Integral polytopes and polynomial factorization

Fatih KOYUNCU[*]

Department of Computer Engineering, Faculty of Engineering and Natural Sciences,
Yıldırım Beyazıt University, 06050, Ankara, Turkey

Abstract: For any field F, there is a relation between the factorization of a polynomial $f \in F[x_1, ..., x_n]$ and the integral decomposition of the Newton polytope of f. We extended this result to polynomial rings $R[x_1, ..., x_n]$ where R is any ring containing some elements which are not zero-divisors. Moreover, we have constructed some new families of integrally indecomposable polytopes in $\mathbb{R}^n$ giving infinite families of absolutely irreducible multivariate polynomials over arbitrary fields.

Key words: Integral polytopes, integral indecomposability, multivariate polynomials, absolute irreducibility

1. Introduction

Throughout this study, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. Let S be a subset of $\mathbb{R}^n$. The smallest convex set containing S, denoted by $\mathrm{conv}(S)$, is called the *convex hull* of S. If $S = \{a_1, a_2, ..., a_n\}$ is a finite set, then we denote $\mathrm{conv}(S)$ by $\mathrm{conv}(a_1, ..., a_n)$.

Definition 1.1 *For any two sets A and B in $\mathbb{R}^n$, the sum*

$$A + B = \{a + b : a \in A, b \in B\}$$

is called the Minkowski sum or, shortly, the sum of A and B.

The convex hull of finitely many points in $\mathbb{R}^n$ is called a *polytope*. A point in $\mathbb{R}^n$ is called *integral* if its coordinates are integers. A polytope in $\mathbb{R}^n$ is called *integral* if all of its vertices are integral. An integral polytope C is called *integrally decomposable* if there exist integral polytopes A and B such that $C = A + B$, where both A and B have at least two points. Otherwise, C is called *integrally indecomposable*.

Definition 1.2 *Let F be any field and*

$$f(x_1, x_2, ..., x_n) = \sum c_{e_1 e_2 ... e_n} x_1^{e_1} x_2^{e_2} ... x_n^{e_n} \in F[x_1, ..., x_n].$$

The Newton polytope of f, which is denoted by P_f, is defined as the convex hull of the set $S = \{(e_1, ..., e_n) : c_{e_1 e_2 ... e_n} \neq 0\}$ in $\mathbb{R}^n$.

[*]Correspondence: fatih@ybu.edu.tr

A polynomial over a field F is called *absolutely irreducible* if it is irreducible over every algebraic extension of F.

Lemma 1.3 *[7]. Let $f, g, h \in F[x_1, ..., x_n]$ with $f \neq 0$ and $f = gh$. Then $P_f = P_g + P_h$.*

Proof See e.g. [3, Lemma 2.1]. □

Corollary 1.4 *[3, page 507]. Let F be any field and f a nonzero polynomial in $F[x_1, ..., x_n]$ not divisible by any x_i. If the Newton polytope of f is integrally indecomposable, then f is absolutely irreducible over F.*

Infinitely many integrally indecomposable polytopes in $\mathbb{R}^n$ and infinite families of absolutely irreducible polynomials which are associated to these polytopes are presented in [3], [4] and [6] over any field F.

We recall some terminologies. For details, see [2].

Definition 1.5 *For $\alpha \in \mathbb{R}, \beta \in \mathbb{R}^n$ the set*

$$H = \{x \in \mathbb{R}^n : \beta \cdot x = \alpha\}$$

is called a hyperplane, where

$$\beta \cdot x = \beta_1 x_1 + ... + \beta_n x_n$$

is the dot product of the vectors $\beta = (\beta_1, ..., \beta_n)$, $x = (x_1, ..., x_n)$. The closed halfspaces formed by H are defined as

$$H^- = \{x \in \mathbb{R}^n : \beta \cdot x \leq \alpha\}, \quad H^+ = \{x \in \mathbb{R}^n : \beta \cdot x \geq \alpha\}.$$

A hyperplane H_K is called a supporting hyperplane of a closed convex set $K \subset \mathbb{R}^n$ if $K \subset H_K^+$ or $K \subset H_K^-$ and $K \cap H_K \neq \emptyset$, i.e. H_K contains a boundary point of K. A supporting hyperplane H_K of K is called nontrivial if K is not contained in H_K. The halfspace that contains K is called a supporting halfspace of K.

