characterizations and properties are given. As applications, the connections among Baer rings, p.p. rings and von Neumann regular rings are studied.

Key words: Baer ring, p.p. ring, c.p. module, von Neumann regular ring

1. Introduction

Throughout this paper, all rings are associative with unity and all modules are unitary modules. For a ring R, ${}_RM$ (M_R) denotes a left (right) R-module (in this article, we abbreviate ${}_RM$ to M if no ambiguity arises), and module homomorphisms are written on the right (left) of their arguments. The notations $N \subseteq M$, $N \leq M$ or $N \leq^{\oplus} M$ mean that N is a subset, a submodule or a direct summand of M, respectively. M^I ($M^{(I)}$) is the direct product (sum) of copies of M indexed by a set I. For a module ${}_RM$ and each subset S of M, set $l_R(S) = \{r \in R : rs = 0, \forall s \in S\}$. Let m, n be two positive integers. The set of $m \times n$ matrices over R is denoted by $M_{m\times n}(R)$, in particular, $M_n(R)$ denotes the $n \times n$ matrices ring over R. Write $T_n(R)$ for the ring of all $n \times n$ upper triangular matrices over R. $\mathbb{Q}$ and $\mathbb{Z}$ denote the ring of rational and integer numbers, respectively. By $\mathbb{Z}_n$ we denote the ring of integers modulo n. General background material can be found in [1, 15]. In this paper, we extend the upper triangular matrix ring $T_n(R)$ to the general module theoretic setting. For a left R-module M, write $T_n(M)$ for the set of all $n \times n$ formal upper triangular matrices over M and define the addition and scalar multiplication via $(x_{ij}) + (y_{ij}) = (x_{ij} + y_{ij})$, $(r_{ij})(x_{ij}) = (\sum_{k=1}^{n} r_{ik}x_{kj})$ for any $(x_{ij}), (y_{ij}) \in T_n(M)$ and $(r_{ij}) \in T_n(R)$. Then $T_n(M)$ is a left $T_n(R)$-module relative to the addition and scalar multiplication. Denote $(R^nX : \xi) = \{r \in R : r\xi \in R^nX\}$ for any $X \in T_n(M)$ and $\xi \in M^n$, where the operations of R^nX are induced naturally by the ones of ${}_RM$.

A ring R is called a Baer ring [13] if the left annihilator of every nonempty subset of R is a direct summand of ${}_RR$. The study of such rings has its roots in functional analysis with close links to C^*-algebras and von Neumann algebras and it is well known that the concept is left-right symmetric. Closely related to the notion of Baer rings is the more general concept of left p.p. rings. R is called left p.p. [7] if each principal left ideal of R is projective; equivalently, if the left annihilator $l_R(a)$ is a direct summand of ${}_RR$ for each $a \in R$. A number of interesting results on these rings have been obtained (see, for example, [2, 3, 4, 5, 9, 11, 12]). In 1972,

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Evans introduced the notion of c.p. modules in [8], which puts the concept of p.p. rings in the general module theoretic setting, but not much work on the modules was done. A left R-module M is called a c.p. module [8] if every cyclic submodule of M is projective, equivalently if the left annihilator $l_R(x)$ is a direct summand of $_RR$ for each $x \in M$. In 2010, Lee, Rizvi and Roman [16] introduced the concept of Rickart modules as another generalization of left p.p. rings by exploiting the connections between a module $_RM$ and its endomorphism ring $S = \mathrm{End}_R(M)$. A left R-module M is called a Rickart module if the left annihilator in M of any single element of S is generated by an idempotent of S. Equivalently, $\forall \varphi \in S$, $l_M(\varphi) = \mathrm{Ker}\varphi = Me$ for some $e^2 = e \in S$.

In this paper, we consider the notion of c.p. modules. Apply the results on modules to investigate p.p. property of rings and the connections among Baer rings, p.p. rings and (von Neumann) regular rings (a ring R is called regular if every principal left ideal is a direct summand of $_RR$). In Section 2, some characterizations and properties of c.p. modules are obtained. Such as, direct sums of c.p. left R-modules are c.p. left R-modules. It is proved that R is Baer if and only if R is p.p. and the direct products of c.p. left R-modules are again c.p. left R-modules. We provide an example to show that the notion of c.p. modules is distinct from that of Rickart modules, and prove that every cyclic c.p. left R-module is Rickart. In Section 3, we consider the p.p. property of the left $T_n(R)$-module $T_n(M)$. An equivalent condition for $_{T_n(R)}T_n(M)$ to be c.p. is obtained. Moreover, applying the result to rings in Section 4, we get that R is a regular ring if and only if $T_n(R)$ is left p.p. for each $n \geq 2$ if and only if $T_n(R)$ is left p.p. for some $n \geq 2$. An intimate connection among c.p. modules, p.p. rings and regular rings is established.

