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Approximation by iterates of Beta operators

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Abstract: In this paper, we study the degree of approximation by an iterative combination $T_{n,k}$ of the beta operators introduced by Upreti [8].

Key words: Beta operators, iterative combination, modulus of smoothness

1. Introduction

Let $M[0,\infty)$ be the linear space of functions $f(t)$ defined for all $t \geq 0$ and bounded and Lebesgue measurable in every interval $[r,R]$ $(0 < r < R < \infty)$.

For $f \in M[0,\infty)$, Upreti [8] proposed a sequence of linear positive operators, e.g. Beta operators, defined as

$$B_n(f;x) = \int_0^\infty b_n(x,u) f(1/u)\, du, \tag{1.1}$$

where

$$b_n(x,u) = \frac{x^n}{B(n,n)} \frac{u^{n-1}}{(1+xu)^{2n}}$$

and $B(n,n) = ((n-1)!)^2/(2n-1)!$ is the beta function.

Next, let $H[0,\infty)$ be the linear space of functions $f(x) \in M[0,\infty)$, for which $|f(x)| \leq Px^\alpha$ $(P > 0, \alpha > 0, x > 0)$.

Upreti [8] studied some approximation properties of the operators (1.1). Later on, Zhou [9] obtained the direct and inverse theorems for these operators in $L_p[0,\infty), 1 \leq p \leq \infty$, using the K-functional technique.

It turns out that the Beta operators are saturated with $O(n^{-1})$. So, in order to improve the rate of convergence by these operators we apply the technique of iterative combinations to these operators. Several researchers have used these combinations to improve the order of approximation for some other sequences of linear positive operators (e.g. see [1], [2], [5] and [6]).

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For $f \in H[0,\infty)$, following [3], the iterative combination $T_{n,k}(f;x)$ of the operators $B_n(f;x)$ is defined as

$$\begin{aligned} T_{n,k}(f(t);x) &= \big(I-(I-B_n)^k\big)(f;x) \\ &= \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}B_n^r(f(t);x), \end{aligned}$$

where $B_n^0 = I$ and $B_n^r = B_n(B_n^{r-1})$ for $r \in \mathbb{N}$.

In the present paper, we establish some direct results in the ordinary approximation by the operators $T_{n,k}(.\,;x)$. Further, C denotes a constant which is not the same at each occurrence.

2. Preliminary results

In this section, we give some definitions and auxiliary results which are useful in establishing our main results.

For $m \in \mathbb{N}^0$ (the set of non-negative integers), the m^{th} order moment for the operators B_n is defined as

$$\begin{aligned} \mu_{n,m}(x) &= B_n((t-x)^m;x) \\ &= \int_0^\infty b_n(x,t)\left(\frac{1}{t}-x\right)^m dt. \end{aligned}$$

Lemma 1 *For the function $\mu_{n,m}(x)$, we have the following:*
(i) it is a polynomial in x of degree exactly m;
(ii) for each $x \in [0,\infty)$,

$$\mu_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right), \text{ as } n \to \infty,$$

where $[\beta]$ is the integer part of β.

Proof It is easy to see that

$$\sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}j^m = \begin{cases} 0, & m = 0,1,2,...r-1 \\ r!, & m = r. \end{cases} \tag{2.2}$$

We know that

$$B_n(t^j;x) = \frac{n(n+1)...(n+j-1)}{(n-1)(n-2)...(n-j)}x^j, \text{ for } j = 1,2,3,...$$

Hence,

$$\begin{aligned} \mu_{n,m}(x) &= B_n((t-x)^m;x) \\ &= \sum_{j=0}^{m}\binom{m}{j}(-1)^{m-j}x^{m-j}B_n(t^j;x) \\ &= (-x)^m + \sum_{j=1}^{m}\binom{m}{j}(-1)^{m-j}x^{m-j}\frac{n(n+1)...(n+j-1)}{(n-1)(n-2)...(n-j)}x^j. \end{aligned}$$

