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A nonlocal parabolic problem in an annulus for the Heaviside function in Ohmic heating

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Abstract: In this paper, we consider the nonlocal parabolic equation

$$u_t = \Delta u + \frac{\lambda H(1-u)}{\left(\int_{A_{\rho,R}} H(1-u)dx\right)^2}, \quad x \in A_{\rho,R} \subset \mathbb{R}^2,\ t > 0,$$

with a homogeneous Dirichlet boundary condition, where λ is a positive parameter, H is the Heaviside function and $A_{\rho,R}$ is an annulus. It is shown for the radial symmetric case that: there exist two critical values λ_* and λ^*, so that for $0 < \lambda < \lambda_*$, $u(x,t)$ is global in time and the unique stationary solution is globally asymptotically stable; for $\lambda_* < \lambda < \lambda^*$ there also exists a steady state and $u(x,t)$ is global in time; while for $\lambda > \lambda^*$ there is no steady state and $u(x,t)$ "blows up" (in some sense) for any appropriate $(u_0(x) \le 1)$ initial data.

Key words: Nonlocal parabolic equation, steady state, stability, blow-up

1. Introduction

In this paper we study the radially symmetric solutions to the nonlocal parabolic problem

$$\begin{cases} u_t = \Delta u + \dfrac{\lambda H(1-u)}{\left(\int_{A_{\rho,R}} H(1-u)dx\right)^2}, & x \in A_{\rho,R},\ t > 0, \\ u(x,t) = 0, & x \in \partial A_{\rho,R},\ t > 0, \\ u(x,0) = u_0(x), & x \in A_{\rho,R}, \end{cases} \tag{1.1}$$

where $u(x,t) = u(x,t;\lambda) = u(|x|,t)$ stands for the dimensionless temperature of a conductor when an electric current flows through it [7, 14, 15, 17, 23], H denotes the Heaviside function:

$$H(s) = \begin{cases} 1, & s > 0, \\ 0, & s \le 0, \end{cases}$$

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$A_{\rho,R}$ is the annulus

$$A_{\rho,R} = \{x \in \mathbb{R}^2 : \ 0 < \rho <| x |< R\},$$

and $u_0(x) = u_0(r)(r = |x|)$ is a radial symmetric function which will be specified later.

For the derivation of the model of a nonlocal problem, we refer to [15, 16, 23] and references therein. In 1995, Lacey [15] derived the following nonlocal parabolic model related to Ohmic, or Joule heating

$$\begin{cases} u_t - \Delta u = \dfrac{\lambda f(u)}{(\int_\Omega f(u)dx)^2}, & x \in \Omega, \ t > 0, \\ u = 0, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.2}$$

The equation comes from a more general parabolic-elliptic system,

$$\begin{cases} u_t - \nabla(\kappa(u)\nabla u) = \sigma(u)|\nabla\phi|^2, & x \in \Omega, \ t > 0, \\ \nabla \cdot (\sigma(u)\nabla\phi) = 0, & x \in \Omega, \ t > 0, \end{cases} \tag{1.3}$$

where ϕ is the electric potential. For the study of system (1.3), we refer to [1, 4, 5, 10, 11, 25] and references therein. For problem (1.2) in one dimension $(-1 < x < 1)$, in [15, 16], among other things, it was proved that for the case of decreasing $f(s)$, (i) if $\int_0^\infty f(s)ds = \infty$, there is a unique steady state which is globally asymptotically stable; (ii) if $\int_0^\infty f(s)ds = 1$ (which is scaled from $\int_0^\infty f(s)ds < \infty$), (a) there is a unique steady state which is globally asymptotically stable if $\lambda < 8$, (b) there is no steady state and u is unbounded if $\lambda = 8$, (c) there is no steady state and u blows up in finite time for all $-1 < x < 1$, if $\lambda > 8$. For problem (1.2) in one dimensions $(0 < x < 1)$ radially symmetric case, Tzanetis [23] first studied the case of $f(s) = H(1-s)$, and obtained that there exist two critical values $\lambda_* = 4\pi^2$ and $\lambda^* = 8\pi^2$, so that for $0 < \lambda < \lambda^*$, $u(x,t) = u(r,t)$ is global in time and the unique stationary solution is globally asymptotically stable; for $\lambda > \lambda^*$ the solution u "blows up" (in some sense, i.e., it ceases to be less than 1) in finite time.

In [12], Kavallaris and Tzanetis considered the problem

$$\begin{cases} u_t - u_{xx} + u_x = \dfrac{\lambda f(u)}{(\int_0^1 f(u)dx)^2}, & 0 < x < 1, \ t > 0, \\ \mathcal{B}(u) = 0, & x = 0, 1, \ t > 0, \\ u(x,0) = u_0(x), & 0 < x < 1, \end{cases} \tag{1.4}$$

and its associated steady-state problem

$$\begin{cases} w'' - w' + \dfrac{\lambda f(w)}{(\int_0^1 f(w)dx)^2} = 0, & 0 < x < 1, \\ \mathcal{B}(w) = 0, & x = 0, 1, \end{cases}$$

where $f(s)$ is positive and monotonic and $\mathcal{B}$ is a suitable linear boundary operator. They have obtained similar results as in [15, 16]. Problem (1.4) was first considered in [19], where the stability of different models was studied. Related material can be found in [2, 6, 8, 12, 20, 22, 26]. In [13], Kavallaris and Tzanetis considered problem (1.4) for the case of $f(s) = H(1-s)$, and found that there exist two critical values λ_* and λ^*, so that for $0 < \lambda < \lambda_*$, $u(x,t)$ is global in time and the unique stationary solution is globally asymptotically stable; for $\lambda_* < \lambda < \lambda^*$ there exist two steady states, while for $\lambda > \lambda^*$ there is no steady state. They also proved that for $\lambda > \lambda^*$ or for $\lambda_* < \lambda < \lambda^*$ and initial data sufficiently large, the solution u "blows up" (in some sense).

