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On pseudo semi-projective modules

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Abstract: A right R-module M is called semi-projective if, for any submodule N of M, every epimorphism $\pi : M \to N$ and every homomorphism $\alpha : M \to N$, there exists a homomorphism $\beta : M \to M$ such that $\pi\beta = \alpha$ (see [11]). In this paper, we consider some generalizations of semi-projective module, that is quasi pseudo principally projective module. Some properties of this class of module are studied.

Key words: Semi-projective module, pseudo principally projective module

1. Introduction

Throughout the paper, R represents an associative ring with identity $1 \neq 0$ and all modules are unitary R-modules. We write M_R (resp., ${}_RM$) to indicate that M is a right (resp., left) R-module. We also write $J(R)$ for the Jacobson radical of R. If N is a submodule of M (resp., proper submodule) we denote by $N \leq M$ (resp., $N < M$). Moreover, we write $N \leq^{e} M$, $N \ll M$ to indicate that N is an essential submodule, a small submodule, respectively. A module M is called uniform if $M \neq 0$ and every non-zero submodule of M is essential in M. A module M has finite uniform dimension if M has an essential submodule which is a finite direct sum of uniform submodules or, equivalently, M contains no infinite direct sum of nonzero submodules. In case that $\oplus_{i=1}^{n} M_i \leq^{e} M$ for each M_i uniform, we write $\dim(M) = n$. A right R-module N is called M-generated if there exists an epimorphism $M^{(I)} \to N$ for some index set I. If I is finite, then N is called finitely M-generated. In particular, N is called M-cyclic if it is isomorphic to M/L for some submodule L of M. Hence, any M-cyclic submodule X of M can be considered as the image of an endomorphism of M. Following Wisbauer ([11]), $\sigma[M]$ denotes the full subcategory of Mod-R, whose objects are the submodules of M-generated modules.

A right R-module N is called pseudo M-principally injective if every monomorphism from an M-cyclic submodule of M to N can be extended to a homomorphism from M to N. Equivalently, for any homomorphism $\alpha \in \mathrm{End}(M)$, every monomorphism from $\alpha(M)$ to N can be extended to a homomorphism from M to N (see [9]). A module M is called pseudo semi-injective if M is pseudo M-principally injective. A ring R is called right pseudo semi-injective if R_R is pseudo semi-injective. Some characterizations of pseudo semi-injective module are studied and developed.

Next we will introduce the dual notion of pseudo M-principally injective. Following Clark et al. (see [2] or [5]), a right R-module N is called epi-M-projective if for any submodule A of M, every epimorphism

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$N \to M/A$ can be lifted to a homomorphism $N \to M$. A module M is called epi-projective if M is epi-M-projective. Authors studied some properties and characterizations of class epi-projective modules. Following Wisbauer ([11]), a right R-module M is called semi-projective if, for any submodule N of M, every epimorphism $\pi : M \to N$ and every homomorphism $\alpha : M \to N$, there exists a homomorphism $\beta : M \to M$ such that $\pi\alpha = \beta$ or equivalently, for any endomorphism γ of M, and every homomorphism $\alpha : M \to \gamma(M)$, there exists a homomorphism $\beta : M \to M$ such that $\gamma\beta = \alpha$. Naturally we consider module M with the following property: For any endomorphism γ of M, and every epimorphism $\alpha : M \to \gamma(M)$, there exists a homomorphism $\beta : M \to M$ such that $\gamma\beta = \alpha$.

$$\begin{tikzcd}
& M \arrow[dl, dotted, "\beta"'] \arrow[d, "\alpha"] & \\
M \arrow[r, "\gamma"] & \gamma(M) \arrow[r] \arrow[d] & 0 \\
& 0 &
\end{tikzcd}$$

If module M has this property, M is said to be quasi pseudo principally projective (or pseudo semi-projective). Thus the notion pseudo semi-projective is generalization notion of semi-projective and dual notion of pseudo semi-injective. In this paper, we study some properties and characterizations of pseudo semi-projective module. Moreover, we consider relations of pseudo semi-projective module with its endomorphism ring.

General background material can be found in [1], [3], [6], [7] and [11].