2. C.P. modules

In this section, we investigate the notion of a c.p. module which is a generalization of a left (or right) p.p. ring in the general module theoretic setting. A left R-module M is called a c.p. module [8] if every cyclic submodule of M is projective, equivalently if $l_R(x)$ is a direct summand of $_RR$ for each $x \in M$. It is easy to see that a ring R is a left p.p. ring if $_RR$ is a c.p. module. The right analogs are defined similarly.

Each submodule of a c.p. module is again a c.p. module by the definition of c.p. modules. But it is not true for a factor module. For example, we consider the ring $\mathbb{Z}$ of integers. $_{\mathbb{Z}}\mathbb{Z}$ is a c.p. module, but $\mathbb{Z}/2\mathbb{Z}$ is not c.p. as a $\mathbb{Z}$-module.

Example 2.1 (1) *Every projective left module over a left semihereditary ring is a c.p. module.*

(2) *Every regular module (in the sense of Zelmanowitz [20]) is a c.p. module by [20, Theorem 2.2].*

Lemma 2.2 *Let R be a ring and M a left R-module. Then $Ra \bigcap l_R(X) = l_R(aX)a$ for any $a \in R$ and any subset X of M.*

Proof Suppose $a \in R$ and X is a subset of M. For any $x \in X$, we have $l_R(aX)ax = 0$, so $l_R(aX)a \subseteq l_R(X)$. Clearly, $l_R(aX)a \subseteq Ra$. Thus $l_R(aX)a \subseteq Ra \bigcap l_R(X)$. Conversely, if $t = ra \in l_R(X)$, then $raX = 0$, i.e., $r \in l_R(aX)$, so $t = ra \in l_R(aX)a$. Thus $Ra \bigcap l_R(X) = l_R(aX)a$. □

Lemma 2.3 *Let $_RM$ be a c.p. module. Then $Re \bigcap l_R(x)$ is a direct summand of $_RR$ for any $e^2 = e \in R$ and $x \in M$.*

Proof Set $e^2 = e \in R$ and $x \in M$. By Lemma 2.2, $Re \bigcap l_R(x) = l_R(ex)e$. Since $_RM$ is a c.p. module, $l_R(ex) = Rf$ for some idempotent $f \in R$. It is easy to prove that $l_R(ex)e \subseteq l_R(ex)$, so $Rfe =$

$l_R(ex)e \subseteq l_R(ex) = Rf$. Thus $fe = fef$ and $(fe)^2 = fefe = fee = fe$ is an idempotent of R. Hence $Re \bigcap l_R(x) = l_R(ex)e = Rfe$ is a direct summand of ${}_RR$. $\square$

Theorem 2.4 *Let R be a ring and M a left R-module. Then the following are equivalent:*

(1) *M is a c.p. module.*

(2) *$l_R(K)$ is a direct summand of ${}_RR$ for each finite subset K of M.*

Proof (1) $\Rightarrow$ (2). Suppose $K = \{x_1, x_2, \cdots, x_n\} \subseteq M$. Denote $T = \{x_1, x_2, \cdots, x_{n-1}\}$. We now proceed by induction on n. If $n = 1$, then it is clear that $l_R(K)$ is a direct summand of ${}_RR$. Let $n > 1$ and assume that the result is true for $n-1$, then $l_R(T) = Re$ for some idempotent $e \in R$. By Lemma 2.3, $l_R(K) = l_R(T) \bigcap l_R(x_n) = Re \bigcap l_R(x_n)$ is a direct summand of ${}_RR$.

(2) $\Rightarrow$ (1) is trivial. $\square$

Corollary 2.5 *A ring R is a left p.p. ring if and only if the left annihilator of each finitely generated right ideal is a direct summand of ${}_RR$.*

Corollary 2.6 *If R is a left p.p. ring, then the intersection of two direct summands is again a direct summand of ${}_RR$.*

Proof Suppose e, f are two idempotents in R, then $Re \bigcap Rf = l(1-e) \bigcap l(1-f) = l((1-e)R + (1-f)R)$ is a direct summand of ${}_RR$ by Corollary 2.5. $\square$

Theorem 2.7 *Let $(M_\alpha)_{\alpha \in A}$ be an indexed set of left R-modules. Then $\bigoplus_A M_\alpha$ is a c.p. module if and only if each M_α is a c.p. module.*

Proof " $\Rightarrow$ ". It is clear since every submodule of a c.p. module is again a c.p. module.

