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Generating systems of differential invariants and the theorem on existence for curves in the pseudo-Euclidean geometry

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Abstract: Let $M(n,p)$ be the group of all motions of an n-dimensional pseudo-Euclidean space of index p. It is proved that the complete system of M(n,p)-invariant differential rational functions of a path (curve) is a generating system of the differential field of all $M(n,p)$-invariant differential rational functions of a path (curve), respectively. A fundamental system of relations between elements of the complete system of M(n,p)-invariant differential rational functions of a path (curve) is described.

Key words: Curve, differential invariant, pseudo-Euclidean geometry, Minkowski geometry

1. Introduction

The present paper is a continuation of our paper [18]. Let E_p^n be the n-dimensional pseudo-Euclidean space of index p (that is the space R^n with the scalar product $<x,y>= -x_1y_1 - \cdots - x_py_p + x_{p+1}y_{p+1} + \cdots + x_ny_n$), $O(n,p)$ is the group of all pseudo-orthogonal transformations of E_p^n, $M(n,p)=\{F : E_p^n \to E_p^n \mid Fx = gx + b$, $g \in O(n,p),\ b \in E_p^n\}$ and $SM(n,p) = \{F \in M(n,p) : \det g = 1\}$.

Here, for groups $G = M(n,p)$ and $G = SM(n,p)$, we prove that the complete system of G-invariant differential rational functions of a path (curve) obtained in [18, Theorems 2–3 and Corollaries 1–2] is a generating system of the differential field of all G-invariant differential rational functions of a path (respectively, curve). We describe a fundamental system of relations between elements of the complete system of G-invariant functions of a path (curve) (i.e., global existence theorems for a path and a curve).

For groups $G = M(n,0)$ and $G = SM(n,0)$, the generating system of the differential field of all G-invariant differential rational functions of a path in the Euclidean space E_0^n was obtained in [16]. The Frenet-Serret equation for both time-like and space-like curves in spaces E_1^3 and E_1^4 is given in [12, 13, 22]. In papers [1, 4, 5, 8, 14, 19, 20], the Frenet-Serret equation is extended from non-null curves in E_1^3, E_1^4 and E_2^4 to null (lightlike, isotropic) curves. For arbitrary n, the Frenet-Serret equation is obtained for the Lorentz space E_1^n in [2], [9, pp. 52–76]. The Frenet-Serret equation in E_p^n for arbitrary n and index p is considered in [3, 6, 7]. Existence and rigidity (that is uniqueness) theorems for curves in spaces E_1^3 and E_1^4 are studied in [5] and thesis [13] (in the case with constant coefficients). In papers [5, 14], existence and rigidity theorems are

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extended from non-null curves in E_1^3 and E_1^4 to null curves. For arbitrary n, existence and rigidity theorems are extended to the Lorentz space E_1^n and to the space E_2^n in [9, pp. 52–76]; and [10, 11]. For arbitrary n and index p, existence and rigidity theorems for curves in E_p^n are considered in the paper [6]. In these papers, existence theorems are local. The rigid group in the rigidity theorem is given in [6, 12, 14]. The rigid groups in papers [14, 12, 6] are $SM(3,1)$, $SM(4,1)$ and $SM(n,p)$, respectively.

This paper is organized as follows. In Section 2, a definition of the differential field of all G-invariant differential rational functions of a path (curve) is given. For groups $G = M(n,p), SM(n,p)$, it is proved that the complete system of G-invariant differential rational functions of a path (curve) obtaining in [18, Theorems 2 and 3 and Corollaries 1 and 2] is a generating system of the differential field of all G-invariant differential rational functions of a path (curve), respectively. (Theorems 1, 2). In Section 3, the description of a fundamental system of relations between elements of the complete system of G-invariant functions of a path (curve) is given (Theorems 3–4 and Corollary 4).

In this paper we use definitions and notations of the paper [18].

2. Invariant differential rational functions of paths and curves

Below we cite some notation and facts from the differential algebra (see [15–17]) in a form which is convenient for our considerations. Let R be a field of real numbers. Consider the ring $R[y_0, y_1 \ldots, y_n, \ldots]$ of polynomials with real coefficients in the countable set of variables $\{y_0, y_1 \ldots, y_n, \ldots\}$. We let $y_0 = y, y_1 = y^{'}, \ldots, y_{m+1} = (y_m)^{'} = y^{(m+1)}$. The operation $'$: $y_m \to y_m^{'}$ will be called the differentiation of an element y_m. Using the Leibniz rule, this operation can be uniquely extended to the ring $R[y_0, y_1 \ldots, y_n, \ldots]$. As a result, we obtain a differential R-algebra (d-algebra), which will be denoted by $R\{y\}$. Elements of this d-algebra are called differential polynomials in y with coefficients from R. We denote elements of $R\{y\}$ by $f\{y\}$. The element y is called the differential variable (unknown).

Differential polynomials $f\{z_1, \ldots, z_n\}$ and the d-algebra $R\{z_1, \ldots, z_n\}$ in a finite number of differential variables $z_1, \ldots, z_n$ are defined in a similar manner.

We denote by $C^\infty(J)$ the set of all infinitely differentiable functions on an interval $J = (a,b)$. Let $f\{y\}$ be a differential polynomial in a differential variable, and let $y(t) \in C^\infty(J)$. In the expression $f\{y\}$, let y to $y(t)$ and polynomial term $y^{(n)}$ to $\frac{d^n y(t)}{dt^n}$ (n = 1,2,...). We denote the obtained expression by $f\{y(t)\}$. The expression $f\{y(t)\}$ is a polynomial in $y(t)$ and a finite number of derivatives of $y(t)$. For $f_1, f_2 \in R\{y\}$, $f_1 = f_2$ if and only if $f_1\{y(t)\} = f_2\{y(t)\}$ for all $y(t) \in C^\infty(J)$.