2. On pseudo M-principally projective

Definition 2.1 *A right R-module N is called pseudo M-principally projective if, for any endomorphism ε of M, every epimorphism $p : M \to \varepsilon(M)$ and every epimorphism $f : N \to \varepsilon(M)$, there exists a homomorphism $h : N \to M$ such that $ph = f$.*

$$\begin{tikzcd}
& N \arrow[dl, dotted, "h"'] \arrow[d, "f"] & \\
M \arrow[r, "p"] & \varepsilon(M) \arrow[r] \arrow[d] & 0 \\
& 0 &
\end{tikzcd}$$

or equivalently if, for any endomorphism ε of M and every epimorphism $f : N \to M/Ker\varepsilon$, there exists a homomorphism $h : N \to M$ such that $\pi h = f$ with $\pi : M \to M/Ker\varepsilon$ the natural projection.

A module M is called quasi pseudo principally projective (or pseudo semi-projective) if M is pseudo M-principally projective. A module M is called pseudo principally projective if M is pseudo N-principally projective for all right R-module N.

Then we have the relations:

$$\text{self-projective} \Rightarrow \text{ semi-projective} \Rightarrow \text{ pseudo semi-projective.}$$

Note that there is the pseudo semi-projective module but not self-projective module (see [2, Exercise 4.45(8)]). Until now, we do not know a discriminate example of pseudo semi-projective module and semi-projective module.

Next we will give some characterizations of pseudo M-principally projective modules.

Lemma 2.2 *Let M, N be right R-modules and $S = \mathrm{End}(M)$. Then the following are equivalent:*

1. *N is pseudo M-principally projective.*
2. *For all $\alpha \in S$,*
$$\{\beta \in Hom(N, M)|\ \mathrm{Im}(\alpha) = \mathrm{Im}(\beta)\} \subseteq \alpha\mathrm{Hom}(N, M).$$
3. *For all $\alpha \in S$,*
$$\{\beta \in \mathrm{Hom}(N, M)|\ \mathrm{Im}\beta = \mathrm{Im}\alpha\} = \alpha\{\beta \in \mathrm{Hom}(N, M)|\mathrm{Im}\beta + \mathrm{Ker}\alpha = M\}.$$

Proof $(1) \Rightarrow (2)$. Assume that N is pseudo M-principally projective and for each $\alpha \in S$. Let $\beta \in \mathrm{Hom}(N, M)$ with $\mathrm{Im}\alpha = \mathrm{Im}\beta$. We consider the epimorphism $\beta : N \to \mathrm{Im}\beta = \mathrm{Im}\alpha$.

$$\begin{tikzcd}
 & N \arrow[dl, dotted, "h"'] \arrow[d, "\beta"] & \\
M \arrow[r, "\alpha"] & \mathrm{Im}\alpha \arrow[r] & 0
\end{tikzcd}$$

By our hypothesis, there exists $h \in \mathrm{Hom}(N, M)$ such that $\beta = \alpha h$. Therefore $\beta = \alpha\mathrm{Hom}(N, M)$.

$(2) \Rightarrow (3)$. It is easy to see that

$$\alpha\{\beta \in \mathrm{Hom}(N, M)|\ \mathrm{Im}\beta + \mathrm{Ker}\alpha = M\} \subseteq \{\gamma \in \mathrm{Hom}(N, M)|\ \mathrm{Im}\gamma = \mathrm{Im}\alpha\}.$$

Conversely, for each $\gamma \in \mathrm{Hom}(N, M)$ such that $\mathrm{Im}\gamma = \mathrm{Im}\alpha$. Then by (2) there exists $h \in \mathrm{Hom}(N, M)$ such that $\gamma = \alpha h$. It follows that

$$h \in \{\beta \in \mathrm{Hom}(N, M)|\ \mathrm{Im}\beta + \mathrm{Ker}\alpha = M\},$$

which implies

$$\{\gamma \in \mathrm{Hom}(N, M)|\ \mathrm{Im}\gamma = \mathrm{Im}\alpha\} \subseteq \alpha\{\beta \in \mathrm{Hom}(N, M)|\ \mathrm{Im}\beta + \mathrm{Ker}\alpha = M\}.$$

$(3) \Rightarrow (1)$. For any endomorphism $\gamma \in S$, every epimorphism $\alpha : M \to \gamma(M)$ and every epimorphism $\phi : N \to \gamma(M)$.