Let $C \subset \mathbb{R}^n$ be a compact convex set. Then for any nonzero vector $v \in \mathbb{R}^n$, the real number $s = \sup_{x \in C}(x \cdot v)$ is defined as the maximum value of the set $S = \{x \cdot v : x \in C\}$.

Let $K \subset \mathbb{R}^n$ be a nonempty convex compact set. The map

$$h_K : \mathbb{R}^n \to \mathbb{R}, \qquad u \to \sup_{x \in K}(x \cdot u)$$

is called the support function of K.

Let $K \subset \mathbb{R}^n$ be a nonempty convex compact set. For every fixed nonzero vector $u \in \mathbb{R}^n$, the hyperplane having outer normal vector u defined as

$$H_K(u) = \{x \in \mathbb{R}^n : x \cdot u = h_K(u)\}$$

is a supporting hyperplane of K. We know that every supporting hyperplane of K has a representation of this form, see [2, Page 19].

Let P be a polytope. The intersection of P with a supporting hyperplane H_P is called a face of P. A vertex of P is a face of dimension zero. An edge of P is a face of dimension 1, which is a line segment. A face F of P is called a facet if dim $(F)=$ dim $(P) -1$. If u is any nonzero vector in $\mathbb{R}^n$, $F_P(u) = H_P(u) \cap P$ shows the face of P in the direction of u, that is the intersection of P with its supporting hyperplane $H_P(u)$ having outer normal vector u.

Theorem 1.6 *(i) Let K and L be polytopes in $\mathbb{R}^n$ such that $M = K + L$.*

(1) M is also a polytope in $\mathbb{R}^n$.

(2) If u is any nonzero vector in $\mathbb{R}^n$, then

$$F_M(u) = F_K(u) + F_L(u) \quad and \quad H_M(u) = H_K(u) + H_L(u).$$

(3) If h_K and h_L are the support functions of K and L, respectively, then $h_K + h_L$ is the support function of $K + L$ and $h_M = h_K + h_L$.

(4) If F_M is a face of M, then there exist unique faces F_K and F_L of K and L, respectively, such that

$$F_M = F_K + F_L.$$

In particular, each vertex of M is the sum of unique vertices of K and L, respectively.

(ii) If P is a polytope in $\mathbb{R}^n$ with $P = Q + R$, then so are Q and R, which are called summands of P.

Proof See, e.g., the proof of [2, Chapter IV-Theorem 1.5]. □

2. Polytope method over rings containing elements which are not zero-divisors

We have observed that Lemma 1.3 works also for rings without zero-divisors, especially for integral domains, instead of fields.

Theorem 2.1 *Let R be a ring without zero-divisors and $f, g, h \in R[x_1, x_2, ..., x_n]$ with $f \neq 0$ and $f = gh$. Then $P_f = P_g + P_h$.*

Proof If R is an integral domain, the result follows from Lemma 1.3 since any integral domain R is contained in a field F of quotients of R and $f, g, h \in F[x_1, x_2, ..., x_n]$.

Now, suppose that R is a ring without zero-divisors and let

$$f(x_1, x_2, ..., x_n) = \sum c_{e_1 e_2 ... e_n} x_1^{e_1} x_2^{e_2} ... x_n^{e_n},$$

$$g(x_1, x_2, ..., x_n) = \sum c'_{e'_1 e'_2 ... e'_n} x_1^{e'_1} x_2^{e'_2} ... x_n^{e'_n},$$

$$h(x_1, x_2, ..., x_n) = \sum c''_{e''_1 e''_2 ... e''_n} x_1^{e''_1} x_2^{e''_2} ... x_n^{e''_n}.$$

Then we have

$$f = \sum_{(e'_1, e'_2, ..., e'_n)} \sum_{(e''_1, e''_2, ..., e''_n)} c'_{e'_1 e'_2 ... e'_n} c''_{e''_1 e''_2 ... e''_n} x_1^{e'_1 + e''_1} x_2^{e'_2 + e''_2} ... x_n^{e'_n + e''_n}. \tag{1}$$