The set of all expressions $f\{y(t)\}$, where $f \in R\{y\}$, will be denoted by $R\{y(t)\}$. $R\{y(t)\}$ is an R-algebra with respect to the standard operations of addition and multiplication of functions and multiplication of a function by a real number. $R\{y(t)\}$ becomes a differential R-algebra if $\frac{d}{dt}$ is taken as the operation of differentiation. One can easily see that the mapping $f\{y\} \to f\{y(t)\}$ is an isomorphism of differential R-algebras $R\{y\}$ and $R\{y(t)\}$. A similar fact takes place for differential polynomials $f\{z_1, \ldots, z_n\}$ in several variables $z_1, \ldots, z_n$. Let us replace in $f\{z_1, \ldots, z_n\}$ the element z_i $(i = 1, 2, \ldots, n)$ by $z_i(t) \in C^\infty(J)$ and the element $z_i^{(m)}$ by the function $\frac{d^m z_i(t)}{dt^m}$ $(m = 1, 2, 3, \ldots)$. Denote the obtained expression by $f\{z_1(t), \ldots, z_n(t)\}$. We denote by $R\{z_1(t), \ldots, z_n(t)\}$ the set of all $f\{z_1(t), \ldots, z_n(t)\}$, where $f \in R\{z_1, \ldots, z_n\}$. $R\{z_1(t), \ldots, z_n(t)\}$ is a differential R-algebra with respect to the standard operations over functions and the operation $\frac{d}{dt}$. The differential algebras $R\{z_1,\ldots,z_n\}$ and $R\{z_1(t),\ldots,z_n(t)\}$ are isomorphic, and to the operation of differentiation in $R\{z_1,\ldots,z_n\}$ the operation $\frac{d}{dt}$ in $R\{z_1(t),\ldots,z_n(t)\}$ corresponds.

The transition from $f\{z_1,\ldots,z_n\}$ to $f\{z_1(t),\ldots,z_n(t)\}$ will be called a parametric representation of a differential polynomial $f\{z_1,\ldots,z_n\}$. The inverse transition will be called the abstract representation of $f\{z_1(t),\ldots,z_n(t)\}$. The system $(z_1,z_2,\ldots,z_n)$ of differential variables $z_1,z_2,\ldots,z_n$ will be called an n-dimensional differential vector. For brevity, an ordered system $(x_1,x_2,\ldots,x_m)$ of differential vectors $x_1,x_2,\ldots,x_m$ will be denoted by x. We let $R\{x_1,\ldots,x_m\}=R\{x\}$. $R\{x\}$ is an integral domain. We denote its field of quotients by $R\langle x\rangle$. The differentiation in $R\{x\}$ is uniquely extended to a differentiation in $R\langle x\rangle$, and $R\langle x\rangle$ is a differential field. An element of $R\langle x\rangle$ is called a differential rational function of x and denoted by $h\langle x\rangle$.

Let G be a subgroup of $M(n,p)$.

Definition 1 *A differential rational function $h<x>$ will be called G-invariant if $h<gx>=h<x>$ for all $g\in G$.*

The set of all G-invariant differential rational functions of x forms a differential subfield of $R<x>$. We denote it by $R<x>^G$.

Definition 2 *Let α be a curve in R^n and x is a G-invariant parametrization of α. An element $h\in R<x>^G$ is called a G-invariant differential rational function of a curve α.*

Let $\langle x,y\rangle$ be the inner product of vectors $x,y\in E_p^n$.

Definition 3 *A subset S of $R<x>^G$ will be called a system of generators of differential field $R<x>^G$ if the least differential subfield in $R<x>^G$ containing S coincides with $R<x>^G$.*

Theorem 1 *The system*

$$\left\{<x^{(k)}(t),x^{(k)}(t)>,k=1,2,\ldots,n\right\} \tag{1}$$

is a system of generators of $R<x>^{M(n,p)}$.

Proof Let $R<x'>$ be the differential field of all differential rational functions of $x'=\frac{\partial}{\partial t}x$ and $O(n,p)$ is the group of all pseudo-orthogonal $n\times n$-matrices. □

First we prove several lemmas.

Lemma 1 $R<x>^{M(n,p)}=R<x'>^{O(n,p)}$.

Proof Let $h<x>=h(x,x',\ldots,x^{(m)})\in R<x>^{M(n,p)}$. Then it is invariant with respect to parallel translations in E_p^n. This implies that $h(x,x',\ldots,x^{(m)})=h<x'>$. It is also $O(n,p)$-invariant. Hence it is an $O(n,p)$-invariant differential rational function of x'. Conversely, assume that h is an $O(n,p)$-invariant differential rational function of x'. Then it is invariant with respect to parallel translations in E_p^n. Hence it is $M(n,p)$-invariant. □

Lemma 2 *Let* $f \in R < x^{'} >^{O(n,p)}$. *Then differential polynomials* $f_1, f_2 \in R < x^{'} >^{O(n,p)}$ *exist such that* $f = f_1/f_2$.

Proof A proof is similar to the proof in ([16], p. 106). □

Let N be the set of all natural numbers.

Lemma 3 *The system of all elements* $\langle x^{(m)}, x^{(q)} \rangle$, *where* $m \geq 1, q \geq 1, m \in N, q \in N$, *is a generating system of* $R < x >^{M(n,p)}$ *as a field.*

Proof Let $R[x^{(m)}, m \in N]^{O(n,p)}$ be the R-algebra of all $O(n,p)$-invariant polynomials of the system $\{x^{(m)}, m \in N\}$. It is obvious that $R[x^{(m)}, m \in N]^{O(n,p)} = R\left\{x^{'}\right\}^{O(n,p)}$. According to the First Main Theorem for $O(n,0)$ ([21, p. 53]) and $O(n,p)$ ([21, p. 65,66]), the system $\left\{\langle x^{(m)}, x^{(q)} \rangle ; m, q \in N\right\}$ is a generating system of the R-algebra $R[x^{(m)}, m \in N]^{O(n,p)} = R\left\{x^{'}\right\}^{O(n,p)}$. Using Lemmas 1 and 2, we obtain that the system $\left\{\langle x^{(m)}, x^{(q)} \rangle ; m, q \in N\right\}$ is a generating system of $R < x^{'} >^{O(n,p)} = R < x >^{M(n,p)}$ as a field. □

Lemma 4 *Let* $1 \leq i, j, i+j \leq 2n+1$. *Then, for each differential polynomial* $< x^{(i)}, x^{(j)} >$, *a differential polynomial* $P_{ij}\{y_1, ..., y_k\}$ *exists such that*

$$< x^{(i)}, x^{(j)} >= P_{ij}\left\{< x^{'}, x^{'} >, ..., < x^{(k)}, x^{(k)} >\right\},$$

where $k = \left[\frac{i+j}{2}\right]$.