$$\begin{tikzcd} & N \arrow[dl, dotted, "h"'] \arrow[d, "\phi"] & \\ M \arrow[r, "\alpha"] & \gamma(M) \arrow[r] & 0 \end{tikzcd}$$

Then $\mathrm{Im}\phi = \mathrm{Im}\alpha = \mathrm{Im}\gamma$. By (3), there exists $h \in \mathrm{Hom}(N, M)$ such that $\phi = \alpha h$. Thus N is pseudo M-principally projective. □

Corollary 2.3 *Module N is pseudo M-principally projective if and only if for any endomorphism ε of M and every epimorphism $f : N \to \varepsilon(M)$, there exists a homomorphism $h : N \to M$ such that $\varepsilon h = f$.*

Next, we have some properties of pseudo M-principally projective modules.

Proposition 2.4 *Let M and N be R-modules.*

1. *If N is pseudo M-principally projective if and only if N is pseudo K-principally projective for each M-cyclic submodule K of M.*

2. *If N is pseudo M-principally projective, P is pseudo M-principally projective for each direct summand P of N.*

3. *Assume that $N = \oplus_{i \in I} N_i$. Then N is pseudo M-principally projective if and only if N_i is pseudo M-principally projective for all $i \in I$.*

4. *If $N \simeq N'$ and N is pseudo M-principally projective, N' is also pseudo M-principally projective.*

Proof (1) ($\Rightarrow$). Let $K = s(M)$ for some $s \in S = \mathrm{End}(M)$. For each $\alpha \in \mathrm{End}(K)$ and $\beta \in \mathrm{Hom}(N, K)$ with $\mathrm{Im}\alpha = \mathrm{Im}\beta$. Then $\alpha s \in S$, $\iota\beta \in \mathrm{Hom}(N, M)$ and $\mathrm{Im}\alpha s = \mathrm{Im}\iota\beta$, with $\iota : s(M) \to M$ the inclusion monomorphism. It follows that $\iota\beta = (\alpha s)g$ for some $g \in \mathrm{Hom}(N, M)$ by Lemma 2.2. Thus $\beta \in \alpha\mathrm{Hom}(N, s(M))$. That means N is pseudo K-principally projective.

($\Leftarrow$) is obvious.

(2), (3) and (4) are clear. □

Theorem 2.5 *Let M and N be modules and $X = M \oplus N$. The following conditions are equivalent:*

1. *N is pseudo M-principally projective.*

2. *For each submodule K of X such that $X/K \simeq A$ with $A \leq M$ and $K + M = K + N = X$, there exists $C \leq K$ such that $M \oplus C = X$.*

Proof (1) $\Rightarrow$ (2). Let $f : N \to M/(M \cap K)$ via $f(n) = m + M \cap K$ for all $n = k + m \in N$ with $k \in K, m \in M$. Then f is an epimorphism. We get $M/(M \cap K) \simeq (M + K)/K \simeq X/K \simeq A$ with $A \leq M$, then we may regard $M/(M \cap K)$ as a M-cyclic submodule of M. Since N is pseudo M-principally projective, there exists $h : N \to M$ such that $\pi h = f$ with $\pi : M \to M/(M \cap K)$ the natural projection. Let $C = \{n - h(n) |\ n \in N\}$. Then $C \leq K$ and $M \oplus C = X$.

(2) $\Rightarrow$ (1). Let $\alpha \in \mathrm{End}(M)$, $f : N \to M/\mathrm{Ker}\alpha$ an epimorphism and $\pi : M \to M/\mathrm{Ker}\alpha$ the natural projection. Let $K = \{n+m|\ f(n) = -\pi(m)\}$. It is easy to see that $K+M = K+N = X$ and $K \cap M = \mathrm{Ker}\alpha$. Then $X/K \simeq M/(M \cap K) = M/\mathrm{Ker}\alpha \simeq \mathrm{Im}\alpha$. By (2), there exists $C \leq K$ such that $M \oplus C = X$. Let $p : M \oplus C \to M$ be the canonical projection. It follows that $\pi p|_N = f$. Thus N is pseudo M-principally projective. □

3. Some results on pseudo semi-projective modules

In this section, we study some properties of pseudo semi-projective module and its endomorphism ring.

Firstly, following Lemma 2.2, we get this next lemma.