(i) If $m = 2r$, for sufficiently large n, we have

$$\begin{aligned}
\mu_{n,2r}(x) &= x^{2r} + x^{2r}\sum_{j=1}^{2r}\binom{2r}{j}(-1)^{2r-j}\frac{(1+\frac{1}{n})(1+\frac{2}{n})...(1+\frac{j-1}{n})}{(1-\frac{1}{n})(1-\frac{2}{n})...(1-\frac{j}{n})} \\
&= x^{2r} + x^{2r}\sum_{j=1}^{2r}\binom{2r}{j}(-1)^{2r-j}\left(1+\frac{1}{n}p_1(j)+\frac{1}{n^2}p_2(j)+...+\frac{1}{n^j}p_j(j)\right)\times \\
&\quad \left(1+\frac{1}{n}q_1(j)+\frac{1}{n^2}q_2(j)+...\right), \qquad (2.3)
\end{aligned}$$

where $p_1(j), q_1(j)$ are polynomials in j of second degree and $p_2(j), q_2(j)$ are polynomials in j of fourth degree, and so on.

Therefore, $\mu_{n,2r}(x)$ is a polynomial in x of degree exactly $2r$ and the coefficient of n^{-s}, $s \in \mathbb{N}$ in the right hand side of (2.3) is

$$x^{2r}\sum_{j=1}^{2r}\binom{2r}{j}(-1)^{2r-j}\sum_{i=0}^{s}p_i(j)q_{s-i}(j),$$

where $p_0(j) = q_0(j) = 1, \forall j = 1, ..., 2r.$

Hence, by applying the identity (2.2) to $\mu_{n,2r}$, we obtain

$$\mu_{n,2r}(x) = O(n^{-r}), \text{ as } n \to \infty.$$

(ii) Similarly, when $m = 2r-1, r \in \mathbb{N}$, it follows that $\mu_{n,2r-1}(x)$ is a polynomial in x of degree exactly $(2r-1)$ and

$$\mu_{n,2r-1}(x) = O(n^{-r}), \text{ as } n \to \infty.$$

Hence, in view of $\mu_{n,0}(x) = 1$, we obtain the required result. □

For every $m \in \mathbb{N}^0$ and $p \in \mathbb{N}$, the m-th order moment $\mu_{n,m}^{[p]}(x)$ for the operators B_n^p is defined as $\mu_{n,m}^{[p]}(x) = B_n^p((t-x)^m; x)$. We denote $\mu_{n,m}^{[1]}(x)$ by $\mu_{n,m}(x)$.

Lemma 2 *There holds the recurrence relation*

$$\mu_{n,m}^{[p+1]}(x) = \sum_{j=0}^{m}\binom{m}{j}\sum_{i=0}^{m-j}\frac{1}{i!}D^i\left(\mu_{n,m-j}^{[p]}(x)\right)\mu_{n,i+j}(x),$$

where D *denotes* $\frac{d}{dx}$.

Proof By binomial expansion, we may write

$$\begin{aligned}
\mu_{n,m}^{[p+1]}(x) &= B_n^{p+1}\left((t-x)^m; x\right) \\
&= B_n\left(B_n^p((t-x)^m; u); x\right) \\
&= B_n\left(B_n^p((t-u+u-x)^m; u); x\right) \\
&= \sum_{j=0}^{m}\binom{m}{j}B_n\left((u-x)^j\mu_{n,m-j}^{[p]}(u); x\right). \qquad (2.4)
\end{aligned}$$

Now, applying Lemma 1, it follows that $\mu_{n,m-j}^{[p]}(u)$ is a polynomial in u of degree $(m-j)$, therefore by Taylor's expansion, we have

$$\mu_{n,m-j}^{[p]}(u) = \sum_{i=0}^{m-j} \frac{(u-x)^i}{i!} D^i \left(\mu_{n,m-j}^{[p]}(x)\right). \tag{2.5}$$

On combining (2.4) and (2.5), the required result follows. □

Lemma 3 *For* $p \in \mathbb{N}, m \in \mathbb{N}^0$ *and* $x \in [0,\infty)$, *we have*

$$\mu_{n,m}^{[p]}(x) = O\left(n^{-[(m+1)/2]}\right). \tag{2.6}$$

Proof For $p=1$, the result follows from Lemma 1. Suppose (2.6) is true for a certain p. Then $\mu_{n,m-j}^{[p]}(x) = O\left(n^{-[(m-j+1)/2]}\right), \forall\, 0 \le j \le m$.