" $\Leftarrow$ ". Suppose that M_α is c.p. for any $\alpha \in A$ and $\xi = (x_\alpha) \in \bigoplus_A M_\alpha$. Set $S = \{x_\alpha : x_\alpha \neq 0\}$, then S is a finite set and $l_R(\xi) = l_R(S)$. Next we show that $l_R(S)$ is a direct summand of ${}_RR$. It suffices to prove the conclusion in the case $S = \{x_\alpha, x_\beta\}$. Since M_α is a c.p. module, $l_R(x_\alpha) = Re$ for some idempotent $e \in R$. Now since M_β is a c.p. module, $l_R(S) = l_R(x_\alpha) \bigcap l_R(x_\beta) = Re \bigcap l_R(x_\beta)$ is a direct summand of ${}_RR$ by Lemma 2.3. It follows that $l_R(\xi)$ is a direct summand of ${}_RR$ and $\bigoplus_A M_\alpha$ is a c.p. left R-module. $\square$

Applying Theorem 2.7 to rings, the following characterizations of left p.p. rings hold.

Corollary 2.8 *[8, Theorem 3.2] For a ring R the following are equivalent:*

(1) *R is a left p.p. ring.*

(2) *Every free left R-module is a c.p. module.*

(3) *Every projective left R-module is a c.p. module.*

As is well known, every Baer ring is a p.p. ring, but the converse is not true. The following result shows that if the direct products of c.p. left R-modules are again c.p. left R-modules, then R being p.p. implies that R is Baer.

Theorem 2.9 *For a ring R the following are equivalent:*

(1) *R is a left p.p. ring and every direct product of c.p. left R-modules is a c.p. left R-module.*

(2) *${}_RR^A$ is a c.p. module for every set A.*

(3) *R is a Baer ring.*

Proof (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3). For any subset S of R, we consider the element $\gamma = (s)_{s\in S} \in R^S$. It is easy to see that $l_R(S) = l_R(\gamma)$. Since ${}_RR^S$ is a c.p. module, $l_R(\gamma) \leq^{\oplus} R$. So $l_R(S) \leq^{\oplus} R$ and R is Baer.

(3) $\Rightarrow$ (1). Clearly, R is left p.p. by hypothesis. Suppose that $(M_\alpha)_{\alpha\in A}$ is an indexed set of c.p. left R-modules and $\xi = (x_\alpha) \in \prod_A M_\alpha$. Since M_α is c.p., $l_R(x_\alpha) = R(1-e_\alpha) = l_R(e_\alpha)$ for some idempotent $e_\alpha \in R$. Set $S = \{e_\alpha : \alpha \in A\}$. We have $l_R(\xi) = \bigcap_A l_R(x_\alpha) = \bigcap_A l_R(e_\alpha) = l_R(S) \leq^{\oplus} R$ because R is Baer. Hence $\prod_A M_\alpha$ is a c.p. left R-module. □

The next proposition characterizes regular rings and semisimple rings in terms of c.p. modules.

Proposition 2.10 *Let R be a ring.*

(1) *The following are equivalent:*

  (*i*) *R is regular.*

  (*ii*) *Every finitely presented left R-module is a c.p. module.*

(2) *The following are equivalent:*

  (*i*) *R is semisimple.*

  (*ii*) *Every left R-module is a c.p. module.*

  (*iii*) *Every simple left R-module is a c.p. module.*

Proof (1). (*i*) $\Rightarrow$ (*ii*). If R is regular, then every left R-module is flat. So each finitely presented left R-module is projective, and thus c.p. by Corollary 2.8.

(*ii*) $\Rightarrow$ (*i*). Assume that I is a finitely generated left ideal of R. Then R/I is a finitely presented left R-module, and so it is a c.p. module. Thus $I = l_R(1_R + I)$ is a direct summand of R. Hence R is regular.

(2). (*i*) $\Rightarrow$ (*ii*). Since every left ideal of a semisimple ring is generated by an idempotent, the result follows.

(*ii*) $\Rightarrow$ (*iii*) is trivial.

(*iii*) $\Rightarrow$ (*i*). Let I be a maximal left ideal of R. Then R/I is a simple left R-module, and thus it is c.p. by hypothesis. So $I = l_R(1_R + I)$ is a direct summand of R. Therefore R is semisimple. □

Recall that a left R-module M is said to be a Rickart module [16] if the left annihilator in M of any single element of S is generated by an idempotent of S. Equivalently, $\forall \varphi \in S$, $l_M(\varphi) = \mathrm{Ker}\varphi = Me$ for some $e^2 = e \in S$. This is another generalization of left p.p. rings. The following example is provided to show that the notion of c.p. modules is distinct from that of Rickart modules.