Lemma 3.1 *Let M be a right R-module and $S = \mathrm{End}(M)$. Then the following are equivalent:*

1. *M is pseudo semi-projective.*
2. *For all $\alpha, \beta \in S$ with $\mathrm{Im}(\alpha) = \mathrm{Im}(\beta)$, $\alpha S = \beta S$.*
3. *For all $\alpha, \beta \in S$, we have:*

$$\{\gamma \in S|\ \mathrm{Im}(\beta\gamma) = \mathrm{Im}(\beta\alpha)\} \subseteq \alpha S + \{\theta \in S|\ \mathrm{Im}\theta \leq \mathrm{Ker}\beta\}.$$

When $M \oplus M$ is pseudo semi-projective, we have

Proposition 3.2 *If $M \oplus M$ is pseudo semi-projective then M is semi-projective.*

Proof Let $\overline{M} = M \oplus M$ be pseudo semi-projective, we show that M is semi-projective. Let $s \in \mathrm{End}(M)$, and $f : M \to s(M)$ be a homomorphism. Let $g : \overline{M} \to s(M)$ with $g(m_1 + m_2) = f(m_1) + s(m_2)$ for all $m_1 \in M, m_2 \in M$. Then g is an epimorphism. By Proposition 2.4, $\overline{M}$ is pseudo M-principally projective, there is a homomorphism $h : \overline{M} \to M$ such that $g = sh$. Let $\iota : M \to \overline{M}$ be the canonical inclusion. Therefore $s(h\iota) = g\iota = f$. Thus M is semi-projective. □

Corollary 3.3 *For any integer $n \geq 2$, if M^n is pseudo semi-projective then M is semi-projective.*

Proposition 3.4 *Let M be pseudo semi-projective and $\alpha \in \mathrm{End}(M)$. Then $\mathrm{Ker}\alpha$ is a direct summand of M if and only if $\alpha(M)$ is pseudo M-principally projective.*

Proof Assume that $\mathrm{Ker}\alpha$ is a direct summand of M. Then $M/\mathrm{Ker}\alpha$ is isomorphic to a direct summand of M. It follows that $\alpha(M) \simeq M/\mathrm{Ker}\alpha$ is pseudo M-principally projective by Proposition 2.4. Conversely, we consider the diagram

$$\begin{tikzcd} & M/\mathrm{Ker}\alpha \arrow[dl, dotted, "h"'] \arrow[d, "id"] & \\ M \arrow[r, "p"] & M/\mathrm{Ker}\alpha \arrow[r] & 0 \end{tikzcd}$$

with p the canonical projection. Since $\alpha(M) \simeq M/\mathrm{Ker}\alpha$ is pseudo M-principally projective, there exists $h : M/\mathrm{Ker}\alpha \to M$ such that $ph = id$. It follows that $\mathrm{Ker}\alpha$ is a direct summand of M. □

A module M is called D2 if, for any submodule A of M for which M/A is isomorphic to a direct summand of M then A is a direct summand of M. From the Proposition 3.4, we get

Corollary 3.5 *If M is pseudo semi-projective then M has D2.*

Proposition 3.6 *Assume that M is pseudo semi-projective and $\alpha, \beta \in S = \mathrm{End}(M)$. If $\alpha(M) \simeq \beta(M)$ then $\alpha S \simeq \beta S$.*

Proof Let $f : \alpha(M) \to \beta(M)$ be an isomorphism. We consider the following diagram

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, dotted, "h"'] & \alpha(M) \arrow[d, "f"] \\ M \arrow[r, "\beta"] & \beta(M) \arrow[d] \\ & 0. \end{tikzcd}$$

It is easy to see that $f\alpha$ is an epimorphism. Since M is pseudo semi-projective, there exists $h : M \to M$ such that $\beta h = f\alpha$. Let $\phi : \alpha S \to \beta S$ via $\phi(\alpha s) = \beta hs$ for all $s \in S$. Then ϕ is a S-monomorphism. On the other hand, $\beta(M) = f(\alpha(M)) = \beta(h(M)) = \beta h(M)$ by Lemma 3.1, whence $\beta S = \beta hS = \mathrm{Im}\phi$. It follows that ϕ is an epimorphism. □

Recall that M_R is a *principal self-generator (briefly, self p-generator)* if every element $m \in M$ has the form $m = \lambda(m_1)$ for some $\lambda : M_R \longrightarrow mR$ and $m_1 \in M$ (see [8]).