In general, the Heaviside function is a good approximation for a number of physical quantities [21, 24]. The existence and uniqueness of a "weak" (classical a.e.) solution to (1.1) is obtained by using an approximating regularized version of this problem, see [13, 15] and the references therein. Hence, taking into account this remark, in the following we can use comparison arguments in the classical sense.

Our main results are as follows.

- If $0 < \lambda < \lambda_*$, there exists a unique stationary solution; if $\lambda_* < \lambda < \lambda^*$ there also exists a unique two-parameter family of steady state given by (2.4) while $\lambda > \lambda^*$ there is no steady state.
- If $0 < \lambda < \lambda_*$, then the solution $u(r,t)$ of problem (2.1) is global in time and the unique steady state is globally asymptotically stable for any initial data $0 \leq u_0(r) \leq 1$ and no $u_0(r) = 1$ for $r \in (\rho, R)$.
- If $\lambda_* < \lambda < \lambda^*$ then the solution $u(r,t)$ of problem (2.1) is global in time for any initial data $0 \leq u_0(r) \leq 1$.
- If $\lambda > \lambda^*$, then the solution $u(r,t)$ of problem (2.1) "blows up" (in some sense, actually ceases to be less than 1 in $(\rho; R)$) in finite time for any initial data $0 \leq u_0(r) \leq 1$.

This work follows the ideas and techniques which have been used in the one-dimensional case [13, 15, 16] and the two-dimensional radially symmetric case [23]. In contrast to [13], we obtain that $\lambda(s_1, s_2)$ is decreasing function with s_1 (see Section 2). Therefore, for $\lambda_* < \lambda < \lambda^*$, the solution $u(r,t)$ of problem (2.1) is global in time for any initial data $0 \leq u_0(r) \leq 1$. Also, in contrast to [13, 15, 16], here we have to modify their arguments because of the extra technical difficulties encountered in this two-dimensional problem; in contrast to [23], we examine an asymmetric case which is connected with a two-parameter family of steady states, resulting in more technical difficulties.

This paper is organized as follows. In Section 2 we consider the steady-state problem corresponding to (1.1). Section 3 is devoted to the stability and "blows up" (in some sense).

2. Steady-state problem

Since we consider radial solutions, we can rewrite problem (1.1) as

$$\begin{cases} u_t - u_{rr} - \dfrac{1}{r}u_r = \dfrac{\lambda H(1-u)}{4\pi^2(\int_\rho^R H(1-u)rdr)^2}, & \rho < r < R,\ t > 0, \\ u(\rho,t) = u(R,t) = 0, & t > 0, \\ u(r,0) = u_0(r), & \rho < r < R, \end{cases} \tag{2.1}$$

where $u_0(r)$ and $u_0'(r)$ are bounded with $u_0(r) \geq 0$ in $[\rho,\ R]$. Concerning the existence of radial solutions, it is worth to notice that this is true for positive and bounded solutions for a circular disk by [9], while in case of an annulus the symmetry may be breakdown [18]. Using the same properties as in [9], it is also possible to obtain, both for the parabolic and elliptic problems, similar results concerning the radial symmetry of solutions. There may exist under some circumstances no radially symmetric (asymmetric) solutions as well. Also, for simplicity, we assume $0 \le u_0(r) \le 1$ for $r \in [\rho,\ R]$, then $0 \le u(r,t) \le 1$ by the maximum principle. In particular, the first equation of problem (2.1) is equivalent to

$$u_t - u_{rr} - \frac{1}{r}u_r = \begin{cases} 0, & \text{for } u \ge 1, \\ \lambda/m^2(t), & \text{for } u < 1, \end{cases}$$

where $m(t)$ is the measure of the subset of the annuli $A_{\rho,R}$ where $u < 1$.

The steady states of the problem (2.1) play an important role in the description of the asymptotic behavior of the solutions of (2.1) and the construction of the lower and upper solutions, hence we first consider the stationary problem of (2.1). Now we distinguish two cases:

1. $u < 1$ for every $r \in [\rho,\ R]$, then the first equation of problem (2.1) becomes

$$u_t - u_{rr} - \frac{1}{r}u_r = \frac{\lambda}{\pi^2(R^2-\rho^2)^2}$$

and the corresponding steady problem is

$$\begin{cases} w'' + \frac{1}{r}w' + \frac{\lambda}{\pi^2(R^2-\rho^2)^2} = 0, & \rho < r < R, \\ w(\rho) = w(R) = 0. \end{cases} \tag{2.2}$$

2. $u = 1$ in a subinterval $(S_1(t), S_2(t))$ of $[\rho,\ R]$, then there exist $\rho < s_1 \le s_2 < R$ such that the corresponding steady problem has the form:

$$\begin{cases} w'' + \frac{1}{r}w' + \frac{\lambda}{\pi^2(R^2-\rho^2+s_1^2-s_2^2)^2} = 0, & \rho < r < s_1, \text{ or } s_2 < r < R, \\ w(r) = 1, & s_1 \le r \le s_2, \\ w(\rho) = w(R) = 0, \end{cases} \tag{2.3}$$

where $S_1 = S_1(t)$, $S_2 = S_2(t)$ are dependent on t and $S_1(t) \to s_1\pm$ and $S_2(t) \to s_2\pm$ as $t \to \infty$. Throughout the paper, we will write S_1 and S_2 to denote the time-dependent variables $S_1(t)$ and $S_2(t)$, respectively.

