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On characterization and stability of alternate dual of g-frames

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Abstract: One of the essential applications of frames is that they lead to expansions of vectors in the underlying Hilbert space in terms of the frame elements. In this decomposition, dual frames have a key role. G-frames, introduced by Sun, cover many other recent generalizations of frames. In this paper, we give some characterizations of dual g-frames. Moreover, we prove that if two g-frames are close to each other, then we can find duals of them which are close to each other.

Key words: Frame, dual frame, g-Riesz sequence, alternate dual, g-frame

1. Introduction and preliminaries

Frames were first introduced by Duffin and Schaeffer [11]. Today they are a very useful tool in wavelet theory, signal processing and many other fields [4, 5, 6, 15]. The main feature of a frame is to represent every element of underlying Hilbert space as a linear combination of the frame elements. Specifically, if $\mathcal{H}$ is a separable Hilbert space and $\{f_i\}_{i=1}^{\infty}$ is a frame for $\mathcal{H}$, then any $f \in \mathcal{H}$ can be expressed as

$$f = \sum_{i=1}^{\infty} \langle f, h_i \rangle f_i,$$

for some dual frame $\{h_i\}_{i=1}^{\infty}$ of $\{f_i\}_{i=1}^{\infty}$. A dual frame in which the coefficients $\langle f, h_i \rangle$ has minimal l^2-norm for all $f \in \mathcal{H}$ is called the canonical dual. Unfortunately, computing the canonical dual is highly non-trivial in general. Moreover, the frame $\{f_i\}_{i=1}^{\infty}$ might have a certain structure which is not shared by the canonical dual. This complication appears, for example, if $\{f_i\}_{i=1}^{\infty}$ is a wavelet frame: there are cases where the canonical dual of a wavelet frame does not have the wavelet structure [10]. Hence, we try to find more general choices of duals.

Recently, various generalizations of frames have been proposed. For example, continuous frames [2, 3, 14], g-frames [19], fusion frames [7], von Neumann-Schatten frames [18] and so on. The notion of g-frames was introduced by Sun and was developed by several authors [1, 17, 22]. One of our results in this paper is to establish some characterizations of dual g-frames.

The question of stability of frames, which plays an important role, states that if $\{f_i\}_{i=1}^{\infty}$ is a frame in $\mathcal{H}$ and $\{g_i\}_{i=1}^{\infty}$ in some sense is close to $\{f_i\}_{i=1}^{\infty}$, does it follow that $\{g_i\}_{i=1}^{\infty}$ is also a frame? The stability of

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frames has been given by Favier and Zalik [12] and subsequently by many other authors [8, 9, 16]. The stability of dual frames is also important in practice. However, most of the known results on this topic are stated about canonical dual; see [13] for frames and [19, 22] for g-frames. They show that if two frames (or g-frames) are close to each other, so are their dual frames in the same sense. Clearly, the argument for alternate dual is more complicated because the structure of alternate duals may be different. In this paper, we show that for a perturbed g-frame, one can construct always an alternate dual which is close to the dual of original g-frame.

2. Basic results of g-frames

Let $\mathcal{U}$ and $\mathcal{V}$ be two Hilbert spaces and $\{\mathcal{V}_j\}_{j\in J}$ be a sequence of closed subspaces of $\mathcal{V}$, where J is a subset of $\mathbb{Z}$. A sequence $\{\Lambda_j\}_{j\in J} \subseteq \mathcal{B}(\mathcal{U}, \mathcal{V}_j)$ of bounded operators from $\mathcal{U}$ to $\mathcal{V}_j$ is said to be a *generalized frame*, or simply a *g-frame*, for $\mathcal{U}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ if there are two positive constants A and B such that

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} \|\Lambda_j f\|^2 \leq B\|f\|^2, \quad \forall f \in \mathcal{U}. \tag{2.1}$$

We call A and B the *lower* and *upper* frame bounds, respectively. If the right-hand side of (2.1) holds, it is said to be a g-Bessel sequence. Note that if $\mathcal{V}_j = \mathbb{C}$ and $\Lambda_j f = \langle f, f_j\rangle, f \in \mathcal{U}, j \in J$, then we obtain the usual definition of a frame.