Lemma 3.7 *Let M be self p-generator and pseudo semi-projective with $S = \mathrm{End}(M)$. If N is essential in L with $L \le M$, $\mathrm{Hom}(M, N)$ is essential in right S-module $\mathrm{Hom}(M, L)$.*

Proof Let $f \in \mathrm{Hom}(M, L)$ and $\mathrm{Hom}(M, N) \cap fS = 0$. Assume that $f(m) \in N \cap \mathrm{Im}f$. Since M is self p-generator, there exist epimorphisms $g : M \to f(m)R$ and $s : M \to mR$. Then $g(M) = fs(M)$. It follows that $gS = fsS$ by Lemma 3.1. Thus $g = fst$ for some $t \in S$, whence $g \in \mathrm{Hom}(M, N) \cap fS = 0$ or $f(m) = 0$. It means we proved that $N \cap \mathrm{Im}f = 0$. However, $N \le^e L$, $\mathrm{Im}f = 0$ or $f = 0$. Thus $\mathrm{Hom}(M, N)$ is essential in right S-module $\mathrm{Hom}(M, L)$. □

Now we consider the relation of finite uniform dimension of M and its endomorphism ring.

Theorem 3.8 *Assume that M is a self p-generator and pseudo semi-projective module with $S = \mathrm{End}(M)$. Then M has finite uniform dimension if and only if S_S has finite uniform dimension. Moreover in this case, $dim(M_R) = dim(S_S)$*

Proof ($\Rightarrow$) Assume that $\dim(M_R) = k$. There exists $U_i \leq M$, $i = 1, 2, ..., k$ such that $U_1 \oplus U_2 \oplus \cdots \oplus U_k \leq^e M$, with U_i uniform. By Lemma 3.7 we have

$$\mathrm{Hom}(M, U_1 \oplus U_2 \oplus \cdots \oplus U_k) \leq^e S_S.$$

Since M is self p-generator, pseudo semi-projective and U_i uniform, $\mathrm{Hom}(M, U_i)$ is uniform as right S-module for each $i = 1, 2, ..., k$. In fact, assume that for elements $f, g \in \mathrm{Hom}(M, U_i)$ such that $fS \cap gS = 0$. Then if $m \in f(M) \cap g(M)$, $m = f(m_1) = g(m_2)$ for some $m_1, m_2 \in M$. Therefore $mR = f(m_1R) = g(m_2R)$. Since M is self p-generator, there exists $h, h' \in S$ such that $m_1R = h(M), m_2R = h'(M)$, whence $mR = fh(M) = gh'(M)$. It follows that $fhS = gh'S \leq fS \cap gS = 0$ or $fh = 0$ and hence $m = 0$. Thus $f(M) \cap g(M) = 0$. But M_i is uniform, $f(M) = 0$ or $g(M) = 0$. Moreover, we also have

$$\mathrm{Hom}(M, U_1 \oplus U_2 \oplus \cdots \oplus U_k) = \mathrm{Hom}(M, U_1) \oplus \mathrm{Hom}(M, U_2) \oplus \cdots \oplus \mathrm{Hom}(M, U_k)$$

and hence $\mathrm{Hom}(M, U_1) \oplus \mathrm{Hom}(M, U_2) \oplus \cdots \oplus \mathrm{Hom}(M, U_k) \leq^e S_S$. It follows that S_S has finite uniform dimension and $\dim(S_S) = k$.

($\Leftarrow$) Assume that M contains a infinite direct sum of nonzero submodules $\oplus_{i \in I} M_i$. Then S contains the infinite direct sum of right ideals $\oplus_{i \in I}\mathrm{Hom}(M, M_i)$, a contradiction. In fact, for all $f \in \mathrm{Hom}(M, M_i) \cap \Sigma_{i \in I, i \neq j}\mathrm{Hom}(M, M_j)$, then $f = f_{j_1} + \cdots + f_{j_n}$, with $j_1, \ldots, j_n \in \{j \in I | \ j \neq i\}$ and $f_{j_l} \in \mathrm{Hom}(M, M_{i_l})$. Hence, for all $m \in M$, $f(m) = (f_{j_1} + \cdots + f_{j_n})(m) = f_{j_1}(m) + \cdots + f_{j_n}(m) \in M_i \cap (M_{j_1} + \cdots + M_{j_n}) = 0$, whence $f(m) = 0$ or $f = 0$. Thus M contains no infinite direct sums of submodules or M has finite uniform dimension. □