The solution of (2.2) is

$$w_1(r) = w_1(r;\lambda) = \begin{cases} \frac{cr_0^2}{2}(\ln r - \ln\rho) - \frac{c}{4}(r^2-\rho^2), & \rho \le r \le r_0, \\ \frac{c}{4}(R^2-r^2) - \frac{cr_0^2}{2}(\ln R - \ln r), & r_0 \le r \le R, \end{cases}$$

where

$$c = \frac{\lambda}{\pi^2(R^2-\rho^2)^2}, \quad r_0 = \sqrt{\frac{R^2-\rho^2}{2(\ln R - \ln\rho)}}.$$

Obviously, $w_1(r)$ takes its unique maximum at r_0 point, that is

$$w_1(r_0) = \max_{\rho \le r \le R} w_1(r) = \frac{\lambda[2r_0^2(\ln r_0 - \ln\rho) - (r_0^2-\rho^2)]}{4\pi^2(R^2-\rho^2)^2}.$$

If $w_1(r_0) < 1$, we have

$$\lambda < \frac{4\pi^2(R^2-\rho^2)^2}{2r_0^2(\ln r_0 - \ln\rho) - (r_0^2-\rho^2)} = \lambda_*.$$

On the other hand, equation (2.3) gives a two-parameter family of stationary solutions of the form

$$w_2(r) = w_2(r;\lambda) = w_2(r;s_1,s_2) = \begin{cases} 1 + \frac{d}{4}(s_1^2 - r^2) - \frac{ds_1^2}{2}(\ln s_1 - \ln r), & \rho \le r < s_1, \\ 1, & s_1 \le r \le s_2, \\ 1 + \frac{ds_2^2}{2}(\ln r - \ln s_2) - \frac{d}{4}(r^2 - s_2^2), & s_2 < r \le R, \end{cases} \tag{2.4}$$

where $d = \lambda/[\pi^2(R^2-\rho^2+s_1^2-s_2^2)^2]$. For $w_2(\rho;s_1,s_2) = w_2(R;s_1,s_2) = 0$, the first branch implies

$$\lambda(s_1,s_2) = \frac{4\pi^2(R^2-\rho^2+s_1^2-s_2^2)^2}{2s_1^2(\ln s_1 - \ln\rho) - (s_1^2-\rho^2)}, \tag{2.5}$$

and the third branch gives

$$\lambda(s_1,s_2) = \frac{4\pi^2(R^2-\rho^2+s_1^2-s_2^2)^2}{R^2 - s_2^2 - 2s_2^2(\ln R - \ln s_2)}. \tag{2.6}$$

The relations (2.5) and (2.6) imply

$$2s_1^2(\ln s_1 - \ln\rho) - (s_1^2-\rho^2) = R^2 - s_2^2 - 2s_2^2(\ln R - \ln s_2), \tag{2.7}$$

provided that $s_1 \ne \rho$ and $s_2 \ne R$.

Lemma 2.1 *Assume s_1, s_2 satisfy (2.7), then $s_1 \to \rho+$ as $s_2 \to R-$, $s_1 \to r_0-$ as $s_2 \to r_0+$ and vice versa.*

The proof is obvious, so we omit it here.

From (2.7) we have $F(s_1,s_2) = R^2 - s_2^2 - 2s_2^2(\ln R - \ln s_2) + s_1^2 - \rho^2 - 2s_1^2(\ln s_1 - \ln\rho) = 0$ for $(s_1,s_2) \in (\rho,r_0)\times(r_0,R)$. Also $\partial F(s_1,s_2)/\partial s_2 = -4s_2(\ln R - \ln s_2) \ne 0$, then from the implicit function theorem we have $s_2 = \varphi(s_1)$ for all $(s_1,s_2) \in (\rho,r_0)\times(r_0,R)$ and

$$\varphi'(s_1) = \frac{s_1(\ln\rho - \ln s_1)}{s_2(\ln R - \ln s_2)} < 0. \tag{2.8}$$

Lemma 2.2 *Assume s_1, s_2 satisfy (2.7), then we have $s_1(\ln s_1 - \ln\rho) - s_2(\ln R - \ln s_2) \ge 0$ for any $(s_1,s_2) \in (\rho,r_0)\times(r_0,R)$, that is to say $\varphi'(s_1) \le -1$ for any $s_1 \in (\rho,r_0)$.*

Proof Let $f(s_1,s_2) = s_1(\ln s_1 - \ln\rho) - s_2(\ln R - \ln s_2)$ and assume that there exist some points such that $f(s_1,s_2) < 0$. Set

$$c = \min\{s_1 \mid s_1 \in (\rho,r_0),\ f(s_1,s_2) = f(s_1,\varphi(s_1)) < 0\}.$$