Also, since $\mu_{n,m-j}^{[p]}(x)$ is a polynomial in x of degree $(m-j)$, we have

$$D^i\left(\mu_{n,m-j}^{[p]}(x)\right) = O\left(n^{-[(m-j+1)/2]}\right), \ \forall\, 0 \le i \le m-j.$$

Now, applying Lemma 2 and Lemma 1, we obtain

$$\begin{aligned} \mu_{n,m}^{[p+1]}(x) &= \sum_{j=0}^{m}\sum_{i=0}^{m-j} O\left(n^{-[(m-j+1)/2]}\right) O\left(n^{-[(i+j+1)/2]}\right) \\ &= O\left(n^{-[(m+1)/2]}\right). \end{aligned}$$

Thus, the result holds for $p+1$. Hence, the lemma is proved by induction $\forall\, p \in \mathbb{N}$. □

Let $0 < a < b < \infty$, $f \in C[a,b]$ and $[a_1,b_1] \subset (a,b)$. Then, for sufficiently small $\eta > 0$, the Steklov mean $f_{\eta,m}$ of the m-th order corresponding to f is defined as follows:

$$f_{\eta,m}(t) = \eta^{-m} \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \left(f(t) + (-1)^{m-1}\Delta^m_{\sum_{i=1}^m t_i} f(t)\right) \prod_{i=1}^{m} dt_i, \ \ t \in [a_1,b_1],$$

where Δ_h^m is the m-th order forward difference operator with step length h.

Lemma 4 *[7] For the function* $f_{\eta,m}$, *we have*

(a) $f_{\eta,m}$ *has derivatives up to order* m *over* $[a_1,b_1]$;

(b) $\|f_{\eta,m}^{(r)}\|_{C[a_1,b_1]} \le C_r\ \eta^{-r}\omega_r(f,\eta,[a,b]), r = 1,2,...,m;$

(c) $\|f - f_{\eta,m}\|_{C[a_1,b_1]} \le C_{m+1}\ \omega_m(f,\eta,[a,b])$;

(d) $\|f^{(m)}_{\eta,m}\|_{C[a_1,b_1]} \leq C_{m+2}\ \eta^{-m}\|f\|_{C[a,b]}$;

(e) $\|f_{\eta,m}\|_{C[a_1,b_1]} \leq C_{m+3}\ \|f\|_{C[a,b]}$,

where $C_i's$ are certain constants that depend on i but are independent of f and η.

Lemma 5 *For l-th moment ($l \in \mathbb{N}$) of $T_{n,k}$, we find that*

$$T_{n,k}((t-x)^l : x) = O(n^{-k}).$$

Proof For $k = 1$, the result holds from Lemma 1. Let us assume that it is true for a certain k, then by the definition of $T_{n,k}$ we get

$$\begin{aligned} T_{n,k+1}\left((t-x)^l;x\right) &= \sum_{r=1}^{k+1}(-1)^{r+1}\binom{k+1}{r}B_n^r((t-x)^l;x) \\ &= T_{n,k}\left((t-x)^l;x\right) \\ &+ \sum_{r=1}^{k+1}(-1)^{r+1}\binom{k}{r-1}B_n^r((t-x)^l;x) \\ &= I_1 + I_2, \text{ say.} \end{aligned} \tag{2.7}$$

Now, by Lemma 2, the second sum on the right of (2.7) is equal to

$$\begin{aligned} \sum_{r=0}^{k}(-1)^r\binom{k}{r}\mu_{n,l}^{[r+1]}(x) &= \mu_{n,l}(x) - \sum_{j=1}^{l}\sum_{i=0}^{l-j}\binom{l}{j}\frac{1}{i!}\left[D^iT_{n,k}\left((t-x)^{l-j};x\right)\right]\mu_{n,i+j}(x) \\ &- \sum_{i=0}^{l}\frac{1}{i!}\left[D^iT_{n,k}\left((t-x)^l;x\right)\right]\mu_{n,i}(x) \\ &= -\sum_{j=1}^{l-1}\sum_{i=0}^{l-j}\binom{l}{j}\frac{1}{i!}\left[D^iT_{n,k}\left((t-x)^{l-j};x\right)\right]\mu_{n,i+j}(x) \\ &- \sum_{i=1}^{l}\frac{1}{i!}\left[D^iT_{n,k}\left((t-x)^l;x\right)\right]\mu_{n,i}(x) - T_{n,k}\left((t-x)^l;x\right). \end{aligned} \tag{2.8}$$