Remark. In [10, Theorem 3.1], authors proved that: Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Then, M has finite uniform dimension if and only if $S = \mathrm{End}(M)$ has finite uniform dimension. Moreover in this case, $\dim(M_R) = dim(S_S)$. It is well known if M is self-generator, quasi-projective, then M is retractable and semi-projective. On the other hand, if M is a semi-projective module, then $\mathrm{Hom}(M, s(M)) = s$ for any s in S. And if M is retractable, then $\mathrm{Hom}(M, N)$ is nonzero for all nonzero submodule N of M. Therefore the Theorem 3.8 is also true in case M retractable semi-projective. Thus we have the following result: "Let M be a semi-projective, right R-module which is retractable. Then, M has finite uniform dimension if and only if $S = \mathrm{End}(M)$ has finite uniform dimension and $\dim(M_R) = dim(S_S)$". This result and Theorem 3.8 are new results. But is Theorem 3.8 true for M retractable, pseudo semi-projective?

The following is an application for the above results.

Example 3.9 *Let R be a ring with $dim(R_R) = k$, n be a positive integer and S be a ring of $n \times n$ matrices with entries in R. Then $dim(S_S) = nk$.*

Proof By the hypothesis, $\dim(R_R^n) = nk$. Since ring S is isomorphic to endomorphism ring of R^n, we also get $\dim(S_S) = nk$. □

It is well known that endomorphism ring of a self-projective, Artinian module is semiprimary. We also have a similar result for pseudo semi-projective module and is given by the following theorem.

Theorem 3.10 *If M is pseudo semi-projective and Artinian then $S = \mathrm{End}(M)$ is semiprimary.*

Proof Assume that

$$s_1S \geq s_2S \geq \cdots$$

with $s_i \in S$. Then we also $s_1(M) \geq s_2(M) \geq \cdots$. Since M is Artinian, there exists $n \in \mathbb{N}$ such that $s_n(M) = s_{n+k}(M),\ \forall k \in \mathbb{N}$. It follows that $s_nS = s_{n+k}S,\ \forall k \in \mathbb{N}$ by pseudo semi-projectivity of M. Thus S is left perfect.

We will claim that $J(S)$ is nilpotent. In fact, we have chain submodules of M

$$J(S)(M) \geq J(S)^2(M) \geq \cdots.$$

Since M is Artinian, there exists $n \in \mathbb{N}$ such that $J(S)^n(M) = J(S)^{n+k}(M),\ \forall k \in \mathbb{N}$. Let $I = J(S)^n$, hence we get $IM = I^2M$. Assume that $J(S)$ is not nilpotent. There exists $s \in I$ such that $sI \neq 0$. Let s_0M be minimal in the set $\{sM|\ s \in I, sI \neq 0\}$. Since $s_0IM = s_0IIM$, there exists $t \in s_0I \leq I$ such that $tI \neq 0$ and $tM \leq s_0IM \leq s_0M$. It follows that $tM = s_0M$ by minimality of s_0M and hence $s_0M = s_0gM$ for some $g \in I$. On the other hand, M is pseudo semi-projective, there exists $f \in S$ with $s_0 = s_0gf$ for some $f \in S$. It follows that $s_0(1 - gf) = 0$. Since $gf \in J(S)$, $s_0 = 0$, a contradiction. Thus S is semiprimary. □

Remark. In [11, 31.11], author proved that endomorphism ring of a self-projective, Artinian module is semiprimary. But in this proof, author used the property "$\mathrm{Hom}(M, s(M)) = s$ for any s in $S = \mathrm{End}(M)$" to show that S is a left perfect. In Theorem 3.10, we only used the property "$f(M) = g(M)$ if and only if $Sf = Sg$ for all $f, g \in S$" to prove that S is a left perfect. Moreover, if M is semi-projective then M is pseudo semi-projective. Thus Theorem 3.10 is extension of [11, 31.11].

Next, we get some characterizations of semisimple ring via pseudo semi-projectivity. The following result is similar to Theorem 2.11 in [4].