In this expanded product, let us assume that there are r terms containing $x_1^{e_1} x_2^{e_2} ... x_n^{e_n}$, and write

$$S = (d_1 + ... + d_r) x_1^{e_1} x_2^{e_2} ... x_n^{e_n} = c_{e_1 e_2 ... e_n} x_1^{e_1} x_2^{e_2} ... x_n^{e_n}. \tag{2}$$

Note that we may have $S = 0$ if $r \geq 2$, but not if $r = 1$.

We will prove that the set $\{(e_1, e_2, ..., e_n)\}$ of exponents of the polynomial f and the set $\{(e'_1 + e''_1, e'_2 + e''_2, ..., e'_n + e''_n)\}$ of exponents of gh determine the same polytope. From relations (1) and (2), we see that every

element of the set $\{(e_1, e_2, ..., e_n)\}$ is contained in the set $\{(e'_1 + e''_1, e'_2 + e''_2, ..., e'_n + e''_n)\}$. Therefore, we have $P_f \subseteq P_g + P_h$.

To prove the other inclusion $P_g + P_h \subseteq P_f$, we must show that every member of the set $\{(e'_1 + e''_1, e'_2 + e''_2, ..., e'_n + e''_n)\}$, which may be a vertex of or in $P_g + P_h$, is a member of the set $\{(e_1, e_2, ..., e_n)\}$. Note that this fact is not necessarily true, in general, since $d_1 + ... + d_r$ may be equal to zero if $r \geq 2$. But, this follows from the fact that each vertex of $P_g + P_h$ is uniquely determined,

Let v be any vertex of $P_g + P_h$. Then, by Theorem 1.6, there are unique vertices v_1 of P_g and v_2 of P_h such that $v = v_1 + v_2$.

Consequently, if the point $v = (e_1, e_2, ..., e_n)$ is a vertex of $P_g + P_h$, there is only one term $c'_{e'_1 e'_2 ... e'_n} x_1^{e'_1} x_2^{e'_2} ... x_n^{e'_n}$ of $g(x_1, x_2, ..., x_n)$ and just one term $c''_{e''_1 e''_2 ... e''_n} x_1^{e''_1} x_2^{e''_2} ... x_n^{e''_n}$ of $h(x_1, x_2, ..., x_n)$ such that $v_1 = (e'_1, e'_2, ..., e'_n)$ is the unique vertex of P_g, and $v_2 = (e''_1, e''_2, ..., e''_n)$ is the unique vertex of P_h satisfying

$$v = (e_1, e_2, ..., e_n) = v_1 + v_2 = (e'_1 + e''_1, e'_2 + e''_2, ..., e'_n + e''_n).$$

Since, by assumption, $c'_{e'_1 e'_2 ... e'_n} \neq 0$ and $c''_{e''_1 e''_2 ... e''_n} \neq 0$, we must have

$$c'_{e'_1 e'_2 ... e'_n} \cdot c''_{e''_1 e''_2 ... e''_n} \neq 0,$$

i.e. $(e'_1 + e''_1, e'_2 + e''_2, ..., e'_n + e''_n)$ is a member of the set $\{(e_1, e_2, ..., e_n)\}$. Therefore, there is a unique term in the expansion of $g \cdot h$ that has v as its exponent vector. Thus, $v \in P_f$. Hence, we have $P_g + P_h \subseteq P_f$. $\quad\square$

As a result of Theorem 2.1, we have the following *irreducibility criterion* for multivariate polynomials over arbitrary rings without zero-divisors.