Since $f(s_1,\varphi(s_1))$ is continuous function for $s_1 \in (\rho,r_0)$ and $f(\rho,\varphi(\rho)) = f(\rho,R) = 0$, c exists and satisfies $\frac{\partial f}{\partial s_1}\mid_{s_1=c} < 0$. On the other hand, by the assumption we have $-1 < \varphi'(c) < 0$. Then

$$\frac{\partial f}{\partial s_1}\mid_{s_1=c} = [\ln c - \ln\rho + 1 - (\ln R - \ln\varphi(c))\varphi'(c) + \varphi'(c)] > 0,$$

which yields a contradiction. □

Theorem 2.3 *Assume s_1, s_2 satisfy (2.7), then we have*
1. $\partial\lambda(s_1,s_2)/\partial s_1 < 0$ *and* $\partial\lambda(s_1,s_2)/\partial s_2 > 0$ *for* $(s_1,s_2)\in(\rho,r_0)\times(r_0,R)$.
2.

$$\lim_{s_1\to\rho+}\lambda(s_1,s_2) = 8\pi^2(\rho^2+R^2)^2 = \lambda^*,$$

and

$$\lim_{s_1\to r_0-}\lambda(s_1,s_2) = \frac{4\pi^2(R^2-\rho^2)^2}{2r_0^2(\ln r_0-\ln\rho)-(r_0^2-\rho^2)} = \lambda_*.$$

Proof 1. By using (2.8), we have

$$\begin{aligned}\frac{\partial\lambda(s_1,s_2)}{\partial s_1} &= \frac{16\pi^2 s_1(R^2-\rho^2+s_1^2-s_2^2)}{[2s_1^2(\ln s_1-\ln\rho)-(s_1^2-\rho^2)]^2}\\ &\quad [\big(2s_1^2(\ln s_1-\ln\rho)-s_1^2+\rho^2\big)(1+\frac{\ln s_1-\ln\rho}{\ln R-\ln s_2})\\ &\quad -(R^2-\rho^2+s_1^2-s_2^2)(\ln s_1-\ln\rho)]\\ &= K(s_1,s_2)G(s_1,s_2),\end{aligned} \tag{2.9}$$

where

$$K(s_1,s_2) = \frac{16\pi^2 s_1(R^2-\rho^2+s_1^2-s_2^2)}{[2s_1^2(\ln s_1-\ln\rho)-(s_1^2-\rho^2)]^2} > 0$$

for $(s_1,s_2)\in(\rho,r_0)\times(r_0,R)$, and

$$\begin{aligned}G(s_1,s_2) =&[2s_1^2(\ln s_1-\ln\rho)-s_1^2+\rho^2](1+\frac{\ln s_1-\ln\rho}{\ln R-\ln s_2})\\ &-(R^2-\rho^2+s_1^2-s_2^2)(\ln s_1-\ln\rho).\end{aligned}$$

By (2.7) and (2.8), we get

$$\begin{aligned}\frac{\partial G}{\partial s_1} &= 4s_1(\ln s_1-\ln\rho)(1+\frac{\ln s_1-\ln\rho}{\ln R-\ln s_2})\\ &\quad +[2s_1^2(\ln s_1-\ln\rho)-s_1^2+\rho^2]\frac{s_2^2(\ln R-\ln s_2)^2-s_1^2(\ln s_1-\ln\rho)^2}{s_1 s_2^2(\ln R-\ln s_2)^3}\\ &\quad -[4s_1(\ln s_1-\ln\rho)+\frac{2s_2^2(\ln R-\ln s_2)}{s_1}-2s_2\varphi'(s_1)(\ln s_1-\ln\rho)]\\ &= \frac{2s_1^2(\ln s_1-\ln\rho)^2-2s_2^2(\ln R-\ln s_2)^2}{s_1(\ln R-\ln s_2)}\\ &\quad +[(2s_1^2(\ln s_1-\ln\rho)-s_1^2+\rho^2]\frac{s_2^2(\ln R-\ln s_2)^2-s_1^2(\ln s_1-\ln\rho)^2}{s_1 s_2^2(\ln R-\ln s_2)^3}\\ &= \frac{s_1^2(\ln s_1-\ln\rho)^2-s_2^2(\ln R-\ln s_2)^2}{s_1 s_2^2(\ln R-\ln s_2)^3}G_1(s_1,s_2),\end{aligned} \tag{2.10}$$

where $G_1(s_1,s_2)=2s_2^2(\ln R-\ln s_2)^2-2s_1^2(\ln s_1-\ln\rho)+s_1^2-\rho^2$. Moreover,

$$\begin{aligned}\frac{\partial G_1}{\partial s_1}&=4s_2\varphi'(s_1)(\ln R-\ln s_2)^2-4s_2\varphi'(s_1)(\ln R-\ln s_2)-4s_1(\ln s_1-\ln\rho)\\&=-4s_1(\ln s_1-\ln\rho)(\ln R-\ln s_2)<0,\end{aligned}$$

and

$$\frac{\partial G_1}{\partial s_2}=\frac{\partial G_1}{\partial s_1}\frac{\partial s_1}{\partial s_2}=\frac{\partial G_1}{\partial s_1}\frac{1}{\varphi'(s_1)}>0,$$

which imply $G_1(s_1,s_2)<G_1(\rho,R)=0$ for $(s_1,s_2)\in(\rho,r_0)\times(r_0,R)$. From Lemma 2.2 and (2.10), we have $\partial G/\partial s_1<0$ which yields $G(s_1,s_2)<G(\rho,R)=0$ for $(s_1,s_2)\in(\rho,r_0)\times(r_0,R)$. Hence we have

$$\partial\lambda(s_1,s_2)/\partial s_1<0 \text{ and } \partial\lambda(s_1,s_2)/\partial s_2=\partial\lambda(s_1,s_2)/\partial s_1\frac{1}{\varphi'(s_1)}>0.$$