Proof We will prove the existence of P_{ij} by induction on $q = i+j$. Since $i \geq 1, j \geq 1$, we have $i+j \geq 2$. In the case $i+j = 2$, the desired existence of P_{11} is obvious. Assume that a differential polynomial P_{ij} exists for all i, j such that $i+j < q$. Let $i \leq j$ and $q = 2b$, where b is an integer. Then $< x^i, x^j >=< x^{(b-h)}, x^{(b+h)} >$ for some $h \geq 0$. Using the equality

$$< x^{(b-h)}, x^{(b+h)} >=< x^{(b-h-1)}, x^{(b+h)} >^{'} - < x^{(b-h-1)}, x^{(b+h+1)} >$$

and applying the inductions on $q = i+j$ and h, we conclude that $< x^{(i)}, x^{(j)} >$ is a differential polynomial in $< x^{'}, x^{'} >, ..., < x^{(k)}, x^{(k)} >$, where $k \leq b$.

Let $q = 2b+1$. Then $< x^{(b)}, x^{(b)} >^{'} = 2 < x^{(b)}, x^{(b+1)} >$. Using the equality

$$< x^{(b-h)}, x^{(b+h+1)} >=< x^{(b-h-1)}, x^{(b+h+1)} >^{'} - < x^{(b-h-1)}, x^{(b+h+2)} >$$

and applying the inductions on $q = i+j$ and h, we conclude that $< x^{(i)}, x^{(j)} >$ is a differential polynomial of $< x^{'}, x^{'} >, ..., < x^{(k)}, x^{(k)} >$, where $k \leq b$. □

Denote by $\Delta = \Delta_x$ the determinant $\det \left\| < x^{(i)}, x^{(j)} > \right\|_{i,j=1,2,...,n}$. Let V be the system equation (1). Denote by $R\{V\}$ the differential R-subalgebra of $R < x^{'} >^{O(n,p)}$ generated by elements of the system V.

Lemma 5 $\Delta \in R\{V\}$.

Proof By the definition of V, $< x^{(i)}, x^{(i)} >\in V$ for all $1 \le i \le n$. According to Lemma 4, $< x^{(i)}, x^{(j)} >\in V$ for all $1 \le i, j \le n$. Hence $\Delta \in R\{V\}$. □

Denote by $R\left\{V, \Delta^{-1}\right\}$ the differential R-subalgebra of $R < x^{'} >^{O(n,p)}$ generated by elements of the system V and the function Δ^{-1}. According to Lemmas 1 and 3, for a proof of our theorem, it is enough to prove that $< x^{(m)}, x^{(q)} >\in R\left\{V, \Delta^{-1}\right\}$ for all $m, q \in N$.

Denote by $\mathrm{Gr}(y_1, y_2, \ldots, y_m; z_1, z_2, \ldots, z_m)$ the Gram matrix $\|\langle y_i, z_j\rangle\|_{i,j=1,2,\ldots,m}$ of vectors $y_1, y_2, \ldots, y_m;$ $z_1, z_2, \ldots, z_m$ in E^n_p. Let det $\mathrm{Gr}(y_1, y_2, \ldots, y_m; z_1, z_2, \ldots, z_m)$ be the determinant of $\mathrm{Gr}(y_1, y_2, \ldots, y_m; z_1, z_2,$ $\ldots, z_m)$. The following is known.

Lemma 6 *The equality,*

$$detGr(y_1, y_2, \ldots, y_{n+1}; z_1, z_2, \ldots, z_{n+1}) = det|| < y_i, z_j > ||_{i,j=1,2,\ldots,n+1} = 0$$

holds for all vectors $y_1, y_2, \ldots, y_{n+1}, z_1, z_2, \ldots, z_{n+1}$ *in* R^n.

Proof A proof is given in [16, p. 106–107], [21, p. 75]. □

Lemma 7 *Let* $b, c \in N$ *such that* $< x^{(b)}, x^{(i)} >\in R\left\{V, \Delta^{-1}\right\}$ *and* $< x^{(c)}, x^{(i)} >\in R\left\{V, \Delta^{-1}\right\}$ *for all* $1 \le i \le n$. *Then* $< x^{(b)}, x^{(c)} >\in R\left\{V, \Delta^{-1}\right\}$.

Proof Using Lemma 6 to vectors

$$y_1 = z_1 = x^{'}, y_2 = z_2 = x^{(2)}, \ldots, y_n = z_n = x^{(n)}, y_{n+1} = x^{(b)}, z_{n+1} = x^{(c)},$$

we obtain the equality $\det A = 0$, where

$$A = \|< y_i, z_j >\|_{i,j=1,2,\ldots,n+1}.$$

Let D_{n+1j} be the cofactor of the element $< y_{n+1}, z_j >$ of the matrix A for $j = 1, 2, \ldots, n+1$. The equality $\det A = 0$ implies the equality

$$< y_{n+1}, z_1 > D_{n+11}+ < y_{n+1}, z_2 > D_{n+12} + \cdots + < y_{n+1}, z_n > D_{n+1n} + \\ < y_{n+1}, z_{n+1} > D_{n+1n+1} = 0. \tag{2}$$

Since $\Delta = D_{n+1n+1}$, equation (2) implies the equality

$$< y_{n+1}, z_{n+1} >=< x^{(b)}, x^{(c)} >= \\ -\frac{< y_{n+1}, z_1 > D_{n+11}+ < y_{n+1}, z_2 > D_{n+12} + \cdots + < y_{n+1}, z_n > D_{n+1n}}{\Delta}. \tag{3}$$

In equation (3), by the assumption of the lemma, $< y_{n+1}, z_j >=< x^{(b)}, x^j >\in R\left\{V, \Delta^{-1}\right\}$ for each $j : 1 \le j \le n$. We prove that $D_{n+1s} \in R\left\{V, \Delta^{-1}\right\}$ for every $s : 1 \le s \le n$. We have $D_{n+1s} = (-1)^{n+1+s}$ det $\mathrm{Gr}(y_1, y_2, \ldots, y_n;\ z_1, z_2,\ \ldots, z_{s-1}, z_{s+1}, \ldots, z_n, z_{n+1}))$. By the definition of V, $< y_i, z_j >\in V \subset R\{V\}$ for all $i, j : 1 \le i, j \le n$. By the assumption of our lemma, we have $< y_i, z_{n+1} >=< x^{(i)}, x^c >\in R\left\{V, \Delta^{-1}\right\}$

for every $i : 1 \leq i \leq n$. Hence $D_{n+1s} \in R\left\{V, \Delta^{-1}\right\}$ for every $s : 1 \leq s \leq n$ and equation (3) implies $< y_{n+1}, z_{n+1} >\in R\left\{V, \Delta^{-1}\right\}$. □

Lemma 8 *$< x^{(b)}, x^{(i)} >\in R\left\{V, \Delta^{-1}\right\}$ for all $b \in N$ and $1 \leq i \leq n$.*

Proof We prove this assertion by induction on b. By the definition of V and Lemma 4, we obtain that $< x^{(c)}, x^{(i)} >\in R\left\{V, \Delta^{-1}\right\}$ for all $1 \leq c \leq n+1, 1 \leq i \leq n$. This implies that the assertion holds for all $b = c = 1, 2, \ldots, n+1$.