Let $\{\Lambda_j\}_{j\in J}$ be a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ with bounds A and B. Then the g-frame operator $S : \mathcal{U} \to \mathcal{U}$ defined by

$$Sf = \sum_{j\in J} \Lambda_j^* \Lambda_j f, \quad \forall f \in \mathcal{U} \tag{2.2}$$

is a bounded invertible operator. Take $\widetilde{\Lambda_j} = \Lambda_j S^{-1}$. Then $\{\widetilde{\Lambda_j}\}_{j\in J}$ is also a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ with bounds B^{-1} and A^{-1}. This leads to the generalized reconstruction formula

$$f = \sum_{j\in J} \Lambda_j^* \widetilde{\Lambda_j} f = \sum_{j\in J} \widetilde{\Lambda_j}^* \Lambda_j f, \quad \forall f \in \mathcal{U}.$$

A sequence $\{\Lambda_j\}_{j\in J}$ is called a *g-Riesz basis* for $\mathcal{U}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ if the sequence $\{\Lambda_j\}_{j\in J}$ is g-complete, i.e., $\{f \in \mathcal{U} : \Lambda_j f = 0 \quad \text{for all} \quad j \in J\} = \{0\}$ and there are two positive constants A and B such that

$$A \sum_{j\in J_1} \|g_j\|^2 \leq \left\| \sum_{j\in J_1} \Lambda_j^* g_j \right\|^2 \leq B \sum_{j\in J_1} \|g_j\|^2,$$

for any finite subset $J_1 \subseteq J$ and $g_j \in \mathcal{V}_j$, $j \in J_1$. The constants A and B are called the lower and upper g-Riesz bounds, respectively. For a more general statement of this fact see [19, 22]. In order to obtain our main results, we need the following operator.

Definition 2.1 *Let $\{\Lambda_j\}_{j\in J} \subseteq \mathcal{B}(\mathcal{U}, \mathcal{V}_j)$ be a sequence of bounded operators. Define the synthesis operator by*

$$T : \oplus\mathcal{V}_j \to \mathcal{U}; \quad T\{g_j\}_{j\in J} := \sum_{j\in J} \Lambda_j^* g_j.$$

A straightforward calculation gives that $\{\Lambda_j\}_{j\in J}$ *is a g-Bessel with bound* B *if and only if* T *is well-defined and* $\|T\| \leq \sqrt{B}$. *Furthermore,* $\{\Lambda_j\}_{j\in J}$ *is a g-frame if* T *is onto.*

We can summarize the properties of T as follows:

Proposition 2.2 *[22] Let* $\{\Lambda_j\}_{j\in J} \subseteq \mathcal{B}(\mathcal{U}, \mathcal{V}_j)$. *Then the following assertions hold.*

1. $\{\Lambda_j\}_{j\in J}$ *is a g-Bessel sequence with a bound* B *if and only if* T *is well-defined and* $\|T\| \leq \sqrt{B}$.
2. $\{\Lambda_j\}_{j\in J}$ *is a g-Riesz basis with bounds* A, B *if and only if* T *is a linear homeomorphism and*
$$A\|g\|^2 \leq \|Tg\|^2 \leq B\|g\|^2, \quad (g = \{g_j\}_{j\in J} \in \oplus\mathcal{V}_j).$$
3. *If* $\{\Lambda_j\}_{j\in J}$ *is a g-frame and* S, *its g-frame operator, is defined by (2.2), then* $S = TT^*$, *where* $T^* : \mathcal{U} \to \oplus\mathcal{V}_j; f \mapsto \{\Lambda_j f\}_{j\in J}$ *is the adjoint of* T.

The following proposition is a criterion for a g-frame to be a g-Riesz basis. One can find a frame version of this proposition in [21].

Proposition 2.3 *Let* $\{\Lambda_j\}_{j\in J}$ *be a g-frame for* $\mathcal{U}$ *with respect to* $\{\mathcal{V}_j\}_{j\in J}$. *Then* $\{\Lambda_j\}_{j\in J}$ *is a g-Riesz basis if and only if* $rang(T^*) = \oplus\mathcal{V}_j$.