2. From Lemma 2.1 we get

$$\lim_{s_1\to\rho}\varphi'(s_1)=\lim_{s_1\to\rho}\frac{\ln s_1-\ln\rho-1}{\varphi'(s_1)(\ln R-\ln s_2-1)}=\lim_{s_1\to\rho}\frac{1}{\varphi'(s_1)}. \tag{2.11}$$

Combining (2.8) with (2.11), we have $\lim_{s_1\to\rho}\varphi'(s_1)=-1$. Hence

$$\begin{aligned}\lim_{s_1\to\rho+}\lambda(s_1,s_2)&=4\pi^2(\rho+R)\lim_{s_1\to\rho+}\frac{R^2-\rho^2+s_1^2-s_2^2}{s_1(\ln s_1-\ln\rho)}\\&=4\pi^2(\rho+R)\lim_{s_1\to\rho+}\frac{2s_1-2s_2\varphi'(s_1)}{1+\ln s_1-\ln\rho}=8\pi^2(\rho+R)^2=\lambda^*.\end{aligned}$$

By using Lemma 2.1, we have $\lim_{s_1\to r_0-}s_2=r_0$ which implies

$$\lim_{s_1\to r_0-}\lambda(s_1,s_2)=\frac{4\pi^2(R^2-\rho^2)^2}{2r_0^2(\ln r_0-\ln\rho)-(r_0^2-\rho^2)}=\lambda_*.$$

The proof is completed. □

If we denote by $\| w'\|=\sup w'$, then $\| w'\|=w'(\rho)$. Thus

$$w_1'(\rho;\lambda)=\frac{\lambda(r_0^2-\rho^2)}{2\pi^2\rho(R^2-\rho^2)^2}\quad\text{for}\quad 0<\lambda<\lambda_*,$$

$$w_2'(\rho;r_0,r_0)=w_2'(\rho;\lambda_*)=\frac{\lambda_*(r_0^2-\rho^2)}{2\pi^2\rho(R^2-\rho^2)^2},$$

$$w_2'(\rho;\rho,R)=w_2'(\rho;\lambda^*)=\frac{1}{2\pi^2\rho}\lim_{s_1\to\rho+}\frac{s_1^2-\rho^2}{(R^2-\rho^2+s_1^2-s_2^2)^2}=\infty.$$

According to the above analysis, we have the existence theorem for the steady-state problem (2.1).

Theorem 2.4 *If $0<\lambda<\lambda_*$, there exists a unique stationary solution; if $\lambda_*<\lambda<\lambda^*$ there also exists a unique radial symmetric two-parameter family of steady state given by (2.4) while for $\lambda\geq\lambda^*$ there is no steady state.*

3. Stability and "blow-up"

Firstly we consider problem (2.1) with $0 < \lambda < \lambda_*$. From Theorem 2.4, $w_1(r)$ is the unique radial symmetric stationary solution.

Theorem 3.1 *If $0 < \lambda < \lambda_*$, then the solution $u(r,t)$ of problem (2.1) is global in time and the unique steady state is globally asymptotically stable for any initial data $0 \le u_0(r) \le 1$.*

Proof In the case of $0 < u_0(r) \le w_1(r)$, the function

$$z(r,t) = \begin{cases} \dfrac{\alpha(t)[2r_0^2(\ln r - \ln\rho) - (r^2-\rho^2)]}{4\pi^2(R^2-\rho^2)^2}, & \rho \le r \le r_0,\ t>0, \\ \dfrac{\alpha(t)[R^2 - r^2 - 2r_0^2(\ln R - \ln r)]}{4\pi^2(R^2-\rho^2)^2}, & r_0 \le r \le R,\ t>0, \end{cases}$$

is a lower solution to problem (2.1) provided that $\alpha(t)$ satisfies

$$\alpha'(t) = a(\lambda - \alpha(t)), \quad t>0; \quad \alpha(0) = \alpha_0, \tag{3.1}$$

where $a = 4/[2r_0^2(\ln r_0 - \ln\rho) - (r_0^2 - \rho^2)]$ and α_0 is a suitable chosen constant so that $0 \le \alpha_0 \le \lambda$ and $z(r,0) \le u_0(r)$. The solution to (3.1) is

$$\alpha(t) = \lambda + (\alpha_0 - \lambda)e^{-at} \to \lambda- \quad \text{as} \ \ t\to\infty.$$

Hence $z(r,t) \to w_1(r)$ as $t\to\infty$ uniformly for $r\in[\rho,\ R]$. Since $z(r,t) \le u(r,t) \le w_1(r)$ and $z(r,t)\to w_1(r)$ as $t\to\infty$ uniformly for $r\in[\rho,\ R]$, we see that $u(r,t)$ exists globally in time and $u(r,t)\to w_1(r)$ as $t\to\infty$ uniformly for $r\in[\rho,\ R]$.