Assume that the assertion of the theorem holds for $b-1$. Then $< x^{(b-1)}, x^{(i)} >\in R\left\{V, \Delta^{-1}\right\}$ for all $1 \leq i \leq n$. Using $< x^{(b-1)}, x^{(i)} >\in R\left\{V, \Delta^{-1}\right\}$ and $< x^{(c)}, x^{(i)} >\in R\left\{V, \Delta^{-1}\right\}$ for all $1 \leq c \leq n+1, 1 \leq i \leq n$, by Lemma 7, we obtain $< x^{(b-1)}, x^{(c)} >\in R\left\{V, \Delta^{-1}\right\}$ for all $1 \leq c \leq n+1$. Since $< x^{(b-1)}, x^{(i)} >\in R\left\{V, \Delta^{-1}\right\}$ for all $1 \leq i \leq n$, the equality

$$\frac{\partial}{\partial t} < x^{(b-1)}, x^{(i)} >=< x^{(b)}, x^{(i)} > + < x^{(b-1)}, x^{(i+1)} >$$

and $< x^{(b-1)}, x^{(i+1)} >\in R\left\{V, \Delta^{-1}\right\}$ for all $1 \leq i \leq n$ imply $< x^{(b)}, x^{(i)} >\in R\{V\}$ for all $1 \leq i \leq n$. This means that the assertion is proved for b. □

We complete the proof of our theorem. Using Lemmas 8 and 7, we obtain $< x^{(b)}, x^{(c)} >\in R\left\{V, \Delta^{-1}\right\}$ for all $b, c \in N$. By Lemma 5, $\Delta \in R\{V\}$. Since $R < V >$ is a field, we obtain $\Delta^{-1} \in R < V >$. Hence $R\left\{V, \Delta^{-1}\right\} \subset R < V >$. By Lemma 3, the system of all elements $< x^{(b)}, x^{(c)} >$, where $b, c \in N$, is a generating system of $R < x >^{M(n,p)}$ as a field. Hence $R < V >= R < x >^{M(n,p)}$. The theorem is completed. □

Remark 1 *In the paper [18] was proved that the system* (1) *in Theorem 1 is a complete system of M(n,p)-invariants of a paths ([18, Theorem 2]). Then, by Theorem 1 in [18], the system* (1) *in Theorem 1, where $x = x(t_s(x))$ is an invariant parametrization of a curve α, is a complete system of M(n,p)-invariants of a curve α ([18, Corollary 1]). There are relations in the form of inequalities between elements of the system* 1*. These relations will be found later.*

For vectors $a_k \in R^n$, where $a_k = (a_{k1}, \ldots, a_{kn})$ and $k = 1, \ldots, n$, the determinant $\det(a_{kj})$ will be denoted by $[a_1 a_2 \ldots a_n]$. So $\left[x^{'}(t) x^{(2)}(t) \ldots x^{(n)}(t)\right]$ is the determinant of derivatives of a path $x(t)$.

Theorem 2 *The system*

$$\left\{\left[x^{'}(t) x^{(2)}(t) \ldots, x^{(n)}(t)\right], < x^{(k)}(t), x^{(k)}(t) >, k = 1, \ldots, n-1\right\} \tag{4}$$

is a generating system of $R < x >^{SM(n,p)}$.

Proof Let $SO(n,p) = \{F \in O(n,p) : \det F = 1\}$. First we prove several lemmas. □

Lemma 9 *$R < x >^{SM(n,p)}= R < x^{'} >^{SO(n,p)}$.*

Proof A proof is similar to the proof of Lemma 1. □

Lemma 10 *Let* $f \in R < x^{'} >^{SO(n,p)}$. *Then* $SO(n,p)$*-invariant differential polynomials* f_1, f_2 *exist such that* $f = f_1/f_2$.

Proof A proof is similar to the proof in [16, p. 106]. □

Lemma 11 *The system of all elements*

$$\left[x^{(m_1)}x^{(m_2)}\cdots x^{(m_n)}\right], < x^{(q)}, x^{(r)} >, \tag{5}$$

where $m_i, q, r \in N$, *is a generating system of* $R < x^{'} >^{SO(n,p)}$ *as a field.*

Proof Let $R[x^{(m)}; m \in N]^{SO(n,p)}$ be the R-algebra of all $SO(n,p)$-invariant polynomials of the system $\left\{x^{(m)}; m \in N\right\}$. According to the First Main Theorem for $SO(n,p)$ ([21, p.p. 53; 65–66]), the system equation (5) is a generating system of $R[x^{(m)}; m \in N]^{SO(n,p)}$. Lemma 10 implies that the system equation (5) is a generating system of $R < x^{'} >^{SO(n,p)}$ as a field. □

Denote by Z the system equation (4) of differential polynomials. Let $R\{Z\}$ be the differential R-subalgebra of $R < x^{'} >^{SO(n,p)}$ generated by elements of the system Z.

Let $\delta = \delta_x$ be the determinant of the matrix $Gr(y_1, y_2, \ldots, y_{n-1}; z_1, z_2, \ldots, z_{n-1})$, where $y_1 = z_1 = x^{'}, y_2 = z_2 = x^{(2)}, \cdots, y_{n-1} = z_{n-1} = x^{(n-1)}$.

Lemma 12 $< y_i, z_j > \in R\{Z\}$ *for all* $1 \leq i, j, i+j \leq 2n-1$, $\delta \in R\{Z\}$ *and* $\delta^{-1} \in R < Z >$.

Proof Using Lemma 4, we get $< x^{(i)}, x^{(j)} > \in R\{Z\}$ for all $1 \leq i, j, i+j \leq 2n-1$. The element $< y_i, z_j >$ of the determinant δ is the functions $< x^{(i)}, x^{(j)} >$, where $1 \leq i, j \leq n-1$. Hence $\delta \in R\{Z\}$ and $\delta^{-1} \in R < Z >$. □

In sequel, we need the following lemma.