Corollary 2.2 *Let R be a ring without zero-divisors and $f \in R[x_1, x_2, ..., x_n]$ a nonzero polynomial not divisible by any x_i. If the Newton polytope P_f of f is integrally indecomposable then f is irreducible over every ring extension R' of R.*

Proof Since f is not divisible by any x_i, it has no factor having only one term. Let f be reducible over a ring extension R' of R. This means that $f = gh$ over R', where both g and h have at least two nonzero terms. Therefore, the Newton polytopes of g and h have at least two points. By Theorem 2.1, we have $P_f = P_g + P_h$, which is a contradiction. $\quad\square$

Using Theorem 1.6 and Theorem 2.1, we have obtained the following results. All of these results are valid for any ring R which contains at least one element which is not a zero-divisor. These results are still true if we take elements which are not left or right zero-divisors instead of elements which are not zero-divisors.

Proposition 2.3 *Let R be a ring containing at least one element which is not a zero-divisor and f, g, h nonzero polynomials in $R[x_1, x_2, ..., x_n]$ with $f = gh$. If the coefficients of the terms of g which are forming the vertices of P_g or the coefficients of the terms of h which are forming the vertices of P_h are not zero-divisors in R (in particular, if they are units in R), then $P_f = P_g + P_h$.*

Proof Let R be a ring and a and b nonzero elements in R. If a or b is not a zero-divisor in R then $ab \neq 0$. By using this fact and the proof of Theorem 2.1, we see that $P_g + P_h \subseteq P_f$. By the multiplication property of polynomials, obviously we have $P_f \subseteq P_g + P_h$. $\quad\square$

We have the following result of Proposition 2.3.

Corollary 2.4 *Let R be a ring containing at least one element which is not a zero-divisor and $f \in R[x_1, x_2, ..., x_n]$ a nonzero polynomial not divisible by any x_i. Suppose that the coefficients of all the terms forming the vertices of Newton polytope P_f of f are not zero-divisors in R. If P_f is integrally indecomposable then f is irreducible over every ring extension R' of R.*

Proof If a is not a zero-divisor in R and $a = bc$ for some $b, c \in R$, then both b and c cannot be zero-divisors in R. With respect to this fact, the result follows directly from Proposition 2.3. □

The following result is a special case of Corollary 2.4.

Proposition 2.5 *Let $f \in \mathbb{Z}[x_1, x_2, ..., x_m]$ be a nonzero polynomial not divisible by any x_i and n a positive integer. Suppose that Newton polytope of f is integrally indecomposable. If the coefficients of all terms of f which are forming the vertices of P_f are relatively prime to n, then f is irreducible over $\mathbb{Z}_n$.*

Proof The zero-divisors in $\mathbb{Z}_n$ are precisely the elements which are relatively prime to n. Therefore, the result is a consequence of Corollary 2.4. □

Corollary 2.6 *Let $f \in \mathbb{Z}[x_1, x_2, ..., x_m]$ be a nonzero polynomial not divisible by any x_i. Assume that Newton polytope P_f of f is integrally indecomposable. For the prime numbers p_i in $\mathbb{Z}$ such that p_i do not divide the coefficients of all the terms forming the vertices of P_f of f, f is irreducible over the ring $\mathbb{Z}_{p_i^k}$ for any positive integer k.*

Proof For $k = 1$, the result follows from Corollary 1.4 since $\mathbb{Z}_p$ is a field if p is a prime number.

Let $k \geq 2$ and p be a prime number. In this case, the zero-divisors in $\mathbb{Z}_{p^k}$ are the elements which are not divisible by p. Under this condition, the result follows directly from Corollary 2.4. □

We know that Eisenstein-Dumas and Stepanov-Schmidt criteria are special cases of the polytope method, see [3]. Similarly, [1, Lemma 6.2] is a special case of Corollary 2.6.

Example 2.7 *The polynomial*

$$f = 6x^{375} + 21y^{154} + 22x^2y^8 + 13x^{78} + 9y^6 + 10 + \sum c_{ij}x^iy^j \in \mathbb{Z}[x, y],$$

having Newton polytope $P_f = conv((0,0), (375,0), (0,154))$, is irreducible over $\mathbb{Z}_n$ by Proposition 2.5 if $gcd(6, n) = gcd(21, n) = gcd(10, n) = 1$, where the operator gcd stands for greatest common divisor. Because, P_f is an integrally indecomposable triangle in $\mathbb{R}^2$ by [3, Corollary 4.12] while $gcd(375, 154) = 1$.