For $w_1(r) < u_0(r) \le 1$, our prospective comparison function $V(r,t)$ is

$$V(r,t) = w_2(r;S_1,S_2) = \begin{cases} 1 + \dfrac{S_1^2 - r^2 - 2S_1^2(\ln S_1 - \ln r)}{2S_1^2(\ln S_1 - \ln\rho) - S_1^2 + \rho^2}, & \rho \le r < S_1, 0<t<t_1, \\ 1, & S_1 \le r \le S_2, 0<t<t_1, \\ 1 + \dfrac{2S_2^2(\ln r - \ln S_2) - r^2 + S_2^2}{R^2 - S_2^2 - 2S_2^2(\ln R - \ln S_2)}, & S_2 < r \le R, 0<t<t_1, \end{cases}$$

and

$$V(r,t) = w_1(r;\beta(t)) = \begin{cases} \dfrac{\beta(t)[2r_0^2(\ln r - \ln\rho) - (r^2-\rho^2)]}{4\pi^2(R^2-\rho^2)^2}, & \rho \le r \le r_0,\ t>t_1, \\ \dfrac{\beta(t)[R^2 - r^2 - 2r_0^2(\ln R - \ln r)]}{4\pi^2(R^2-\rho^2)^2}, & r_0 \le r \le R,\ t>t_1, \end{cases}$$

where S_1 and S_2 are functions of t which satisfy $\rho < S_1(t) \le S_2(t) < R$ and relation (2.7). For $0<t<t_1$, let $S_1'(t) \ge 0$, then by (2.7) and (2.8) we have

$$\begin{aligned} V_t &= \frac{4S_1S_1'[(\ln S_1 - \ln r)(S_1^2-\rho^2) - (\ln S_1 - \ln\rho)(S_1^2 - r^2)]}{[2S_1^2(\ln S_1 - \ln\rho) - S_1^2 + \rho^2]^2} \\ &\ge -\frac{4S_1S_1'(\ln S_1 - \ln\rho)(S_1^2 - \rho^2 + R^2 - S_2^2)}{[2S_1^2(\ln S_1 - \ln\rho) - S_1^2+\rho^2]^2},\ \rho \le r < S_1,\ 0<t<t_1, \end{aligned}$$

and

$$\begin{aligned} V_t &= \frac{4S_2S_2'[(\ln r - \ln S_2)(R^2 - S_2^2) - (\ln R - \ln S_2)(r^2 - S_2^2)]}{[R^2 - S_2^2 - 2S_2^2(\ln R - \ln S_2)]^2} \\ &\geq \frac{4S_2S_2'(\ln R - \ln S_2)(R^2 - S_2^2)}{[2S_1^2(\ln S_1 - \ln \rho) - S_1^2 + \rho^2]^2} = -\frac{4S_1S_1'(\ln S_1 - \ln \rho)(R^2 - S_2^2)}{[2S_1^2(\ln S_1 - \ln \rho) - S_1^2 + \rho^2]^2}, \\ &\geq -\frac{4S_1S_1'(\ln S_1 - \ln \rho)(S_1^2 - \rho^2 + R^2 - S_2^2)}{[2S_1^2(\ln S_1 - \ln \rho) - S_1^2 + \rho^2]^2}, \ S_2 < r \leq R, \ 0 < t < t_1, \end{aligned}$$

which imply that $V(r,t)$ is an upper solution to problem (2.1) as long as $S_1(t)$ satisfies

$$S_1'(t) = \frac{(\lambda(S_1, \varphi(S_1)) - \lambda)[2S_1^2(\ln S_1 - \ln \rho) - S_1^2 + \rho^2]^2}{4\pi^2 S_1(\ln S_1 - \ln \rho)(S_1^2 - \rho^2 + R^2 - \varphi^2(S_1))^3}, 0 < t < t_1; S_1(0) = s_0, \tag{3.2}$$

where $s_0 > \rho$ so that $\lambda(s_0, \varphi(s_0)) > \lambda$ and $V(r,0) = w_2(r; s_0, \varphi(s_0)) \geq u_0(r)$. Problem (3.2) has a unique solution, since the same holds for its equivalent transcendental equation for $S_1(t)$:

$$\int_{s_0}^{S_1(t)} \frac{4\pi^2\sigma(\ln \sigma - \ln \rho)(\sigma^2 - \rho^2 + R^2 - \varphi^2(\sigma))^3}{(\lambda(\sigma) - \lambda)[2\sigma^2(\ln \sigma - \ln \rho) - \sigma^2 + \rho^2]^2} d\sigma = t, \quad 0 < t < t_1. \tag{3.3}$$

Note that the function

$$G(\xi) = \int_{s_0}^{\xi} \frac{4\pi^2\sigma(\ln \sigma - \ln \rho)(\sigma^2 - \rho^2 + R^2 - \varphi^2(\sigma))^3}{(\lambda(\sigma) - \lambda)[2\sigma^2(\ln \sigma - \ln \rho) - \sigma^2 + \rho^2]^2} d\sigma$$

is a C^1-diffeomorphism from $[s_0, \ r_0]$ to $[0, \ T]$ (see [3]), where

$$T = \int_{s_0}^{r_0} \frac{4\pi^2\sigma(\ln \sigma - \ln \rho)(\sigma^2 - \rho^2 + R^2 - \varphi^2(\sigma))^3}{(\lambda(\sigma) - \lambda)[2\sigma^2(\ln \sigma - \ln \rho) - \sigma^2 + \rho^2]^2} d\sigma < \infty.$$