Lemma 13 *The equality*

$$(-1)^p[y_1 \ldots y_n][z_1 \ldots z_n] = det\|\langle y_i, z_j\rangle\|_{i,j=1,2,\ldots,n}$$

holds for all vectors $y_1, \ldots, y_n, z_1, \ldots, z_n$ *in* E_p^n.

Proof A proof of the this lemma is similar to the proof in ([16], p.72). □

Let Δ be the function in the proof of Theorem 2.

Lemma 14 $\Delta \in R\{Z\}$ *and* $\Delta^{-1} \in R < Z >$.

Proof Using Lemma 13 to vectors $y_1 = z_1 = x^{'}, y_2 = z_2 = x^{(2)}, \cdots, y_n = z_n = x^{(n)}$, we obtain

$$(-1)^p \left[x^{'}x^{(2)} \ldots x^{(n)}\right]^2 = \det\| < y_i, z_j > \|_{i,j=1,2,\ldots n} = \Delta. \tag{6}$$

Since $\left[x^{'}x^{(2)} \ldots x^{(n)}\right] \in Z$, we have $\Delta \in R\{Z\}$ and $\Delta^{-1} \in R < Z >$. □

Denote by $R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ the differential R-subalgebra of $R < x^{'} >^{SM(n,p)}$ generated by Z and functions δ^{-1}, Δ^{-1}. By Lemmas 10 and 11, for a proof of our theorem, it is enough to prove that $\left[x^{(m_1)}x^{(m_2)}\ldots x^{(m_n)}\right] \in R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ and $\left\langle x^{(b)}, x^{(c)}\right\rangle \in R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ for all $m_i, b, c \in N$.

Let V be the system in the proof of Theorem 2.

Lemma 15 $< x^{(n)}, x^{(n)} > \in R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ *and* $R\left\{V, \Delta^{-1}\right\} \subset R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$.

Proof For $i = 1, 2, \ldots, n$, denote by D_{ni} the cofactor of the element $< y_n, z_j >$ of the matrix $A = \| < y_i, z_j > \|_{i,j=1,2,\ldots n}$ in equation (6). Then we obtain the equality

$$\Delta = < y_n, z_1 > D_{n1} + < y_n, z_2 > D_{n2} + \cdots + < y_n, z_{n-1} > D_{nn-1} + < y_n, z_n > D_{nn}. \tag{7}$$

Since $\delta = D_{nn} \neq 0$, equalities equation (6) and equation (7) imply

$$< y_n, z_n > = < x^{(n)}, x^{(n)} > = \Delta\delta^{-1} - < y_n, z_1 > D_{n1}\delta^{-1} - < y_n, z_2 > D_{n2}\delta^{-1} - \cdots - < y_n, z_{n-1} > D_{nn-1}\delta^{-1}. \tag{8}$$

By Lemma 12, $< y_n, z_j > = < x^{(n)}, x^{(j)} > \in R\{Z\}$ for each $1 \leq j \leq n-1$. We prove that $D_{ns} \in R\{Z\}$ for every $1 \leq s \leq n-1$. We have $D_{ns} = (-1)^{n+s}\det Gr(y_1, y_2, \ldots, y_{n-1}; z_1, z_2, \ \ldots, z_{s-1}, z_{s+1}, \ldots, z_n)$. Elements of D_{ns} have the following forms $< y_i, z_j >$ and $< y_i, z_n >$, where $i < n, j < n$. Since $< y_i, z_n > = < y_n, z_i > \in R\{Z\}$, we have $D_{ns} \in R\{Z\}$. Hence equation (8) implies $< y_n, z_n > \in R\left\{Z, \delta^{-1}\right\}$. Using $V \subset Z \cup \{\langle y_n, z_n\rangle\}$, we obtain $R\left\{V, \Delta^{-1}\right\} \subset R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$. □

Lemma 16 $\left\langle x^{(b)}, x^{(c)}\right\rangle \in R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ *for all* $b, c \in N$.

Proof By Lemma 15, we have $R\left\{V, \Delta^{-1}\right\} \subset R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$. Since $\left\langle x^{(b)}, x^{(c)}\right\rangle \in R\left\{V, \Delta^{-1}\right\}$ for all $b, c \in N$, we obtain $< x^{(b)}, x^{(c)}) \in R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ for all $b, c \in N$. □

Lemma 17 $\left[x^{(m_1)}x^{(m_2)}\cdots x^{(m_n)}\right] \in R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ *for all* $m_i \in N$.

Proof Using Lemma 13 to vectors $y_1 = x^{'}, y_2 = x^{(2)}, \cdots, y_n = x^{(n)}, z_1 = x^{(m_1)}, z_2 = x^{(m_2)} \ldots, z_n = x^{(m_n)}$, we obtain that

$$(-1)^p[y_1 \ldots y_n][z_1 \ldots z_n] = \det|| < y_i, z_j > ||_{i,j=1,2,\ldots,n}. \tag{9}$$

Since $\Delta = (-1)^p[y_1 \ldots y_n]^2$, equation (9) implies

$$[z_1 \ldots z_n] = \Delta^{-1}[y_1 \ldots y_n]\det|| < y_i, z_j > ||_{i,j=1,2,\ldots,n}.$$

By Lemma 16, $< y_i, z_j > = < x^{(i)}, x^{(m_j)} > \in R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$ for all $i, j = 1, 2, \ldots, n$. Since $[y_1 \ldots y_n] \in Z \subset R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$, we obtain $[z_1 \ldots z_n] \in R\left\{Z, \delta^{-1}, \Delta^{-1}\right\}$. □

We complete the proof of our theorem. By Lemmas 12 and 14, $\delta^{-1}, \Delta^{-1} \in R < Z >$. Hence $R\left\{Z, \delta^{-1}, \Delta^{-1}\right\} \subset R < Z >$. By Lemma 16, $< x^{(b)}, x^{(c)} > \in R\left\{Z, \delta^{-1}, \Delta^{-1}\right\} \subset R < Z >$ for all $b, c \in N$. By Lemma 17, $\left[x^{(m_1)} x^{(m_2)} \cdots x^{(m_n)}\right] \in R\left\{Z, \delta^{-1}, \Delta^{-1}\right\} \subset R < Z >$ for all $m_i \in N$. Hence Lemmas 9-11 imply that $R < Z >= R < x >^{SM(n,p)}$. The theorem is completed. □

Remark 2 *In the paper [18] was proved that the system* (4) *in Theorem 2 is a complete system of SM(n,p)-invariants of paths ([18, Theorem 3]). Then, by Theorem 1 in [18], the system* (4) *in Theorem 1, where* $x = x(t_s(x))$ *is an invariant parametrization of a curve* α, *is a complete system of SM(n,p)-invariants of a curve* α *([18, Corollary 2]). There are relations in the form of inequalities between elements of the system* 4. *These relations will be found below.*

3. Relations between elements of complete systems of invariants of a curve in E_p^n

Definition 4 *A system of differential polynomials* $p_1\{x\}, \ldots, p_m\{x\} \in R\{x\}$ *is called differential algebraically independent if there is no nonzero differential polynomial* $f\{y_1, \ldots, y_m\} \in R\{y_1, \ldots, y_m\}$ *such that* $f\{p_1\{x\}, \ldots, p_m\{x\}\} = 0$ *for all paths* x.