Proof Let $\{\mathcal{V}_j\}_{j\in J}$ be a g-frame and T^* be onto. By Theorem 2.8 of [22] it is sufficient to show that if $\sum_{j\in J} \Lambda_j^* g_j = 0$, for $\{g_j\}_{j\in J} \in \oplus\mathcal{V}_j$, then $g_j = 0$ for all $j \in J$. To see this, consider $\{g_j\}_{j\in J} \in \oplus\mathcal{V}_j$ and choose $f \in \mathcal{U}$ such that $T^* f = \{g_j\}_{j\in J}$. Then

$$0 = \sum_{j\in J} \Lambda_j^* g_j = \sum_{j\in J} \Lambda_j^* \Lambda_j f = Sf.$$

So $f = 0$. This follows that $\{g_j\}_{j\in J} = T^* f = 0$. To prove the converse, we note that the definition of g-Riesz basis implies that T is bounded below. □

3. Characterizations of duals

In this section we first review the definitions of canonical and alternate duals of g-frames then we give some characterizations of them.

Definition 3.1 *Two g-Bessel sequences* $\{\Lambda_j\}_{j\in J}$ *and* $\{\Gamma_j\}_{j\in J}$ *for* $\mathcal{U}$ *with respect to* $\{\mathcal{V}_j\}_{j\in J}$ *are called dual g-frames if*

$$f = \sum_{j\in J} \Lambda_j^* \Gamma_j f, \quad \forall f \in \mathcal{U}. \tag{3.1}$$

It is not difficult to see that two such g-Bessel sequences indeed are g-frames, and we will say that the g-frame $\{\Gamma_j\}_{j\in J}$ *is dual to* $\{\Lambda_j\}_{j\in J}$, *and vice versa. If* S *is the g-frame operator* $\{\Lambda_j\}_{j\in J}$, *then* $\{\widetilde{\Lambda_j}\} = \{\Lambda_j S^{-1}\}$ *is a dual for* $\{\Lambda_j\}_{j\in J}$; *it is called the canonical dual, a dual which is not the canonical dual is called an alternate dual, or simply a dual.*

The following lemma gives some elementary characterizations of duals in terms of the synthesis operators:

Lemma 3.2 *Let $\{\Lambda_j\}_{j\in J}$ and $\{\Gamma_j\}_{j\in J}$ be two g-frames with the synthesis operators T and U, respectively. Then the following are equivalent:*

1. $\{\Gamma_j\}_{j\in J}$ *is a dual for* $\{\Lambda_j\}_{j\in J}$.
2. $TU^* = I$.
3. $UT^* = I$.
4. $(T^*U)^2 = T^*U$.

W. Sun has shown in [19, Theorem 3.1] that every g-frame can be considered as a frame. More precisely, if $\{\Lambda_j\}_{j\in J}$ is a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$, then there exists a frame $\{u_{j,k}\}_{j\in J,k\in K_j}$ of $\mathcal{U}$ such that

$$u_{j,k} = \Lambda_j^* e_{j,k}, \quad j \in J, k \in K_j \tag{3.2}$$

where $\{e_{j,k} : k \in K_j\}$ is an orthonormal basis for $\mathcal{V}_j$ and K_j is a subset of $\mathbb{Z}$, $j \in J$. Now all properties of g-frame $\{\Lambda_j\}_{j\in J}$ can be read as the same properties of the frame $\{u_{j,k}\}_{j\in J,k\in K_j}$ of $\mathcal{U}$. Such as, $\{\Lambda_j\}_{j\in J}$ is a g-Riesz basis, tight g-frame if and only if $\{u_{j,k}\}_{j\in J,k\in K_j}$ is a Riesz basis, tight frame, respectively. Furthermore, a dual of $\{\Lambda_j\}_{j\in J}$ can be obtained from a dual of $\{u_{j,k}\}_{j\in J,k\in K_j}$. Indeed, let $\{u'_{j,k}\}_{j\in J,k\in K_j}$ be a dual frame of $\{u_{j,k}\}_{j\in J,k\in K_j}$, and define the g-frame $\{\Gamma_j\}_{j\in J}$ by

$$\Gamma_j f = \sum_k \langle f, u'_{j,k}\rangle e_{j,k}, \quad \forall f \in \mathcal{U}.$$

Then,

$$\begin{aligned}\sum_j \Lambda_j^*\Gamma_j f &= \sum_{j\in J}\sum_k \Lambda_j^*\langle f, u'_{j,k}\rangle e_{j,k} \\ &= \sum_j \sum_k \langle f, u'_{j,k}\rangle u_{j,k} = f\end{aligned}$$

for all $f \in \mathcal{U}$. That is, $\{\Gamma_j\}_{j\in J}$ is a dual for $\{\Lambda_j\}_{j\in J}$.