Example 2.11 *The $\mathbb{Z}$-module $\mathbb{Q} \oplus \mathbb{Z}_2$ is a Rickart module by [16, Example 2.5]. However, it is not a c.p. $\mathbb{Z}$-module: $l_{\mathbb{Z}}(x) = 2\mathbb{Z}$ is not a direct summand of ${}_{\mathbb{Z}}\mathbb{Z}$ where $x = (0,1) \in \mathbb{Q} \oplus \mathbb{Z}_2$.*

Proposition 2.12 *Every cyclic c.p. left R-module is a Rickart module and the ring of its endomorphisms is a left p.p. ring.*

Proof Suppose ${}_RM$ is a cyclic c.p. module. For each $f \in \mathrm{End}_R(M)$, Imf is a cyclic submodule of M, and thus Imf is projective. So $\mathrm{Ker}f$ is a direct summand of M. It follows that M is a Rickart module, and then $\mathrm{End}_R(M)$ is a left p.p. ring by [16, Proposition 3.2]. □

We don't know whether every c.p. left R-module is a Rickart module.

3. C.P. property of ${}_{T_n(R)}T_n(M)$

The triangular matrix ring $T_n(R)$ $(n \geq 1)$ is an important extension of R. A number of interesting papers have been published on the extension (see [6], [10] et al.). In this section, we extend the extension to the general module theoretic setting, i.e., for a left R-module M, let $T_n(M)$ be the set of all $n \times n$ formal upper triangular matrices over M. It is easy to see that $T_n(M)$ is a left $T_n(R)$-module relative to the addition and scalar multiplication defined via

$$(x_{ij}) + (y_{ij}) = (x_{ij} + y_{ij}),\ (r_{ij})(x_{ij}) = (\sum_{k=1}^{n} r_{ik}x_{kj}).$$

We next explore when the left $T_n(R)$-module $T_n(M)$ is a c.p. module, and then apply the results to investigate the p.p. property of $T_n(R)$. Firstly the following lemma is given.

Lemma 3.1 *Let M be a left R-module, $n \geq 2$, and $X = \begin{pmatrix} x & \xi \\ 0 & X_1 \end{pmatrix} \in T_n(M)$ with $x \in M, \xi \in M^{n-1}, X_1 \in T_{n-1}(M)$. Then $l_{T_n(R)}(X)$ is a direct summand of $T_n(R)$ if and only if $l_{T_{n-1}(R)}(X_1)$ is a direct summand of $T_{n-1}(R)$ and $l_R(x)\bigcap(R^{n-1}X_1 : \xi)$ is a direct summand of R.*

Proof Denote $S_n = T_n(R)$ for each $n \geq 1$.

" $\Rightarrow$ ". Suppose $l_{T_n(R)}(X)$ is a direct summand of $T_n(R)$, i.e., $l_{S_n}(X) = S_nE$ for some idempotent $E = \begin{pmatrix} e & \alpha \\ 0 & E_1 \end{pmatrix} \in S_n$, where $e \in R,\ \alpha \in R^{n-1},\ E_1 \in S_{n-1}$. Then $e^2 = e,\ E_1^2 = E_1$ and

$$\begin{pmatrix} ex & e\xi+\alpha X_1 \\ 0 & E_1X_1 \end{pmatrix} = \begin{pmatrix} e & \alpha \\ 0 & E_1 \end{pmatrix}\begin{pmatrix} x & \xi \\ 0 & X_1 \end{pmatrix} = EX = 0.$$

It follows that $ex = 0,\ e\xi + \alpha X_1 = 0,\ E_1X_1 = 0$. Hence $Re \subseteq l_R(x)\bigcap(R^{n-1}X_1 : \xi)$ and $S_{n-1}E_1 \subseteq l_{S_{n-1}}(X_1)$.

Next we show that $l_R(x)\bigcap(R^{n-1}X_1 : \xi) \subseteq Re$ and $l_{S_{n-1}}(X_1) \subseteq S_{n-1}E_1$.

For any $r \in l_R(x)\bigcap(R^{n-1}X_1 : \xi)$, we have $rx = 0$ and $r\xi + \delta X_1 = 0$ for some $\delta \in R^{n-1}$. Then

$$\begin{pmatrix} r & \delta \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x & \xi \\ 0 & X_1 \end{pmatrix} = \begin{pmatrix} rx & r\xi+\delta X_1 \\ 0 & 0 \end{pmatrix} = 0.$$

So $\begin{pmatrix} r & \delta \\ 0 & 0 \end{pmatrix} \in l_{S_n}(X) = S_nE$ and $\begin{pmatrix} r & \delta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} r & \delta \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e & \alpha \\ 0 & E_1 \end{pmatrix}$. Thus $r = re \in Re$, and so

$$l_R(x)\bigcap(R^{n-1}X_1 : \xi) \subseteq Re.$$

We have proved that $l_R(x)\bigcap(R^{n-1}X_1 : \xi) = Re$ is a direct summand of R.