Theorem 3.11 *The following conditions are equivalent for ring R.*

1. *R is semisimple.*
2. *Every pseudo semi-projective module is projective.*
3. *Every direct sum of any family of pseudo semi-projective modules is projective.*
4. *The direct sum of two pseudo semi-projective modules is projective.*

Proof (1) $\Rightarrow$ (2) by [1, Exercise 16.9] and (2) $\Rightarrow$ (3) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (1). Let M be a simple right R-module. It follows that M is pseudo semi-projective. Then $M \oplus R_R$ is projective by our assumption and hence M is projective. Thus R is semisimple by [1, Exercise 16.9]. □

Note that the direct sum of two pseudo semi-projective modules need not be semi-projective. For example $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is direct sum of two pseudo semi-projective, but M is not pseudo semi-projective (see [5, Example 3.1]).

It is well known a ring R is right perfect if and only if every right R-module has projective cover. We also have a similar result for pseudo semi-projective modules in the following theorem.

Theorem 3.12 *The following conditions are equivalent for ring R:*

1. *R is right perfect.*

2. *For any right R-module M, there exists an epimorphism $f : N \to M$ such that N is pseudo semi-projective and $Ker f \ll N$.*

Proof (1) $\Rightarrow$ (2) is obviously.

(2) $\Rightarrow$ (1) Let M be a right R-module. There exists a free module F and an epimorphism $\psi : F \to M$. By (2), there exists an epimorphism $\phi : S \to F \oplus M$ such that S is pseudo semi-projective and $\mathrm{Ker}\phi \ll S$. Denote $p_1 : F \oplus M \to F$ and $p_2 : F \oplus M \to M$ the natural projections. Then $p_1\phi : S \to F$ is an epimorphism. By projectivity of F, $S = \mathrm{Ker}(p_1\phi) \oplus T$ with $T \leq S$. Let $M' = \mathrm{Ker}(p_1\phi)$. We get $S/M' \simeq F$ and $S/M' \simeq T$ and hence $F \simeq T$. From this, we can regard $S = M' \oplus F$. We get $f = \phi|_{M'} : M' \to M$ is an epimorphism. Now we will show that M' is a projective cover of M. Assume that $A + \mathrm{Ker}f = M'$. Since $\mathrm{Ker}f \leq \mathrm{Ker}\phi$, $F + A + \mathrm{Ker}\phi = M' + F = S$ whence $F + A = F + M'$. Hence $A = M'$ or $\mathrm{Ker}f \ll M'$.

On the other hand, F is projective, there exists $\overline{\psi} : F \to M'$ such that $f\overline{\psi} = \psi$. But $\mathrm{Ker}f \ll M'$ and so $\overline{\psi}$ is an epimorphism. Let $\pi_1 : S \to F$, $\pi_2 : S \to M'$ the natural projections. We consider the diagram

$$\begin{tikzcd} & S \arrow[dl, dotted, "h"'] \arrow[d, "\pi_2"] & \\ S \arrow[r, "\overline{\psi}\pi_1"] & M' \arrow[r] & 0. \end{tikzcd}$$

Since M' is a direct summand of S (and so M' is a S-cyclic submodule of S) and S is pseudo semi-projective, there exists $h : S \to S$ such that $\overline{\psi}\pi_1 h = \pi_2$. Let $g = \pi_1 h\iota$ with $\iota : M' \to S$ the natural inclusion. Then $\overline{\psi}g = id$, and M' is isomorphic to a direct summand of F and hence M' is projective. Thus M' is the projective cover of M. □

From the Theorem 3.12, we get the following corollaries:

Corollary 3.13 *The following conditions are equivalent for ring R:*

1. *R is semiperfect.*

2. *For any finitely generated right R-module M, there exists an epimorphism $f : N \to M$ such that N is pseudo semi-projective and $Ker f \ll N$.*

Corollary 3.14 *For ring R. The following conditions are equivalent:*

1. *R is semiregular.*

2. *For any finitely presented right R-module M, there exists an epimorphism $f : N \to M$ such that N is pseudo semi-projective and $Ker f \ll N$.*

Proof Note that in proof of Theorem 3.12, if M is finitely presented, $M \simeq F/K$ with F free and both F and K finitely generated. Then $F \oplus M$ is also finitely presented. Thus M has a projective cover. It follows that R is semiregular by [7, Theorem B.56]. □

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