Hence, on combining (2.7–2.8) and then using Lemma 1, we obtain

$$T_{n,k+1}\left((t-x)^l;x\right) = O\left(n^{-(k+1)}\right).$$

Thus, the result holds for $k+1$. Hence, the lemma follows by induction for all $k \in \mathbb{N}$. □

3. Main results

First, we establish Voronovskaja type asymptotic formula for the operators $T_{n,k}(.;x)$.

Theorem 1 *Let f be bounded and integrable on $[0,\infty)$, admitting a derivative of order $2k$ at a point $x \in [0,\infty)$. Then we have*

$$\lim_{n\to\infty} n^k[T_{n,k}(f;x) - f(x)] = \sum_{\nu=1}^{2k} \frac{f^{(\nu)}(x)}{\nu!} Q(\nu,k,x) \tag{3.9}$$

and

$$\lim_{n\to\infty} n^k[T_{n,k+1}(f;x) - f(x)] = 0, \tag{3.10}$$

where $Q(\nu,k,x)$ are certain polynomials in x of degree at most ν. Further, the limits in (3.9) and (3.10) hold uniformly in $[a,b]$ if $f^{(2k)}(x)$ is continuous on $(a-\eta, b+\eta) \subset [0,\infty), \eta > 0$.

Proof By a partial Taylor's expansion of f, we have

$$f(t) = \sum_{\nu=0}^{2k} \frac{f^{(\nu)}(x)}{\nu!}(t-x)^\nu + \epsilon(t,x)(t-x)^{2k}, \tag{3.11}$$

where $\epsilon(t,x) \to 0$ as $t \to x$.

Operating by $T_{n,k}$ on both sides of (3.11), we get

$$\begin{aligned} n^k\left[T_{n,k}(f;x) - f(x)\right] &= n^k \sum_{\nu=1}^{2k} \frac{f^{(\nu)}(x)}{\nu!} T_{n,k}\left((t-x)^\nu; x\right) \\ &+ n^k T_{n,k}\left(\epsilon(t,x)(t-x)^{2k}; x\right) \\ &= I_1 + I_2, \text{ say.} \end{aligned}$$

Making use of Lemma 5, we obtain

$$I_1 = \sum_{\nu=1}^{2k} \frac{f^{(\nu)}(x)}{\nu!} Q(\nu,k,x) + o(1),$$

where $Q(\nu,k,x)$ is the coefficient of n^{-k} in $T_{n,k}\left((t-x)^\nu;x\right), \nu = 1,2,...,2k$.

For a given $\epsilon' > 0$, we can find a $\delta > 0$ such that $|\epsilon(t,x)| < \epsilon'$ whenever $0 < |t-x| < \delta$ and for $|t-x| \geq \delta$, $|\epsilon(t,x)| \leq K$ for some $K > 0$. Suppose $\chi(t)$ is the characteristic function of the interval $(x-\delta, x+\delta)$, then

$$\begin{aligned} |I_2| &\leq n^k \sum_{r=1}^{k} \binom{k}{r} B_n^r\left(|\epsilon(t,x)|(t-x)^{2k}\chi(t); x\right) \\ &+ n^k \sum_{r=1}^{k} \binom{k}{r} B_n^r\left(|\epsilon(t,x)|(t-x)^{2k}(1-\chi(t)); x\right) \\ &= I_3 + I_4, \text{ say.} \end{aligned}$$

In view of Lemma 3, we have $I_3 = \epsilon' O(1)$.
Again, using Lemma 3,

$$\begin{aligned} I_4 &\leq Kn^k \sum_{r=1}^{k} \binom{k}{r} B_n^r \left((t-x)^{2s} \delta^{2k-2s}; x\right) \\ &\leq Kn^k \delta^{2k-2s} \sum_{r=1}^{k} \binom{k}{r} B_n^r \left((t-x)^{2s}; x\right) \\ &= O(n^{k-s}) = o(1), \text{ for any integer } s > k. \end{aligned}$$