Assume $L_1 \in l_{S_{n-1}}(X_1)$, then $L_1X_1 = 0$. We get that

$$\begin{pmatrix} 0 & 0 \\ 0 & L_1 \end{pmatrix}\begin{pmatrix} x & \xi \\ 0 & X_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & L_1X_1 \end{pmatrix} = 0.$$

Then $\left(\begin{smallmatrix} 0 & 0 \\ 0 & L_1 \end{smallmatrix}\right) \in l_{S_n}(X) = S_nE$. So $\left(\begin{smallmatrix} 0 & 0 \\ 0 & L_1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & L_1 \end{smallmatrix}\right)\left(\begin{smallmatrix} e & \alpha \\ 0 & E_1 \end{smallmatrix}\right)$, and $L_1 = L_1E_1 \in S_{n-1}E_1$. Thus $l_{S_{n-1}}(X_1) \subseteq S_{n-1}E_1$. Hence $l_{S_{n-1}}(X_1) = S_{n-1}E_1$ is a direct summand of S_{n-1}.

" $\Leftarrow$ ". Let $l_{S_{n-1}}(X_1) = S_{n-1}E_1$ and $l_R(x)\bigcap(R^{n-1}X_1 : \xi) = Re$ for some $E_1^2 = E_1 \in S_{n-1}$, $e^2 = e \in R$. We have that $ex = 0$, $E_1X_1 = 0$ and $e\xi + \alpha X_1 = 0$ for some $\alpha \in R^{n-1}$. Set $\beta = e\alpha(I - E_1)$, where I is the identity of S_{n-1}. Then

$$e\xi + \beta X_1 = e\xi + e\alpha(I - E_1)X_1 = e\xi + e\alpha X_1 = e(e\xi + \alpha X_1) = 0$$

and

$$e\beta + \beta E_1 = e^2\alpha(I - E_1) + e\alpha(I - E_1)E_1 = e\alpha(I - E_1) = \beta.$$

Let $E = \begin{pmatrix} e & \beta \\ 0 & E_1 \end{pmatrix}$, then $E^2 = \begin{pmatrix} e & \beta \\ 0 & E_1 \end{pmatrix}\begin{pmatrix} e & \beta \\ 0 & E_1 \end{pmatrix} = \begin{pmatrix} e^2 & e\beta+\beta E_1 \\ 0 & E_1^2 \end{pmatrix} = \begin{pmatrix} e & \beta \\ 0 & E_1 \end{pmatrix} = E$.

Next we show that $l_{S_n}(X) = S_nE$. Since $EX = \begin{pmatrix} e & \beta \\ 0 & E_1 \end{pmatrix}\begin{pmatrix} x & \xi \\ 0 & X_1 \end{pmatrix} = \begin{pmatrix} ex & e\xi+\beta X_1 \\ 0 & E_1X_1 \end{pmatrix} = 0$, $S_nE \subseteq l_{S_n}(X)$.

For any $T = \left(\begin{smallmatrix} t & \gamma \\ 0 & T_1 \end{smallmatrix}\right) \in l_{S_n}(X)$,

$$\begin{pmatrix} tx & t\xi+\gamma X_1 \\ 0 & T_1X_1 \end{pmatrix} = \left(\begin{smallmatrix} t & \gamma \\ 0 & T_1 \end{smallmatrix}\right)\begin{pmatrix} x & \xi \\ 0 & X_1 \end{pmatrix} = TX = 0.$$

It follows that $tx = 0$, $t\xi + \gamma X_1 = 0$, $T_1X_1 = 0$. So $t \in l_R(x)\bigcap(R^{n-1}X_1 : \xi) = Re$ and $T_1 \in l_{S_{n-1}}(X_1) = S_{n-1}E_1$. Then $t = te$, $T_1 = T_1E_1$ and

$$(\gamma - t\beta)X_1 = \gamma X_1 - t\beta X_1 = \gamma X_1 + te\xi = \gamma X_1 + t\xi = 0.$$

Thus

$$\left(\begin{smallmatrix} \gamma-t\beta \\ 0 \end{smallmatrix}\right) X_1 = 0,$$

where the first $0 \in R^{(n-2)\times(n-1)}$. So $\left(\begin{smallmatrix} \gamma-t\beta \\ 0 \end{smallmatrix}\right) \in l_{S_{n-1}}(X_1) = S_{n-1}E_1$, and then

$$\left(\begin{smallmatrix} \gamma-t\beta \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \gamma-t\beta \\ 0 \end{smallmatrix}\right) E_1.$$