For $t > t_1$, we require $\beta(t)$ to satisfy

$$\beta'(t) = a(\lambda - \beta(t)), \quad t > t_1; \quad \beta(t_1) = \lambda_*, \tag{3.4}$$

then $V(r,t)$ is an upper solution to problem (2.1) for $t > t_1$. (3.4) is equivalent to $\beta(t) = \lambda + (\lambda_* - \lambda)e^{a(t_1 - t)} \to \lambda+$ as $t \to \infty$, which implies $V(r,t) \to w_1(r)-$ as $t \to \infty$ uniformly for $r \in [\rho, \ R]$. Since $w_1(r) \leq u(r,t) \leq V(r,t)$, we have $u(r,t) \to w_1(r)-$ as $t \to \infty$ uniformly for $r \in [\rho, \ R]$. As this holds for any initial data $0 \leq u_0(r) \leq 1$, it is clear that the unique steady state $w_1(r)$ is a globally asymptotically state. The proof is completed. □

Next we consider problem (2.1) with $\lambda_* < \lambda < \lambda^*$. From Theorem 2.4, there exists a unique two-parameter family of steady state $w_2(r) = w_2(r; \lambda) := w_2(r; S_1, S_2)$. Then we have:

Theorem 3.2 *If $\lambda_* < \lambda < \lambda^*$ and $0 \leq u_0(r) \leq w_2(r)$, then the solution $u(r,t)$ of problem (2.1) is global in time.*

The proof is obvious, we omit it.

Remark 3.1 *Since $\lambda(S_1, \varphi(S_1))$ is strictly decreasing for $S_1 \in (\rho, r_0)$, we cannot construct a lower solution to problem (2.1) which is increasing in time t of a form similar to the steady state. Therefore, it seems difficult to verify that $w_2(r;\lambda)$ is globally asymptotically stable for the case of $\lambda_* < \lambda < \lambda^*$ and $0 \le u_0(r) \le w_2(r)$ as in [13, 15, 16, 23].*

Let $(s_\lambda, \varphi(s_\lambda))$ be the unique solution of $\lambda = \lambda(S_1, S_2)$ (since $\partial\lambda/\partial S_1 < 0$).

Lemma 3.3 *Assume $\lambda_* < \lambda < \lambda^*$, then we have*

$$\lim_{S_1 \to s_\lambda} \frac{\lambda(S_1, S_2) - \lambda}{S_1 - s_\lambda} = \lim_{S_1 \to s_\lambda} \frac{\lambda(S_1, S_2) - \lambda(s_\lambda, \varphi(s_\lambda))}{S_1 - s_\lambda} = C,$$

where C is a negative constant.

Proof From (2.9), we have

$$\lim_{S_1 \to s_\lambda} \frac{\lambda(S_1, S_2) - \lambda}{S_1 - s_\lambda} = \lim_{S_1 \to s_\lambda} K(S_1, S_2)G(S_1, S_2) = K(s_\lambda, \varphi(s_\lambda))G(s_\lambda, \varphi(s_\lambda)) = C.$$

□

Theorem 3.4 *If $\lambda_* < \lambda < \lambda^*$ and $w_2(r) < u_0(r) \le 1$, then the solution $u(r,t)$ of problem (2.1) is global in time.*

Proof Assume the solution $u(r,t)$ of problem (2.1) "blows up" (in some sense) in finite time $t^* < \infty$. We look for comparison function $V(r,t)$ of the form

$$V(r,t) = w_2(r; S_1, S_2) = \begin{cases} 1 + \dfrac{S_1^2 - r^2 - 2S_1^2(\ln S_1 - \ln r)}{2S_1^2(\ln S_1 - \ln \rho) - S_1^2 + \rho^2}, & \rho \le r < S_1, \\ 1, & S_1 \le r \le S_2, \\ 1 + \dfrac{2S_2^2(\ln r - \ln S_2) - r^2 + S_2^2}{R^2 - S_2^2 - 2S_2^2(\ln R - \ln S_2)}, & S_2 < r \le R, \end{cases}$$

where S_1 and S_2 satisfy $\rho < S_1(t) \le S_2(t) < R$ and relation (2.7). If $S_1(t)$ satisfies

$$\begin{cases} S_1'(t) = h(S_1) \equiv \dfrac{(\lambda(S_1, \varphi(S_1)) - \lambda)[2S_1^2(\ln S_1 - \ln \rho) - S_1^2 + \rho^2]^2}{4\pi^2 S_1(\ln S_1 - \ln \rho)(S_1^2 - \rho^2 + R^2 - \varphi^2(S_1))^3}, & t > 0, \\ S_1(0) = \rho_1, \end{cases} \tag{3.5}$$

where $0 < \rho_1 < s_\lambda$ such that $V(r,0) = w_2(r; \rho_1, \varphi(\rho_1)) \ge u_0(r)$, then $V(r,t)$ is an upper solution to problem (2.1).

Now we show that $V(r,t)$ exists globally in time. Indeed, problem (3.5) is equivalent to the transcendental equation for $S_1(t)$:

$$\int_{\rho_1}^{S_1(t)} \frac{4\pi^2\sigma(\ln\sigma - \ln\rho)(\sigma^2 - \rho^2 + R^2 - \varphi^2(\sigma))^3}{(\lambda(\sigma, \varphi((\sigma)) - \lambda)[2\sigma^2(\ln\sigma - \ln\rho) - \sigma^2 + \rho^2]^2} = \int_{\rho_1}^{S_1(t)} \frac{d\sigma}{g(\sigma)} = t,$$

where $g(\sigma) = h(\sigma)$. Let T^* be the value such that $S_1(t)$ becomes s_λ. By Lemma 3.3, we have

$$T^* = \int_{\rho_1}^{s_\lambda} \frac{d\sigma}{g(\sigma)} = \infty,$$

which implies that $V(r,t)$ exists globally in time. This is a contradiction. □

Finally we consider the case of $\lambda > \lambda^*$ where there is no stationary solution, then we prove that $u(r,t)$ "blows up" (in some sense) in finite time.