Theorem 3 . *The system*

$$\left\{< x^{(k)}(t), x^{(k)}(t) >, k = 1, 2, ..., n\right\}$$

is differential algebraically independent.

Proof A proof is similar to the proof of Theorem 12.8 in ([16], p.112). □

Let $A(x(t)) = \left\|x^{'}(t) x^{(2)}(t) \ldots x^{(n)}(t)\right\|$, $A(x)^{\top}$ be the transpose matrix of $A(x)$ and $I_p = \|b_{ij}\|$ be the diagonal $n \times n$-matrix such that $b_{ii} = -1$ for all $i = 1, \ldots, p$ and $b_{jj} = 1$ for all $j = p+1, \ldots, n$. We have the equality $A(x)^{\top} I_p A(x) = \left\|< x^{(i)}, x^{(j)} >\right\|_{i,j=1,2,\ldots,n}$. The matrix $A(x)^{\top} I_p A(x)$ is congruent to the matrix I_p for every non-singular J-path $x(t)$ and all $t \in J$. This fact, in view of the equality $A(x)^{\top} I_p A(x) = || < x^{(i)}, x^{(j)} > ||$, gives some system of relations (inequalities) between $< x^{'}(t), x^{'}(t) >, \ldots, < x^{(n)}(t), x^{(n)}(t) >$ and their derivatives. Below we prove that an arbitrary relation between $< x^{'}(t), x^{'}(t) >, \ldots, < x^{(n)}(t), x^{(n)}(t) >$ and their derivatives is a consequence of the above mentioned relations.

Corollary 1 *Let* $y_1, y_2, \ldots, y_n$ *be differential variables and* $f \in R\{y_1, y_2, \ldots, y_n\}$. *Then the differential polynomial* $f\{y_1, y_2, \ldots, y_n\}$ *is uniquely determined by its values on functions* $y_1(t)$, $y_2(t), \ldots, y_n(t)$ *in the form*

$$y_i(t) =< x^{(i)}(t), x^{(i)}(t) >, \tag{10}$$

where $x(t)$ *run through the space* $(C^{\infty}(J))^n$.

Proof Assume that $f_1, f_2 \in R\{y_1, y_2, \ldots, y_n\}$ exist such that $f_1 \neq f_2$ and

$$f_1\{y_1(t), y_2(t), \ldots, y_n(t)\} = f_2\{y_1(t), y_2(t), \ldots, y_n(t)\} \tag{11}$$

for all $y_1(t)$, $y_2(t), \ldots, y_n(t)$ in the form equation (10). From equation (11), we obtain the equality

$$f\{y_1(t), y_2(t), \ldots, y_n(t)\} = 0 \tag{12}$$

for all $y_1(t),\ y_2(t), \dots, y_n(t)$ in the form equation (10), where $f = f_1 - f_2$ is a nonzero differential polynomial since $f_1 \neq f_2$. Equation (12) means that differential polynomials $< x^{(1)}, x^{(1)} >, < x^{(2)}, x^{(2)} >, \dots < x^{(n)}, x^{(n)} >$ are differential-algebraically dependent, which contradicts Theorem 3. □

Corollary 2 *The differential polynomial* $P_{ij}\{y_1, y_2, \dots, y_k\}$ *in Lemma 4 is unique.*

Proof A proof follows from Theorem 3. □

For convenience, we will write $P_{ij}\{y_1, y_2, \dots, y_n\}$ instead of $P_{ij}\{y_1, y_2, \dots, y_k\}$.

Let the symbol $'$ denotes the differentiation in the differential algebra $R\{y_1, y_2, \dots, y_n\}$.

Corollary 3 *The equality*

$$P_{ij}\{y_1, y_2, \dots, y_n\}' = P_{i+1j}\{y_1, y_2, \dots, y_n\} + P_{ij+1}\{y_1, y_2, \dots, y_n\} \tag{13}$$

holds for all i, j *satisfying the conditions* $1 \leq i, j, i + j \leq 2n$.

Proof From the definition of differential polynomials $P_{ij}\{y_1, y_2, \dots, y_n\}$, we have the equality

$$< x^{(i)}(t), x^{(j)}(t) >= P_{ij}\{y_1(t), y_2(t), \dots, y_n(t)\},$$

where $y_k(t) =< x^{(k)}(t), x^{(k)}(t) >, k = 1, \dots, n, 1 \leq i, j, i + j \leq 2n + 1$. Differentiating this equality, we obtain

$$\frac{d}{dt} < x^{(i)}(t), x^{(j)}(t) >= \frac{d}{dt} P_{ij}\{y_1(t), y_2(t), \dots, y_n(t)\}.$$

Assume that $1 \leq i, j, i + j \leq 2n$. Since

$$\frac{d}{dt} < x^{(i)}(t), x^{(j)}(t) >=< x^{(i+1)}(t), x^{(j)}(t) > + < x^{(i)}(t), x^{(j+1)}(t) >,$$

we have

$$\frac{d}{dt} P_{ij}\{y_1(t), y_2(t), \dots, y_n(t)\} = P_{i+1j}\{y_1(t), y_2(t), \dots, y_n(t)\} + P_{ij+1}\{y_1(t), y_2(t), \dots, y_n(t)\}.$$

This equality takes place for all functions $y_1(t),\ y_2(t), \dots, y_n(t)$ in the form equation (10). Applying the Corollary 3, we conclude that the latest equality takes place for all $y_1(t), y_2(t), \dots, y_n(t) \in C^\infty(J)$. Passing from a parametric representation of differential polynomials in the latest equality to their abstract differential polynomials, we obtain equation (13). □

Let $f_1(t), \dots, f_n(t)$ be arbitrary C^∞-functions on J=(a,b). For convenience, the indexes of these functions will be written in the form $f_i(t) = f_{ii}(t)$. Using functions $f_{ii}(t)$, we define functions

$$f_{ij}(t) = P_{ij}\{f_{11}(t), \dots, f_{nn}(t)\}, 1 \leq i, j \leq n. \tag{14}$$

Proposition 1 *The equality*

$$f_{ij}^{'}(t) = f_{i+1j}(t) + f_{ij+1}(t)$$

holds for all i, j *satisfying the conditions* $1 \leq i, j, i + j \leq 2n$.