Example 2.8 *The polynomial f given in Example 2.7 is irreducible over $\mathbb{Z}_{p^k}$ if $p \neq 2, 3, 5, 7$ since the prime divisors of 6, 21 and 10 are in the set $S = \{2, 3, 5, 7\}$.*

Example 2.9 *Consider the polynomial*

$$f = b_1x^6 + b_2y^4 + b_3x^{14}y^2 + b_4x^{18}y^{11} + b_5x^9y^{12} + \sum c_{ij}x^iy^j \in \mathbb{Z}[x, y],$$

having Newton polytope

$$P_f = conv((0,4), (6,0), (14,2), (18,11), (9,12)),$$

which is an integrally indecomposable pentagon by [4, Lemma 13]. If n is a positive integer relatively prime to coefficients b_i for $i = 1, ..., 5$, then f is irreducible over $\mathbb{Z}_n$. In particular, if p is a prime number relatively prime to $b_1, ..., b_5$, then f is irreducible over $\mathbb{Z}_{p^k}$ for any positive integer k.

Example 2.10 *Let a_1, a_2, a_3, a_4 and a_5 be nonzero integers. Consider the polynomial*

$$f = a_1 x^5 y^{20} + a_2 y^7 z^{18} + a_3 x^{14} y^{11} + a_4 x^8 y^{11} z^6 + a_5 x y^6 z^{35} + \sum c_{ijk} x^i y^j z^k \in \mathbb{Z}[x, y, z]$$

having Newton polytope

$$P_f = conv((5, 20, 0), (0, 7, 18), (14, 11, 0), (8, 11, 6), (1, 6, 35)),$$

which is an integrally indecomposable pyramid in $\mathbb{R}^3$ by [3, Theorem 4.2]. By Proposition 2.5, g is irreducible over $\mathbb{Z}_n$ for any positive integer n such that $gcd(n, a_1) = gcd(n, a_2) = gcd(n, a_3) = gcd(n, a_4) = gcd(n, a_5) = 1$.

We can use absolutely irreducible polynomials over any field F given in [3] and [4] to have examples of irreducible polynomials over $\mathbb{Z}_n$. We only need to play with certain coefficients of the terms, which are forming Newton polytopes of these polynomials, suitably. More precisely, we should change the coefficients of these related terms which are not zero-divisors over mentioned rings.

Remark 2.11 *Let $f(x_1, ..., x_n) \in \mathbb{Z}[x_1, x_2, ...x_n]$ be a polynomial not divisible by any x_i. If p is a prime number not dividing the coefficients of the terms of f which are forming P_f, then f is irreducible over $\mathbb{Z}_{p^k}$ for any positive integer k.*

Consequently, any polynomial $f(x_1, ..., x_n) \in \mathbb{Z}[x_1, x_2, ...x_n]$ is absolutely irreducible over $\mathbb{Z}_p$ for infinitely many primes p, more precisely, for the prime numbers not dividing the coefficients of the terms of f which are forming the vertices of P_f.

3. Some families of integrally indecomposable polytopes

Gao gave a criterion for the integral indecomposability of polytopes lying inside a pyramid with an integrally indecomposable base. Here, we generalized this result to the polytopes lying inside the convex hull of two polytopes, one of which is integrally indecomposable, which lie in two different hyperplanes.

Gao gave the following result.

Theorem 3.1 *[3, Theorem 4.11]. Let Q be an integrally indecomposable polytope in $\mathbb{R}^n$ which is contained in a hyperplane H and having at least two points. Let $v \in \mathbb{R}^n$ be an arbitrary point which is not contained in H. If S is any set of integral points in the pyramid $conv(v, Q)$, then the polytope $P = conv(Q, S)$ is integrally indecomposable.*

Being a more general result, when compared with [5, Theorem 3], our new criterion is given as follows. Note that in [5, Theorem 3], we require that H_1 and H_2 are different parallel hyperplanes. But, this is not necessary in the following result and the former is a special case of the latter.