Thus $\gamma - t\beta = (\gamma - t\beta)E_1$ and $\gamma = t\beta + (\gamma - t\beta)E_1$. Since $e\beta + \beta E_1 = \beta$,

$$t\beta + t\beta E_1 = t(e\beta + \beta E_1) = t\beta.$$

So $t\beta E_1 = 0$ and $\gamma = t\beta + \gamma E_1$. Hence we get that

$$T = \left(\begin{smallmatrix} t & \gamma \\ 0 & T_1 \end{smallmatrix}\right) = \begin{pmatrix} te & t\beta+\gamma E_1 \\ 0 & T_1E_1 \end{pmatrix} = \left(\begin{smallmatrix} t & \gamma \\ 0 & T_1 \end{smallmatrix}\right)\begin{pmatrix} e & \beta \\ 0 & E_1 \end{pmatrix} \in S_nE.$$

Thus $l_{S_n}(X) = S_nE$ is a direct summand of S_n and the result follows. $\square$

Corollary 3.2 *Let M be a left R-module and $X = \left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right) \in T_2(M)$. Then $l_{T_2(R)}(X)$ is a direct summand of $T_2(R)$ if and only if $l_R(z)$ and $l_R(x)\bigcap(Rz : y)$ are two direct summands of R.*

Lemma 3.3 *$T_2(M)$ is a c.p. left $T_2(R)$-module if and only if $(Rx : y)$ is a direct summand of R for any $x, y \in {}_RM$.*

Proof " $\Rightarrow$ ". Suppose $x, y \in M$. Set $X = \left(\begin{smallmatrix} 0 & y \\ 0 & x \end{smallmatrix}\right)$. Then, by hypothesis, $l_{T_2(R)}(X)$ is a direct summand of $T_2(R)$. So $(Rx : y) = l_R(0) \bigcap (Rx : y)$ is a direct summand of R by Corollary 3.2.

" $\Leftarrow$ ". Since $(Rx : y)$ is a direct summand of R for any $x, y \in M$, $l_R(y) = (R0 : y)$ is a direct summand for any $y \in M$. Thus M is a c.p. module. Set $X = \left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right) \in T_2(M)$, then $l_R(z)$ is a direct summand of R. Write $(Rz : y) = Re$ with $e^2 = e \in R$. Since M is c.p., $l_R(x) \bigcap (Rz : y) = l_R(x) \bigcap Re$ is a direct summand of R by Lemma 2.3. By Corollary 3.2, $l_{T_2(R)}(X)$ is a direct summand of $T_2(R)$. Therefore $T_2(M)$ is a c.p. left $T_2(R)$-module. $\square$

Theorem 3.4 *Let n be a positive integer and $n \geq 2$. The following are equivalent for a left R-module M:*

(1) *$T_n(M)$ is a c.p. left $T_n(R)$-module.*

(2) *$(R^{n-1}X : \xi)$ is a direct summand of R for any $X \in T_{n-1}(M)$, $\xi \in M^{n-1}$.*

Proof (1) $\Rightarrow$ (2). Suppose $X \in T_{n-1}(M)$, $\xi \in M^{n-1}$. Set $\bar{X} = \left(\begin{smallmatrix} 0 & \xi \\ 0 & X \end{smallmatrix}\right)$, then $\bar{X} \in T_n(M)$. Since $T_n(M)$ is a c.p. left $T_n(R)$-module, $l_{T_n(R)}(\bar{X})$ is a direct summand of $T_n(R)$. So $(R^{n-1}X : \xi) = l_R(0) \bigcap (R^{n-1}X : \xi)$ is a direct summand of R by Lemma 3.1.

(2) $\Rightarrow$ (1). First we show that, for each $1 \leq m < n-1$ and any $X \in T_m(M)$, $\xi \in M^m$, $(R^m X : \xi)$ is a direct summand of R.

Suppose $X \in T_m(M)$, $\xi \in M^m$. Set $\bar{X} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & X \end{smallmatrix}\right)$ and $\bar{\xi} = (0, \xi)$ such that $\bar{X} \in T_{n-1}(M)$, $\bar{\xi} \in M^{n-1}$. By hypothesis, $(R^{n-1}\bar{X} : \bar{\xi})$ is a direct summand of R. We shall show that $(R^m X : \xi) = (R^{n-1}\bar{X} : \bar{\xi})$. Assume that $r \in (R^m X : \xi)$, then $r\xi = \alpha X$ for some $\alpha \in R^m$. So

$$r\bar{\xi} = r(0, \xi) = (0, r\xi) = (0, \alpha X) = (0, \alpha) \left(\begin{smallmatrix} 0 & 0 \\ 0 & X \end{smallmatrix}\right) = (0, \alpha)\bar{X},$$