Definition 3.1 *We say that the solution to (2.1) "blows up" in finite time $T^* < \infty$ if $u(r;t)$ ceases to be less than 1 in some subinterval of $(\rho;R)$ i.e., there exists $T^* < \infty$ such that $\lim_{t\to T^*} u(r;t) = 1$ for all $r \in (\rho;R)$.*

Theorem 3.5 *If $\lambda > \lambda^*$, then the solution $u(r,t)$ of problem (2.1) "blows up" (in some sense) in finite time for any initial data $0 \le u_0(r) \le 1$.*

Proof We only need to construct a lower solution which "blows up" (in some sense) in finite time, therefore we consider the function

$$z(r,t) = \begin{cases} \dfrac{\alpha(t)[2r_0(\ln r - \ln\rho) - (r^2-\rho^2)]}{4\pi^2(R^2-\rho^2)^2}, & \rho \le r \le r_0,\ 0<t<t_1, \\ \dfrac{\alpha(t)[R^2 - r^2 - 2r_0(\ln R - \ln r)]}{4\pi^2(R^2-\rho^2)^2}, & r_0 \le r \le R,\ 0<t<t_1. \end{cases}$$

The function $z(r,t)$ is a lower solution to problem (2.1) provided $\alpha(t)$ satisfies:

$$\alpha'(t) = a(\lambda - \alpha(t)), \quad 0<t<t_1; \quad \alpha(0)=0,$$

where $a = 4/[2r_0^2(\ln r_0 - \ln\rho) - (r_0^2-\rho^2)]$ and t_1 is such that $\alpha(t_1) = \lambda_*$. Since $\lambda > \lambda_*$, $t_1 = a^{-1}[\ln\lambda - \ln(\lambda - \lambda_*)] < \infty$. If $u(r,t)$ exists $(u<1)$ at $t=t_1$, then we define $z(r,t)$ for $t>t_1$, such that

$$z(r,t) = \begin{cases} 1 + \dfrac{S_1^2 - r^2 - 2S_1^2(\ln S_1 - \ln r)}{2S_1^2(\ln S_1 - \ln\rho) - S_1^2 + \rho^2}, & \rho \le r < S_1, \\ 1, & S_1 \le r \le S_2, \\ 1 + \dfrac{2S_2^2(\ln r - \ln S_2) - r^2 + S_2^2}{R^2 - S_2^2 - 2S_2^2(\ln R - \ln S_2)}, & S_2 < r \le R, \end{cases}$$

where S_1 and S_2 satisfy $\rho < S_1(t) \le S_2(t) < R$ and relation (2.7). If $S_1(t)$ satisfies

$$\begin{cases} S_1'(t) = h(r) \equiv \dfrac{(\lambda(S_1,\varphi(S_1)) - \lambda)[2S_1^2(\ln S_1 - \ln\rho) - S_1^2 + \rho^2]^2}{4\pi^2 S_1(\ln S_1 - \ln\rho)(S_1^2 - \rho^2 + R^2 - \varphi^2(S_1))^3}, & t > t_1, \\ S_1(t_1) = r_0, \end{cases} \tag{3.6}$$

then the function $z(r,t)$ is a lower solution to problem (2.1). Using (3.6), we have

$$T_1^* = \int_\rho^{r_0} \frac{d\sigma}{g(\sigma)} + t_1 < \infty, \tag{3.7}$$

where $g(\sigma) = -h(\sigma)$ and T_1^* satisfies $S_1(T_1^*) = \rho$ (or equivalently $S_2(T_1^*) = R$). (3.7) holds since $\lim_{\sigma\to\rho+} g(\sigma) = (\lambda - \lambda^*)/[8\pi^2(\rho+R)^3]$ is bounded. This implies that $u(r,t) \to 1-$ as $t \to t_1^*- \le T_1^*$ uniformly for every $r \in (\rho,\ R)$, that is $u(r,t)$ "blows up" in finite time. □

4. Discussion

In this paper, we consider the nonlocal parabolic equation

$$u_t = \Delta u + \frac{\lambda H(1-u)}{\left(\int_{A_{\rho,R}} H(1-u)dx\right)^2}, \quad x \in A_{\rho,R} \subset \mathbb{R}^2,\ t > 0,$$

with a homogeneous Dirichlet boundary condition, where H is the Heaviside function, $u(x,t) = u(x,t;\lambda) = u(|x|,t)$ stands for the dimensionless temperature of a conductor when an electric current flows through it [14, 15, 17]. Since $H(1-s)$ is decreasing, comparison techniques can be applied. In this problem there exist two critical values λ_* and λ^*, so that for $\lambda > \lambda^*$ or for $0 < \lambda_* < \lambda < \lambda_*$ and sufficiently "warm" initial conditions the solution "blows up" in the sense that it becomes 1 at a finite time except for the points assigned zero boundary conditions. Regarding the original physical problem, this means that the food (or the substance undergoing the heating) loses all resistivity at temperature $u = 1$, that is the heating ceases across the channel after finite time.

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