Theorem 3.2 *Let H_1 and H_2 be different hyperplanes in $\mathbb{R}^n$, and let Q_1 be an integrally indecomposable polytope lying inside H_1 and having at least two points. Let Q_2 be an integral polytope in $\mathbb{R}^n$ such that*

$Q_2 \subset H_2$ and $Q_2 \subset H_1^+$ or $Q_2 \subset H_1^-$. Assume that the projection of Q_2 onto H_1 is Q_2' and there exists a point $v \in \mathbb{R}^n$ such that $Q_2' + v \subsetneq Q_1$. If S is any set of integral points in the polytope $conv(Q_1, Q_2)$, then the polytope $P = conv(Q_1, S)$ is integrally indecomposable.

Proof   The proof is very similar to the proof of [3, Theorem 4.11]. For the convenience of the reader, in order to emphasize the new situation in this theorem, we present a proof here. Let $P = conv(Q_1, S)$ be a polytope which satisfies the required properties. Observe that, since $Q_1 = P \cap H_1$, Q_1 is also a face of P. If $P = K + L$ for some integral polytopes K and L then, by Theorem 1.6, K and L have unique faces K_1 and L_1 respectively such that $Q_1 = K_1 + L_1$. While Q_1 is integrally indecomposable, K_1 or L_1 must consist of only one point, say $K_1 = \{a\}$ for some point $a \in \mathbb{R}^n$, and hence $L_1 = Q_1 + (-a)$. Shifting K and L suitably, i.e. writing

$$P = (K + (-a)) + (L + a),$$

we may suppose that $K_1 = \{0\}$ and $L_1 = Q_1$. Our aim is to show that K must contain only one point, i.e. $K = K_1 = \{0\}$. But, this is geometrically obvious since, for all nonzero $u \in \mathbb{R}^n$, any shifting $u + Q_1$ cannot lie in the polytope $conv(Q_1, Q_2)$.   □

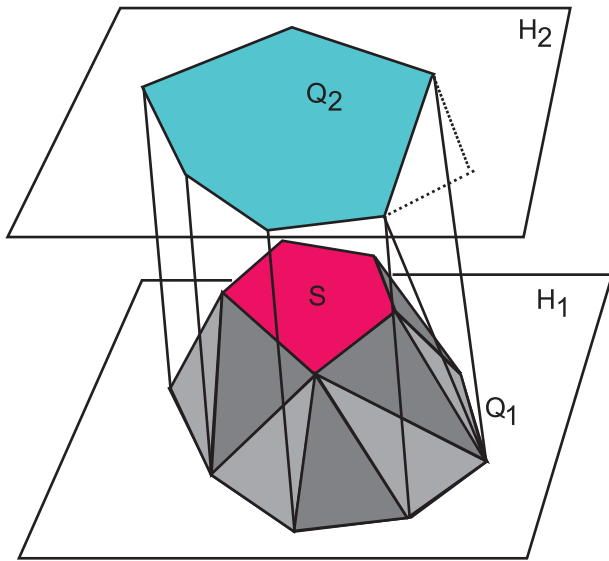We demonstrate some examples of the mentioned situation in Theorem 3.2 in Figure 1, Figure 2 and Figure 3.
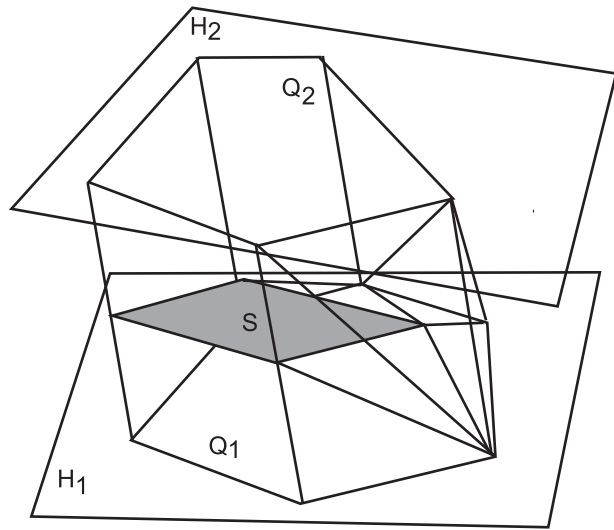

Figure 1.