Now we give a characterization of dual g-frame with respect to left inverse of the synthesis operator. Note that $\{\delta_j\}_{j\in J}$ denotes the canonical basis of $l^2(J)$.

Theorem 3.3 *Let $\{\Lambda_j\}_{j\in J}$ be a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ with the synthesis operator T. Then a g-frame $\{\Gamma_j\}_{j\in J}$ is a dual for $\{\Lambda_j\}_{j\in J}$ if and only if $\Gamma_j^* e_{j,k} = \phi(e_{j,k}\delta_j)$, $k \in K_j$, $j \in J$ where $\phi : \oplus\mathcal{V}_j \to \mathcal{U}$ is a bounded left inverse for T^*.*

Proof First note that if $\{g_j\} \in \oplus\mathcal{V}_j$, then

$$\{g_j\}_{j\in J} = \sum_{j\in J} g_j\delta_j = \sum_{j\in J}\sum_{k\in K_j} \langle g_j, e_{j,k}\rangle e_{j,k}\delta_j.$$

Roughly speaking $\{e_{j,k}\delta_j\}_{j,k}$ is an orthonormal basis of $\oplus\mathcal{V}_j$. Suppose now that $\{\Lambda_j\}_{j\in J}$ is a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ and $\{u_{j,k}\}_{j\in J,k\in K_j}$ be defined as in (3.2). If ϕ is a bounded left inverse of T^* and $\Gamma_j^* e_{j,k} = \phi(e_{j,k}\delta_j)$, $k \in K_j$, $j \in J$ then for all $f \in \mathcal{U}$ we have

$$\begin{aligned} f = \phi T^* f &= \phi(\sum_{k,j}\langle \Lambda_j f, e_{j,k}\rangle e_{j,k}\delta_j) \\ &= \sum_j \sum_k \langle f, \Lambda_j^* e_{j,k}\rangle \phi(e_{j,k}\delta_j) \\ &= \sum_j \sum_k \langle f, u_{j,k}\rangle \Gamma_j^* e_{j,k} \\ &= \sum_j \Gamma_j^* \big(\sum_k \langle f, u_{j,k}\rangle e_{j,k}\big) = \sum_j \Gamma_j^* \Lambda_j f, \end{aligned}$$

which is equivalent to (3.1). To show the converse assume that $\{\Gamma_j\}_{j\in J}$ is an alternate dual of $\{\Lambda_j\}$, then a calculation as above shows that

$$\sum_j \sum_k \langle f, u_{j,k}\rangle \Gamma_j^* e_{j,k} = f = \sum_j \sum_k \langle f, u_{j,k}\rangle \phi(e_{j,k}\delta_j), \quad \forall f \in \mathcal{U}.$$

Combining this with the fact that $\{e_{j,k} : k \in K_j\}$ is an orthonormal basis for $\oplus\mathcal{V}_j$ shows that $\Gamma_j^* e_{j,k} = \phi(e_{j,k}\delta_j)$, $k \in K_j$, $j \in J$. □

The following characterization of alternate duals shows that the difference between an alternate dual and canonical dual can be considered as a bounded operator.

Theorem 3.4 *Let $\{\Lambda_j\}_{j\in J}$ be a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ with the bounds A and B. Then there exists a one-to-one correspondence between duals of $\{\Lambda_j\}_{j\in J}$ and operators $\psi \in B(\mathcal{U}, \oplus\mathcal{V}_j)$ such that $T\psi = 0$.*