Thus, due to the arbitrariness of ϵ', it follows that $|I_2| = o(1)$.
Combining the estimates of I_1 and I_2, we obtain (3.9). Assertion (3.10) follows along the same lines by using the fact that

$$T_{n,k+1}\left((t-x)^{\nu}; x\right) = O\left(n^{-(k+1)}\right), \text{ for all } \nu \in \mathbb{N}.$$

The uniformity assertion follows due to the uniform continuity of $f^{(2k)}$ on $[a,b]$ which enables δ to become independent of x and the uniformness of the term $o(1)$ in the estimate of I_1. □

In our next result, we obtain an estimate of the degree of approximation of a function with specified smoothness.

Theorem 2 *Let $p \in \mathbb{N}, 1 \leq p \leq 2k$ and f be bounded and integrable on $[0,\infty)$. If $f^{(p)}$ exists and is continuous on $(a-\eta, b+\eta) \subset [0,\infty)$, for some $\eta > 0$, then*

$$\|T_{n,k}(f;x) - f(x)\|_{C[a,b]} \leq \max\left\{C_1\, n^{-p/2} \omega\left(f^{(p)}, n^{-1/2}\right), C_2\, n^{-k}\right\}, \tag{3.12}$$

where $C_1 = C_1(k,p)$, $C_2 = C_2(k,p,f)$ and $\omega\left(f^{(p)}, \delta\right)$ is the modulus of continuity of $f^{(p)}$ on $(a-\eta, b+\eta)$.

Proof By our hypothesis, we may write for all $t \in [0,\infty)$ and $x \in [a,b]$

$$\begin{aligned} f(t) &= \sum_{\nu=0}^{p} \frac{f^{(\nu)}(x)}{\nu!}(t-x)^{\nu} + \frac{f^{(p)}(\xi) - f^{(p)}(x)}{p!}(t-x)^p \chi(t) \\ &+ F(t,x)\left(1-\chi(t)\right), \end{aligned} \tag{3.13}$$

where $\chi(t)$ is the characteristic function of $(a-\eta, b+\eta)$, ξ lies between t and x and $F(t,x)$ is defined as

$$F(t,x) = f(t) - \sum_{\nu=0}^{p} \frac{f^{(\nu)}(x)}{\nu!}(t-x)^{\nu},$$

$\forall\ t \in [0,\infty) \setminus (a-\eta, b+\eta)$ and $x \in [a,b]$.

Now, operating by $T_{n,k}$ on both sides of (3.13), we get

$$\begin{aligned} T_{n,k}(f(t);x) &= \sum_{\nu=0}^{p} \frac{f^{(\nu)}(x)}{\nu!} T_{n,k}((t-x)^{\nu};x) + T_{n,k}\left(\frac{f^{(p)}(\xi) - f^{(p)}(x)}{p!}(t-x)^p \chi(t); x\right) \\ &+ T_{n,k}\big(F(t,x)\left(1-\chi(t)\right); x\big) \\ &= I_1 + I_2 + I_3, \text{ say.} \end{aligned}$$

On an application of Lemma 5, we obtain

$$I_1 \quad = f(x) + O(n^{-k}), \text{uniformly in } x \in [a,b].$$

Next, applying Schwarz inequality and Lemma 3, we get

$$\begin{aligned} |I_2| &\leq \sum_{r=1}^{k} \binom{k}{r} \frac{\omega\left(f^{(p)}, \delta\right)}{p!} B_n^r \left(|t-x|^p + \frac{|t-x|^{p+1}}{\delta}; x \right) \\ &= \omega\left(f^{(p)}, \delta\right) [O(n^{-p/2}) + \delta^{-1} O(n^{-(p+1)/2})] \\ &= \omega\left(f^{(p)}, n^{-1/2}\right) O(n^{-p/2}), \end{aligned}$$

uniformly in $x \in [a,b]$, on choosing $\delta = n^{-1/2}$.