Proof Letting $y_1 = f_{11}(t), y_2 = f_{22}(t) \dots, y_n = f_{nn}(t)$ in Corollary 3, we obtain the desired equality. □

Theorem 4 *Let $f_{11}(t), \dots, f_{nn}(t)$ be infinitely differentiable functions on J such that:*

(i) $det||f_{ij}(t)|| \neq 0$ for all $t \in J$, where $f_{ij}(t)$ is defined by (14);

(ii) the matrix $||f_{ij}(t)||$ is congruent to the matrix I_p for some $t_0 \in J$.

Then there exists a non-singular J-path $x(t)$ in E_p^n such that

$$< x^{(i)}(t), x^{(i)}(t) >= f_{ii}(t)$$

for all $t \in J$ and $i = 1, \dots, n$.

Proof Define the $n \times n$-matrix function $Q(t) = \|f_{ij}\|$, where f_{ij} is defined by equation (14). Since the differential polynomials P_{ij} satisfy the relations $P_{ij} = P_{ji}$, we obtain $Q^{\top}(t) = Q(t)$, where $Q(t)^{\top}$ is the transpose matrix of $Q(t)$. Let $Q^{'}(t)$ be the derivative of $Q(t)$.

Lemma 18 *A unique solution $B(t) = \|b_{ij}\|$ of the $n \times n$-matrix equation*

$$Q^{'}(t) = B^{\top}(t)Q(t) + Q(t)B(t) \tag{15}$$

exists which satisfies the conditions

(γ_1) $b_{j+1j}(t) = 1$ for all $t \in J$ and $1 \leq j \leq n-1$;

(γ_2) $b_{ij}(t) = 0$ for all $t \in J$ and $j \neq n, i \neq j+1, 1 \leq i \leq n$.

Proof By (γ_1) and (γ_2), only the elements $b_{1n}(t), \dots, b_{nn}(t)$ of the matrix $B(t)$ are unknown. Using (γ_1) and (γ_2), from equation (15) and $Q^{\top}(t) = Q(t)$, we obtain

$$f_{ij}^{'}(t) = f_{i+1j}(t) + f_{ij+1}(t)$$

for $1 \leq i, j \leq n-1$ and

$$f_{ni}^{'}(t) = f_{in}^{'}(t) = f_{i+1n}(t) + \sum_{k=1}^{n} f_{ik}(t)b_{kn}(t)$$

for $1 \leq i \leq n-1$,

$$f_{nn}^{'}(t) = \sum_{k=1}^{n} b_{kn}(t)f_{kn}(t) + \sum_{k=1}^{n} f_{nk}(t)b_{kn}(t) = 2\sum_{k=1}^{n} f_{nk}(t)b_{kn}(t).$$

Hence, for the elements $b_{1n}(t), \dots, b_{nn}(t)$, we obtain the following system of n linear equations in n unknowns:

$$\sum_{k=1}^{n} f_{ik}(t)b_{kn}(t) = f_{ni}^{'}(t) - f_{i+1n}(t), 1 \leq i \leq n-1,$$

$$\sum_{k=1}^{n} f_{nk}(t)b_{kn}(t) = 0,5f_{nn}^{'}(t).$$

By assumption (i) of the theorem , the determinant of this system is $\det Q(t) = \det \|f_{ij}(t)\| \neq 0$ for all $t \in J$. Consequently, this system has a unique solution $b_{1n}(t), \ldots, b_{nn}(t)$. □

It is obvious that $B(t)$ is infinitely differentiable.

Lemma 19 *Let $B(t)$ be the solution of equation (15). Then an infinitely differentiable $n \times n$-matrix function $A(t)$ on J exists such that*

(δ_1) $A(t) = ||a(t)a^{'}(t) \ldots a^{(n-1)}(t)||$ *for some J-path $a(t)$ in E_p^n;*

(δ_2) $\det A(t) \neq 0$ *for all $t \in J$;*

(δ_3) $A^{'}(t) = A(t)B(t)$*;*

(δ_4) $A^{\top}(t)I_pA(t) = Q(t)$*, where $I_p = \|b_{ij}\|$ be the diagonal $n \times n$-matrix such that $b_{ii} = -1$ for all $i = 1, \ldots, p$ and $b_{jj} = 1$ for all $j = p+1, \ldots, n$..*

Proof From assumptions (γ_1) and (γ_2) of Lemma 19, and from the theory of linear differential equations it follows that a solution $A(t)$ of equation (δ_3) exists such that $\det A(t) \neq 0$ for all $t \in J$. Since the matrix $B(t)$ satisfies relations (γ_1) and (γ_2), it follows that the matrix $A(t)$ is the form (δ_1) for some path $a(t)$ in E_p^n. We have $\det(A^{\top}(t)A(t)) \neq 0$ for all $t \in J$. Let t_0 be such that the matrix $Q(t_0) = ||f_{ij}(t_0)||$ is congruent to the matrix I_p. Since $Q(t_0)$ is congruent to the matrix I_p, $\det Q(t_0) \neq 0$, $\det(A^{\top}(t)A(t)) \neq 0$, and $Q^{\top}(t) = Q(t)$ for all $t \in J$, it follows that a nondegenerate $n \times n$-matrix $g \in GL(n, R)$ exists such that

$$(g^{\top})^{-1}(A^{\top}(t_0))^{-1}Q(t_0)A^{-1}(t_0)g^{-1} = I_p.$$

Hence we have $A^{\top}(t_0)g^{\top}I_pgA(t_0) = Q(t_0)$. The matrix function $gA(t)$ is also solution of (δ_3). The matrix function $H(t) = A^{\top}(t)g^{\top}I_pgA(t)$ satisfies the following conditions: $H^{\top}(t) = H(t), H^{'}(t) = B^{\top}(t)H(t) + H(t)B(t)$ for all $t \in J$. But these conditions are also fulfilled for the function $Q(t)$. Then from the equality $H(t_0) = Q(t_0)$, by the existence and uniqueness theorem of a solution of a system of linear differential equations, it follows that $H(t) = Q(t)$ for all $t \in J$. □