Proof First assume that $\{\Gamma_j\}$ is a dual of $\{\Lambda_j\}_{j\in J}$ with the bounds A_1 and B_1 and S is the g-frame operator of $\{\Lambda_j\}_{j\in J}$. Define $\psi : \mathcal{U} \to \oplus\mathcal{V}_j; f \mapsto \psi f$ by

$$(\psi f)_j = \Gamma_j f - \Lambda_j S^{-1} f, \quad (j \in J).$$

Then ψ is a bounded operator. Indeed,

$$\begin{aligned} \|\psi f\|^2 &= \sum_j \|\Gamma_j f - \Lambda_j S^{-1} f\|^2 \\ &= \sum_j \|\Gamma_j f\|^2 + \sum_j \|\Lambda_j S^{-1} f\|^2 + 2\left(\sum_j \|\Gamma_j f\|^2\right)^{\frac{1}{2}} \left(\sum_j \|\Lambda_j f\|^2\right)^{\frac{1}{2}} \\ &= (B_1 + A^{-1} + 2\sqrt{B_1}\sqrt{A^{-1}})\|f\|^2 \end{aligned}$$

for all $f \in \mathcal{U}$. Moreover, by (3.1) we obtain

$$T\psi f = \sum_j \Lambda_j^* (\psi f)_j = \sum_j \Lambda_j^* \Gamma_j f - \Lambda_j^* \Lambda_j S^{-1} f = 0.$$

Conversely, let $\psi \in B(\mathcal{U}, \oplus \mathcal{V}_j)$ and $T\psi = 0$. Take

$$\Gamma_j f = \Lambda_j S^{-1} f + (\psi f)_j, \quad \forall f \in \mathcal{U}.$$

It is easy to see that $\{\Gamma_j\}_{j\in J}$ is a g-Bessel sequence. Furthermore,

$$\sum_j \Lambda_j^* \Gamma_j f = \sum_j \Lambda_j^* \Lambda_j S^{-1} f + \Lambda_j^* (\psi f)_j = f + T\psi f = f.$$

Thus, $\{\Gamma_j\}_{j\in J}$ is a dual for $\{\Lambda_j\}_{j\in J}$ as a g-Bessel sequence. This immediately follows that $\{\Gamma_j\}_{j\in J}$ is also a g-frame. □

Now we are ready to state our main result about the stability of alternate duals.

Theorem 3.5 *Let $\{\Lambda_j\}_{j\in J}$ and $\{\Lambda'_j\}_{j\in J}$ be two g-frames for $\mathcal{U}$ with respect to $\{\mathcal{V}_j\}_{j\in J}$ with the bounds A_1, B_1 and A_2, B_2, respectively. Also let $\{\Gamma_j\}_{j\in J}$ be a fix alternate dual for $\{\Lambda_j\}_{j\in J}$. If $\{\Lambda_j - \Lambda'_j\}_{j\in J}$ is a g-Bessel sequence with sufficiently small bound $\epsilon > 0$, then there exists an alternate dual $\{\Gamma'_j\}_{j\in J}$ for Λ'_j such that $\{\Gamma_j - \Gamma'_j\}_{j\in J}$ is also g-Bessel and its bound is a multiple of ϵ.*

Proof Denote the synthesis operators of $\{\Lambda_j\}_{j\in J}$ and $\{\Lambda'_j\}_{j\in J}$ with T_1 and T_2, respectively. Also we consider $S_1 = T_1 T_1^*$ and $S_2 = T_2 T_2^*$ as their g-frame operators. By Theorem 3.4 there exists a $\psi \in B(\mathcal{U}, \oplus \mathcal{V}_j)$ such that $T_1 \psi = 0$ and

$$\Gamma_j f = \Lambda_j S_1^{-1} f + (\psi f)_j \tag{3.3}$$

for all $f \in \mathcal{U}$. Define

$$M_j f = \Lambda'_j S_2^{-1} f + (\psi f)_j, \quad \forall f \in \mathcal{U}. \tag{3.4}$$

Clearly, $\{M_j\}_{j\in J}$ is a g-Bessel with the bound $A_2^{-1} + \|\psi\|^2 + 2A_2^{\frac{-1}{2}} \|\psi\|$.

If we denote the synthesis operator of $\{M_j\}_{j\in J}$ by T_3, then

$$\begin{aligned} \|f - T_2 T_3^* f\| &= \left\| f - \sum_j (\Lambda'_j)^* M_j f \right\| \\ &= \left\| f - \sum_j (\Lambda'_j)^* \Lambda'_j S_2^{-1} f - \sum_j (\Lambda'_j)^* (\psi f)_j \right\| \\ &= \|T_2 \psi f\| \\ &= \|T_2 \psi f - T_1 \psi f\| \\ &\leq \|T_1 - T_2\| \|\psi\| \|f\| \leq \epsilon \|\psi\| \|f\|, \end{aligned}$$