Lastly, to estimate $I_3 = T_{n,k}\left(F(t,x)(1-\chi(t))x\right)$, we proceed as follows:

On an application of Lemma 3, for any integer $s > \max\{\frac{p}{2}, k\}$, we obtain

$$\begin{aligned} |I_3| &\leq M \sum_{r=1}^{k} \binom{k}{r} B_n^r \left(|t-x|^p; x\right) \\ &\leq M \sum_{r=1}^{k} \binom{k}{r} B_n^r \left(\frac{(t-x)^{2s}}{\delta^{2s-p}}; x \right) \\ &= \frac{M}{\delta^{2s-p}} \sum_{r=1}^{k} \binom{k}{r} \mu_{n,2s}^{[r]}(x) \\ &= O(n^{-s}) = o(n^{-k}), \text{ uniformly in } x \in [a,b]. \end{aligned}$$

Now, combining the estimates of I_1, I_2 and I_3, (3.12) is established. This completes the proof. □

In our next result, we obtain an estimate of the degree of approximation of a function in terms of higher order modulus of continuity which is an improvement of Theorem 2.

Theorem 3 *Let f be bounded and integrable on $[0,\infty)$. If f is continuous on $(a-\lambda, b+\lambda) \subset (0,\infty)$, for some $\lambda > 0$, then*

$$\left\|T_{n,k}(f;.) - f\right\|_{C[a,b]} \leqslant C_1 n^{-k} + C_2 \omega_{2k}\left(f; n^{-1/2}; (a-\lambda, b+\lambda)\right),$$

where C_1 and C_2 are independent of f and n.

Proof Let $f_{\eta,2k}$ be the Steklov mean of $2k$-th order corresponding to f. Then we can write

$$\begin{aligned} \|T_{n,k}(f;.) - f\|_{C[a,b]} &\leq \|T_{n,k}\left(f - f_{\eta,2k};.\right)\|_{C[a,b]} + \left\|T_{n,k}\left(f_{\eta,2k};.\right) - f_{\eta,2k}\right\|_{C[a,b]} \\ &+ \left\|f - f_{\eta,2k}\right\|_{C[a,b]} \\ &= S_1 + S_2 + S_3, \text{ say.} \end{aligned}$$

By property (c) of the Steklov mean, we get

$$S_3 \leq C\ \omega_{2k}(f, \eta, (a-\lambda, b+\lambda)).$$

Next, applying Theorem 1 and the interpolation property [4], for each $\nu = 1, 2, ..., 2k$, it follows that

$$\begin{aligned} S_2 &\leq C\, n^{-k} \sum_{\nu=1}^{2k} \|f_{\eta,2k}^{(\nu)}\|_{C[a,b]} \\ &\leqslant C\, n^{-k} \left(\|f_{\eta,2k}\|_{C[a,b]} + \|f_{\eta,2k}^{(2k)}\|_{C[a,b]} \right). \end{aligned}$$

Hence, by properties (b) and (e) of Steklov mean, we have

$$S_2 \leqslant Cn^{-k} + C'\eta^{-2k} n^{-k} \omega_{2k}(f, \eta, (a-\lambda, b+\lambda)).$$

Let a^* and b^* be such that $a - \lambda < a^* < a < b < b^* < b + \lambda$ and ψ be the characteristic function of $[a^*, b^*]$. Then, by using Hölder's inequality, Lemma 1 and property (c) of Steklov mean, we get

$$\begin{aligned} S_1 &\leq \|T_{n,k}\big(\psi(t)(f(t) - f_{\eta,2k}(t));.\big)\|_{C[a,b]} + \|T_{n,k}\big((1-\psi(t))(f(t) - f_{\eta,2k}(t));.\big)\|_{C[a,b]} \\ &\leq C\left(\|f - f_{\eta,2k}\|_{C[a^*,b^*]} + n^{-m}\right), \forall m > 0 \\ &\leq C\left(\omega_{2k}(f, \eta, (a-\lambda, b+\lambda)) + n^{-m}\right). \end{aligned}$$

Now, choosing $m \geq k$ and $\eta = n^{-1/2}$ in the estimates of S_1, S_2 and S_3, the result follows. □

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