where $(0, \alpha) \in R^{n-1}$. It follows that $r \in (R^{n-1}\bar{X} : \bar{\xi})$. Conversely, if $r \in (R^{n-1}\bar{X} : \bar{\xi})$, then $r\bar{\xi} = \bar{\alpha}\bar{X}$ for some $\bar{\alpha} = (\alpha_1, \alpha) \in R^{n-1}$, where $\alpha_1 \in R^{n-m-1}$, $\alpha \in R^m$, i.e.,

$$(0, r\xi) = r(0, \xi) = r\bar{\xi} = \bar{\alpha}\bar{X} = (\alpha_1, \alpha) \left(\begin{smallmatrix} 0 & 0 \\ 0 & X \end{smallmatrix}\right) = (0, \alpha X).$$

So $r\xi = \alpha X$, and $r \in (R^m X : \xi)$. This shows that $(R^m X : \xi) = (R^{n-1}\bar{X} : \bar{\xi})$ is a direct summand of R for each $1 \leq m < n-1$ and any $X \in T_m(M)$, $\xi \in M^m$. In particular, $(Rx : y)$ is a direct summand for any $x, y \in M$. So $T_2(M)$ is a c.p. left $T_2(R)$-module by Lemma 3.3.

Since $T_2(M)$ is a c.p. left $T_2(R)$-module and $(R^2 X : \xi)$ is a direct summand of R for any $X \in T_2(M)$, $\xi \in M^2$, using a similar argument as the proof of Lemma 3.3, we have that $T_3(M)$ is a c.p. left $T_3(R)$-module by Lemma 3.1. Proceeding in this way we get that $T_m(M)$ is a c.p. left $T_m(R)$-module for all $1 \leq m \leq n$. In particular, $T_n(M)$ is a c.p. left $T_n(R)$-module. $\square$

Corollary 3.5 *If $T_n(M)$ is a c.p. left $T_n(R)$-module, then $T_m(M)$ is a c.p. left $T_m(R)$-module for all $1 \leq m \leq n$.*

Proof By Theorem 3.4, for any $X \in T_{n-1}(M)$, $\xi \in M^{n-1}$, $(R^{n-1}X : \xi)$ is a direct summand of R. Then we know that, for each $1 \leq m \leq n-1$ and any $X \in T_m(M)$, $\xi \in M^m$, $(R^m X : \xi)$ is a direct summand of R by the proof of Theorem 3.4. So, again by Theorem 3.4, $T_m(M)$ is a c.p. left $T_m(R)$-module for each $2 \leq m \leq n$.

Clearly, $T_2(M)$ being a c.p. left $T_2(R)$-module implies that M is a c.p. left R-module by Lemma 3.3. Hence $T_m(M)$ is a c.p. left $T_m(R)$-module for all $1 \le m \le n$. □

Corollary 3.6 *If R is semisimple, then ${}_{T_n(R)}T_n(M)$ is c.p. for any left R-module M and each positive integer n.*

Corollary 3.7 *If R is semisimple, then $T_n(R)$ is Baer for any $n \ge 1$.*

Proof By Corollary 3.6, ${}_{T_n(R)}T_n(R)^I \cong {}_{T_n(R)}T_n(R^I)$ is a c.p. module for every set I. The result follows by Theorem 2.9. □

The converse of Corollary 3.7 is not true: The ring $R = \mathbb{Z}_2^I$ is not semisimple when I is an infinite set. But $T_n(\mathbb{Z}_2)$ is Baer for all $n \ge 1$ by Corollary 3.7. Therefore $T_n(R) = T_n(\mathbb{Z}_2^I) \cong T_n(\mathbb{Z}_2)^I$ is also Baer for any $n \ge 1$.

Corollary 3.8 *Let $n \ge 2$. The following are equivalent for a ring R:*

(1) *$T_n(R)$ is a left p.p. ring.*

(2) *$(R^{n-1}A : \alpha)$ is a direct summand of R for any $A \in T_{n-1}(R)$, $\alpha \in R^{n-1}$.*

4. Applications

In [21], Zhang and Chen have shown that R is left semihereditary if and only if $M_n(R)$ is left p.p. for all positive integers n. In this section, we consider to characterize regular rings in terms of p.p. property of $T_n(R)$ as an application of Theorem 3.4.