where in the last inequality we have used Proposition 2.2 for the g-Bessel sequence $\{\Gamma_j - \Gamma'_j\}_{j\in J}$. Thus, by assumption, $T_2 T_3^*$ is invertible for sufficiently small $\epsilon > 0$. In particular,

$$\|I - T_2 T_3^*\| \leq \epsilon \|\psi\|. \tag{3.5}$$

Hence,

$$f = (T_2 T_3^*)^{-1} T_2 T_3^* f = \sum (\Lambda'_j)^* (T_2 T_3^*)^{-1} M_j f, \quad \forall f \in \mathcal{U}.$$

This follows that $\{\Gamma'_j\}_{j\in J} := \{(T_2T_3^*)^{-1}M_j\}_{j\in J}$ is a dual g-frame for $\{\Lambda'_j\}_{j\in J}$. We claim that $\{\Gamma'_j\}_{j\in J}$ is the desired g-frame.

First, note that

$$\begin{aligned}\|S_1 - S_2\| &= \|T_1T_1^* - T_1T_2^* + T_1T_2^* - T_2T_2^*\| \\ &= \|T_1 - T_2\|(\|T_1\| + \|T_2\|) \\ &\leq \epsilon(\sqrt{B_1} + \sqrt{B_2}).\end{aligned}$$

Hence by using (3.3) and (3.4) for every $g = \{g_j\}_{j\in J} \in \oplus\mathcal{V}_j$, we have

$$\begin{aligned}\left\|\sum_{j\in J}\Gamma_j^*(g) - M_j^*(g)\right\| &= \left\|\sum_{j\in J}S_1^{-1}\Lambda_j^*(g) - S_2^{-1}(\Lambda'_j)^*(g)\right\| \\ &= \left\|\sum_{j\in J}S_1^{-1}\Lambda_j^*(g) - S_2^{-1}\Lambda_j^*(g) + S_2^{-1}\Lambda_j^*(g) - S_2^{-1}(\Lambda'_j)^*(g)\right\| \\ &\leq \|S_1^{-1} - S_2^{-1}\|\left\|\sum_{j\in J}\Lambda_j^*(g)\right\| + \|S_2^{-1}\|\left\|\sum_{j\in J}\Lambda_j^*(g) - (\Lambda'_j)^*(g)\right\| \\ &\leq \|S_2^{-1}\|\,\|S_2 - S_1\|\,\|S_1^{-1}\|\left\|\sum_{j\in J}\Lambda_j^*(g)\right\| + \|S_2^{-1}\|\,\|T_1(g) - T_2(g)\| \\ &\leq \epsilon\left(\frac{B_1 + \sqrt{B_1B_2}}{A_1} + 1\right)\frac{\|g\|}{A_2}.\end{aligned}$$

If we take $W = (T_3T_2^*)^{-1}$, then we obtain from (3.5),

$$\|W\| \leq \frac{1}{1 - \|I - W^{-1}\|} < \frac{1}{1 - \epsilon\|\psi\|},$$

and

$$\|I - W\| = \|W\|\,\|I - W^{-1}\| < \frac{\epsilon\|\psi\|}{1 - \epsilon\|\psi\|}.$$

Consequently,

$$\begin{aligned}\left\|\sum_{j\in J}\Gamma_j^*(g) - (\Gamma'_j)^*(g)\right\| &= \left\|\sum_{j\in J}\Gamma_j^*(g) - M_j^*(T_3T_2^*)^{-1}(g)\right\| \\ &= \left\|\sum_{j\in J}\Gamma_j^*(g) - \Gamma_j^*W(g) + \Gamma_j^*W(g) - M_j^*W(g)\right\| \\ &\leq \|I - W\|\left\|\sum_{j\in J}\Gamma_j^*(g)\right\| + \left\|\sum_{j\in J}\Gamma_j^* - M_j^*\right\|\|W(g)\| \\ &\leq \epsilon\left(\|\psi\|\sqrt{B'} + \frac{B_1 + \sqrt{B_1B_2}}{A_1A_2} + \frac{1}{A_2}\right)\frac{\|g\|}{1 - \epsilon\|\psi\|},\end{aligned}$$

where B' is the upper bound of $\{\Gamma_j\}_{j\in J}$. This completes the proof. □

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