Figure 2.

Example 3.3 *Let m and n be relatively prime positive integers, and $c \geq 0$ and $d \geq n+1$ be arbitrary integers. Then, the quadrangle*

$$Q = conv((m,0), (m+1, d+c), (0,d), (0,n))$$

is integrally indecomposable by Theorem 3.1 or Theorem 3.2. Consequently, by Theorem 3.2, the integral polytopes

$$A = conv((m,0,0), (m+1, d+c, 0), (0,d,0), (0,n,0), (m,0,p), (0,d,r), (0,n,q)),$$

$$B = conv((m,0,0), (m+1, d+c, 0), (0,d,0), (0,n,0), (m,0,p), (m+1, d+c, q), (0,d,r)),$$

$$C = conv((m, 0, 0), (m + 1, d + c, 0), (0, d, 0), (0, n, 0), (m, 0, p), (m + 1, d + c, r), (0, n, q))$$

are integrally indecomposable, where p, q *and* r *are arbitrary nonnegative integers, see Figure 4.*

For example, for $m = 10$, $n = 21$, $d = 30$, $c = 5$ *and arbitrary nonnegative integers* p, q, r, *the integral polytope*

$$P = conv((10, 0, 0), (11, 35, 0), (0, 30, 0), (0, 21, 0), (10, 0, p), (0, 30, q), (0, 21, r))$$

is integrally indecomposable.

As a result, the family of multivariate polynomials of the form

$$f = a_1 x^{10} + a_2 x^{11} y^{35} + a_3 y^{30} + a_4 y^{21} + a_5 x^{10} z^p + a_6 y^{30} z^q + a_7 y^{21} z^r + \sum c_{ijk} x^i y^j z^k,$$

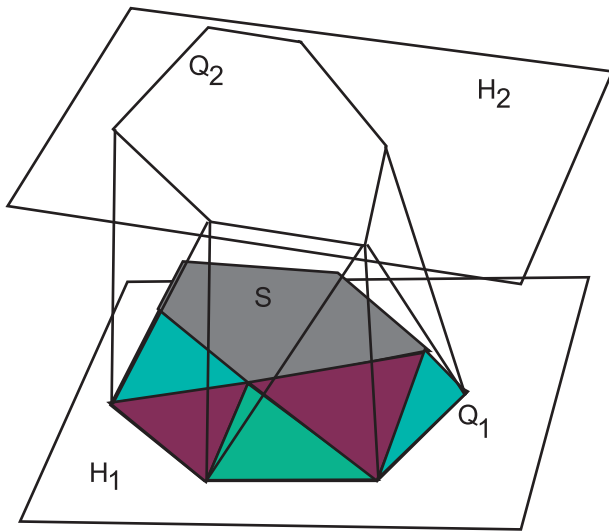*having Newton polytope* $P_f = P$, *are absolutely irreducible over any field* F *by Corollary 1.4.*
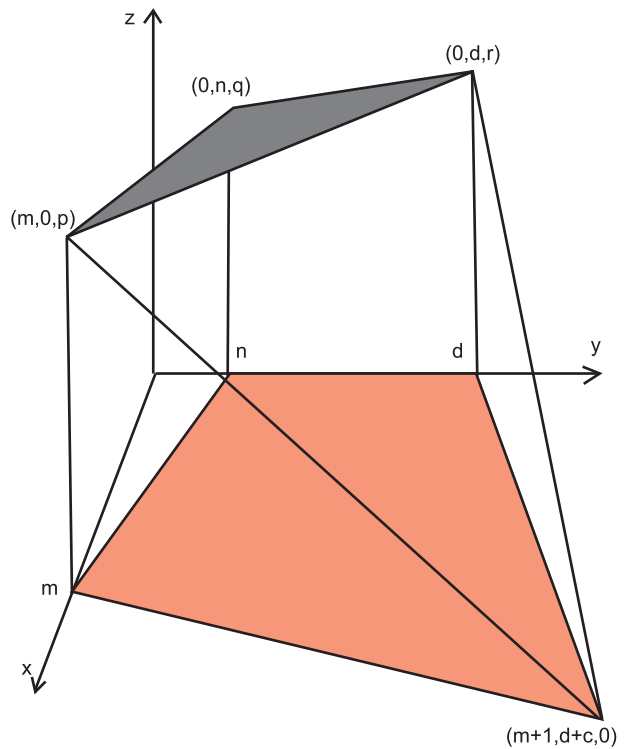

Figure 3.