Now we return to the proof of the theorem. By Lemma 19, a matrix

$$A(t) = ||a(t)a^{'}(t) \ldots a^{(n-1)}(t)||$$

exists such that $A^{'}(t) = A(t)B(t), A^{\top}(t) \cdot I_p \cdot A(t) = Q(t)$. Using the relation $A^{\top}(t) \cdot I_p \cdot A(t) = \left\| < a^{(i)}(t), a^{(j)}(t) > \right\|$, we obtain $< a^{(i)}(t), a^{(j)}(t) >= f_{i+1j+1}$ for all $i, j = 0, 1, \ldots, n-1$. Let $x(t) = \int_t^{t_0} a(t)dt$. Then $< x^{(i)}(t), x^{(i)}(t) >= f_{ii}$ for all $i = 1, \ldots, n$. Since $\left[x^{'}(t)x^{(2)} \ldots x^{(n)}(t)\right]^2 = \det A^{\top}(t) \cdot I_p \cdot A(t) = \det Q(t) \neq 0$ for all $t \in J$, the path $x(t)$ is nondegenerate. □

Let J be one of intervals $(0, l), 0 < l \leq +\infty, (-\infty, 0), (-\infty, +\infty)$.

Corollary 4 *Let J be one of intervals $(0, l), 0 < l \leq +\infty, (-\infty, 0), (-\infty, +\infty)$. Assume that $f_{11}(s), \ldots, f_{nn}(s)$ be infinitely differentiable functions on J such that:*

(i) $|f_{11}(s)| = 1$ *for all* $s \in J$;

(ii) $det||f_{ij}(s)|| \neq 0$ *for all* $s \in J$, *where the function* $f_{ij}(s)$ *is defined by equation* (14);

(iii) *the matrix* $||f_{ij}(s)||$ *is congruent to the matrix* I_p *for some* $t_0 \in J$.

Then a regular non-singular curve α *and its invariant parametrization* x *exist such that* $< x^{(i)}(s), x^{(i)}(s) >= f_{ii}(s)$ *for all* $s \in I$, $i = 1, \ldots, n$.

Proof This corollary is a special case of Theorem 4. □

Let $f_{11}(t), \ldots, f_{n-1n-1}(t)$ and $d(t)$ be C^{∞}-functions on an interval J. We consider the matrix $Q(t) = ||f_{ij}(t)||_{i,j=1,\ldots,n}$, where the function $f_{ij}(t), i+j < 2n$, is defined by $f_{11}, \ldots, f_{n-1n-1}$ as in (14) and the function $f_{nn}(t)$ will be defined below. Let $A_{ni}(t)$ be the cofactor of the element $f_{ni}(t)$ in the matrix $Q(t)$. Since every element of the cofactor $A_{ni}(t)$ consists of all f_{pq} such that $p+q < 2n$, $A_{ni}(t)$ is a differential polynomial of $f_{11}, \ldots, f_{n-1n-1}$. Assume that $A_{nn}(t) = \det||f_{ij}(t)||_{i,j=1,\ldots,n-1} \neq 0$ for all $t \in J$. We define the element f_{nn} as follows:

$$f_{nn} = \frac{(-1)^p d^2(t) - f_{n1}(t)A_{n1} - \cdots - f_{nn-1}(t)A_{nn-1}}{A_{nn}}.$$

This equality implies

$$\det Q(t) = f_{n1}(t)A_{n1} + \cdots + f_{nn-1}(t)A_{nn-1} + f_{nn}(t)A_{nn} = (-1)^p d^2(t) \tag{16}$$

for all $t \in J$.

Theorem 5 *Let* $f_{11}(t), \ldots, f_{n-1n-1}(t)$ *and* $d(t)$ *be infinitely differentiable functions on an interval* J *such that:*

(λ_1) $A_{nn}(t) \neq 0$ *for all* $t \in J$;

(λ_2) *the matrix* $Q(t)$ *is congruent to the matrix* I_p *for some* $t_0 \in J$;

(λ_3) $d(t) \neq 0$.

Then there exists a non-singular path $x(t)$ *in* E_p^n *such that*

$$< x^{(i)}(t), x^{(i)}(t) >= f_{ii}(t), \qquad \left[x^{'}(t) \ldots x^{(n)}(t)\right] = d(t)$$

for all $t \in J$ *and* $i = 1, \ldots, n-1$.

Proof Equation (16) and the condition (λ_3) implies $\det Q(t) \neq 0$ for all $t \in J$. Hence, according to Theorem 4 there exists a non-singular path $x(t)$ in E_p^n such that $< x^{(i)}(t), x^{(i)}(t) >= f_{ii}(t)$ for all $t \in J$ and $i = 1, \ldots, n$. Using these equalities, Lemma 4 and equation (14), we obtain $\det Q(t) = \det \left\|\langle x^{(i)}(t), x^{(i)}(t)\rangle\right\|$. Then using this equality, equation (16) and Lemma 13, we obtain $\left[x^{'}(t) \ldots x^{(n)}(t)\right]^2 = d^2(t)$. Then $\left[x^{'}(t) \ldots x^{(n)}(t)\right] = d(t)$ or $\left[x^{'}(t) \ldots x^{(n)}(t)\right] = -d(t)$. Since $\left[x^{'}(t) \ldots x^{(n)}(t)\right] \neq 0$ and $d(t) \neq 0$ for all $t \in J$, we have $\left[x^{'}(t) \ldots x^{(n)}(t)\right] = d(t)$ or $\left[x^{'}(t) \ldots x^{(n)}(t)\right] = -d(t)$ for all $t \in I$. In the first case, a proof is completed. In the second case, we consider $g \in O(n)$ such that $\det g = -1$. In this case, we put $y(t) = gx(t)$. Then $< y^{(i)}(t), y^{(i)}(t) >= < gx^{(i)}(t), gx^{(i)}(t) >=< x^{(i)}(t), x^{(i)}(t) >= f_{ii}(t,)$ and $\left[y^{'}(t) \dots y^{(n)}(t)\right] = \left[gx^{'}(t) \dots gx^{(n)}(t)\right] = d(t)$. Thus the path $y(t)$ satisfies all conditions of our theorem. $\square$

An analog of Corollary 4 takes place for the complete system of SM(n,p)-invariants of a curve.

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