A left R-module M is called feebly Baer [17] if, whenever $ax = 0$ with $a \in R$ and $x \in M$, there exists $e^2 = e \in R$ such that $ae = a$ and $ex = 0$. A ring R is called feebly Baer if ${}_RR$ is a feebly Baer module. From the definition, we can see that the notion of feebly Baer rings is left-right symmetric. And it is straightforward to check that a left (or right) p.p. ring is feebly Baer. But the converse is not true by [14, Example 5.8]. In [17], Lee and Zhou have proved the following result:

Theorem 4.1 *[17, Theorem 5] Let $n \ge 2$. The following are equivalent for a ring R:*

(1) *R is a regular ring.*

(2) *$T_n(R)$ is a feebly Baer ring.*

(3) *$M_n(R)$ is a feebly Baer right module over $T_n(R)$.*

(4) *$M_{m\times n}(R)$ is a feebly Baer right module over $T_n(R)$ for all $m \ge 1$.*

(5) *$M_n(R)$ is a feebly Baer left module over $T_n(R)$.*

(6) *$M_{n\times m}(R)$ is a feebly Baer left module over $T_n(R)$ for all $m \ge 1$.*

In the above result, "feebly Baer ring" and "feebly Baer module" can be changed into "p.p. ring" and "c.p. module", respectively.

Theorem 4.2 *Let $n \ge 2$. The following are equivalent for a ring R:*

(1) *R is a regular ring.*

(2) *$T_n(R)$ is a left p.p. ring.*

(3) $M_n(R)$ *is a c.p. left module over* $T_n(R)$.

(4) $M_{n\times m}(R)$ *is a c.p. left module over* $T_n(R)$ *for all* $m \geq 1$.

Proof (1) $\Rightarrow$ (2). Since R is regular, R is left coherent. So, by [18, Theorem 2.4], $(R^{n-1}A : \alpha)$ is a finitely generated left ideal of R for each $n \geq 2$ and any $A \in T_{n-1}(R)$, $\alpha \in R^{n-1}$. Again since R is regular, $(R^{n-1}A : \alpha)$ is a direct summand of R. Hence $T_n(R)$ is a left p.p. ring by Corollary 3.8.

(2) $\Rightarrow$ (1). Assume (2), $T_2(R)$ is left p.p. by Corollary 3.5. So, by Lemma 3.3, $(Ra : b)$ is a direct summand of R for any $a, b \in R$. Set $b = 1$, then $Ra = (Ra : 1)$ is a direct summand of R for any $a \in R$. Therefore R is a regular ring.

(1) $\Rightarrow$ (4). From (1) $\Leftrightarrow$ (2), we have $T_k(R)$ is a left p.p. ring for each $k \geq 2$. Suppose $X \in M_{n\times m}(R)$, set $\bar{X} = \left(\begin{smallmatrix} 0 & X \\ 0 & 0 \end{smallmatrix}\right) \in T_{n+m}(R)$. Then there exists an idempotent $\bar{E} = \left(\begin{smallmatrix} E & E_1 \\ 0 & E_2 \end{smallmatrix}\right) \in T_{n+m}(R)$ with $E \in T_n(R)$ such that $l_{T_{n+m}(R)}(\bar{X}) = T_{n+m}(R)\bar{E}$. It follows that E is an idempotent of $T_n(R)$ and $EX = 0$. For any $A \in l_{T_n(R)}(X)$,

$$\left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & X \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & AX \\ 0 & 0 \end{smallmatrix}\right) = 0.$$

Hence $\left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right) \in l_{T_{n+m}(R)}(\bar{X}) = T_{n+m}(R)\bar{E}$ and

$$\left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} E & E_1 \\ 0 & E_2 \end{smallmatrix}\right).$$

So $A = AE \in T_n(R)E$. Therefore $l_{T_n(R)}(X) = T_n(R)E$ is a direct summand of $T_n(R)$ and $M_{n\times m}(R)$ is a c.p. left $T_n(R)$-module.

(4) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (2). Since each submodule of a c.p. module is also c.p. and $T_n(R) \leq M_n(R)$ as left $T_n(R)$-modules, the result follows. □

Recall that a ring R is said to be right P-*injective* if, for each $a \in R$, every right R-homomorphism from aR to R_R can extend to one from R_R to R_R; equivalently [19, Lemma 1.1] if $l_R r_R(a) = Ra$ for each $a \in R$. It is clear that a regular ring is right P-injective.

Proposition 4.3 *For any ring* R *and each integer* $n \geq 2$, $T_n(R)$ *is not regular.*

Proof Denote $S = T_n(R)$ and E_{ij} $(1 \leq i, j \leq n)$ are the matrix units. Set $A = E_{1n} \in S$. It is easy to prove that $SA \subsetneqq l_S r_S(A)$. So S is not right P-injective, and thus S is not regular. □

Remark 4.4 *For a regular ring* R, $T_n(R)$ $(n \geq 2)$ *is a p.p. ring but not regular by Theorem 4.2 and Proposition 4.3. This is an example of the p.p. ring that is not regular.*

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