Figure 4. The polytope A.

Example 3.4 *Let* m *and* n *be any positive relatively prime integers. By [4, Lemma 13], any pentagon*

$$K = conv((m, 0), (m + 1, n + 1), (m, n + m + 1), (0, n + m), (0, n))$$

is integrally indecomposable. For any positive integer k, *we form the following polygons, which are integrally indecomposable by [4, Lemma 13], in* $\mathbb{R}^3$ *as*

$$A = conv((m, 0, 0), (m + 1, n + 1, 0), (m, n + m + 1, 0), (0, n + m, 0), (0, n, 0),$$

$$B = conv(m, 0, k), (m + 1, n + 1, k), (m, n + m + 1, k), (0, n + m, k)),$$

$$C = conv((m+1, n+1, 0), (m, n+m+1, 0), (0, n+m, 0), (0, n, 0),$$
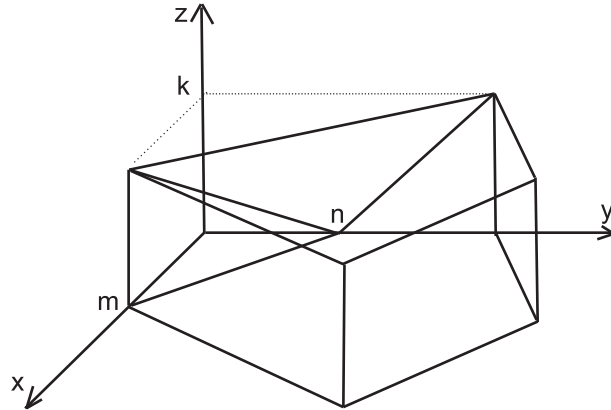
$$D = conv((m, 0, 0), (m+1, n+1, 0), (m, n+m+1, 0), (0, n, 0)).$$

Then, by Theorem 3.2, the polytopes

$$P = conv(A \cup B),$$

$$Q = conv(A \cup C),$$

$$R = conv(A \cup D)$$

are integrally indecomposable in $\mathbb{R}^3$*, see Figure 5.*

Figure 5. The polytope B.

As a result, all multivariate polynomials in $F[x_1, x_2, ..., x_n]$ *having these kinds of integrally indecomposable polytopes are absolutely irreducible over any field* F*.*

Note that [3, Theorem 4.11] does not work for the integral indecomposability of the mentioned polytopes in Example 3.3 and Example 3.4. Because, none of these polytopes lies inside a pyramid in $\mathbb{R}^3$.

Acknowledgments

References

[1] Dinh, H. Q., Permouth, S. R. L.: Cyclic and Negacyclic Codes Over Finite Chain Rings, IEEE Transactions on Information Theory 50, no.8, 1728–1744 (2004).

[2] Ewald G.: Combinatorial Convexity and Algebraic Geometry, GTM 168, Springer 1996.

[3] Gao S.: Absolute irreducibility of polynomials via Newton polytopes, Journal of Algebra 237, no.2, 501–520 (2001).

[4] Gao, S., Lauder, A. G. B.: Decomposition of Polytopes and Polynomials, Discrete and Computational Geometry 26, no.1, 89–104 (2001).

[5] Koyuncu, F.: Integrally indecomposable polytopes, Selcuk J. of App. Math. 7, no. 1, 3–8 (2006).

[6] Koyuncu, F., Özbudak, F.: Integral and homothetic indecomposability with applications to irreducibility of polynomials, Turkish Journal of Mathematics 33, 283–294 (2009).

[7] Ostrowski, A.M.: On multiplication and factorization of polynomials I, Lexicographic orderings and extreme aggregates of terms, Aequationes Math. 13, 201–228 (1975).