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Existence and multiplicity of positive solutions for a class of nonlinear elliptic problems

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Abstract: We study the existence and multiplicity of nonnegative solutions for the nonlinear elliptic problem, $-\Delta u + v(x)u = a(x)u^p + \lambda f(x,u)$ for $x \in \Omega$ and $u = 0$ on $\partial\Omega$, where Ω is a bounded region in $\mathbb{R}^N$, $N > 2$, $1 < p < \frac{N+2}{N-2}$, $\lambda > 0$ and $f(x,u)$ satisfies some suitable conditions. By extracting the Palais-Smale sequences in the Nehari manifold, it is proved that there exists λ^* such that for $\lambda \in (0,\lambda^*)$, the above problem has at least two positive solutions.

Key words: Nehari manifold, critical point, nonlinear elliptic boundary value problem, Palais-Smale sequence

1. Introduction

In this paper the existence and multiplicity of positive solutions for the following nonlinear elliptic problem is discussed

$$\begin{cases} -\Delta u + v(x)u = a(x)u^p + \lambda f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda > 0$, $1 < p < \frac{N+2}{N-2}$ $(N > 2)$, $\Omega \subset \mathbb{R}^N$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $a \in C(\bar\Omega)$ such that $a^+ = max\{a,0\} \not\equiv 0$, $v \in C(\bar\Omega)$ is a bounded function with $\|v\|_\infty > 0$, and $f(x,u)$ satisfies the following conditions:

(f1) $f(x,u) \in C^1(\Omega \times \mathbb{R})$ such that $f(x,0) \geq 0$, $f(x,0) \not\equiv 0$ and there exists $C_1 > 0$ such that, $f(x,u) \leq C_1(1+u^q)$ where $0 < q < 1$ and $(x,u) \in \Omega \times \mathbb{R}^+$.

(f2) $f_u(x,u) \in L^\infty(\Omega \times \mathbb{R})$ and for $u \in W_0^{1,2}(\Omega)$, $\int_{\partial\Omega} \frac{\partial}{\partial u} f(x,t|u|)u^2 dx$ has the same sign for every $t \in (0,\infty)$.

Remark 1.1 *Note that if f satisfies (f1) and (f2), then*

(f3) $pf(x,u) - uf_u(x,u) \leq C_2(1+u)$, for all $(x,u) \in \Omega \times \mathbb{R}^+$,

(f4) $F(x,u) - \frac{1}{p+1} f(x,u)u \leq C_2(1+u^2)$, for all $(x,u) \in \Omega \times \mathbb{R}^+$, where

$$F(x,u) = \int_0^u f(x,s)ds. \tag{1.2}$$

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In recent years, there have been several studies concerning the existence and multiplicity of solutions for elliptic problems. Some of results relating to these problems can be found in [4, 5, 8, 9, 11, 15] and references cited therein. Also it is an often studied problem to find solutions to the Laplace equation $-\Delta u = g(x,u)$, in $\Omega \subseteq \mathbb{R}^N$, $u \in W_0^{1,2}(\Omega)$ for $N \geq 3$ and g satisfying $\lim_{t\to\infty} \frac{g(x,t)}{t^q} = 0$ uniformly on Ω with $q < \frac{N+2}{N-2}$, and there are many results using the compactness of the embedding of the space $W_0^{1,2}(\Omega)$ into $L^r(\Omega)$ with $r \in [1, \frac{2N}{N-2})$ (see a review article by Lions [13] and the references given there).

In most papers concerning the problem (1.1), it is supposed that $f(x,u) = a(x)u^k$, for instance, Ambrosetti-Brezis-Cerami [2] considered problem (1.1) when $f(x,u) = u^q$, $0 < q < 1$ and proved that there is a $\Lambda > 0$ such that problem has at least two positive solutions for every $\lambda \in (0, \Lambda)$, one positive solution for $\lambda = \Lambda$ and no one for $\lambda > \Lambda$.

Also, if $f(x,u) = a(x)u^q$ and $v(x) \equiv 0$ the problem (1.1) becomes

$$\begin{cases} -\Delta u = \lambda a(x)u^q + b(x)u^p & for\ x \in \Omega, \\ u = 0 & for\ x \in \partial\Omega, \end{cases}$$

where $\lambda > 0$, $q < 1 < p < \frac{N+2}{N-2}$ (which is considered by Brown and Wu [5]), where $a, b : \Omega \to \mathbb{R}$ are smooth functions, which are somewhere positive but which may change sign on Ω, and they proved the existence of at least two positive solutions by using the Nehari manifold and fibering maps.

Tarantello [14] considered the Dirichlet problem

$$\begin{cases} -\Delta u = |u|^{p-2}u + f & for\ x \in \Omega, \\ u = 0 & for\ x \in \partial\Omega, \end{cases}$$

where $p = \frac{2N}{N-2}$, $N \geq 3$ is the limiting Sobolev exponent and $\Omega \subset \mathbb{R}^N$ is a bounded set. He showed that this problem has two distinct solutions if $f \in H^{-1}(\Omega)$ (where $H = H_0^1(\Omega)$) satisfies a suitable condition and $f \not\equiv 0$.

It should be stated, problem (1.1) for the P-Laplacian case when $f(x,u) = a(x)|u|^{p-2}u$ has been studied by Drabek and Pohozaev [9] and the existence of solutions via Nehari manifold method is discussed.

In unbounded domain, the semilinear elliptic problem

$$\begin{cases} -\Delta u + \lambda u = g(x,u) + f(x), & x \in \mathbb{R}^N, \\ u(x) > 0,\ u \in H^1(\mathbb{R}^N), \end{cases}$$

where g satisfies some suitable conditions and $f \in H^{-1}(\mathbb{R}^N)\backslash\{0\}$ is nonnegative, has been the focus of a great deal of research by several authors [1, 6, 7, 12] and the existence of at least two positive solutions was proved.

The aim of this paper is to prove the existence of positive solutions for the nonlinear elliptic problem (1.1) by using the Nehari manifold associated with the Euler functional for the problem. The main difficulty will be the nonlinearity of problem (1.1) due to $f(x,u)$. To overcome this difficultly, we need to restrict problem (1.1) to assumption (f2). Here we present some examples satisfying the conditions f1 and f2 and so the proposed method in this paper can be applied to these examples:

$$f_1(x,u) = \frac{-a_1(x)u^{q+r}}{1+a_2(x)u^2} + a_3(x),\ a_i(x) \in C(\overline{\Omega}),\ a_i(x) \geq 0,\ a_3(x) \not\equiv 0,\ \max\{2-q,0\} \leq r \leq 2.$$

$$f_2(x,u) = b_1(x)\tan^{-1}(b_2(x)u^{q+k})\ln[1+u^2] + b_3(x),\ b_i(x) \in C(\overline{\Omega}),\ b_i(x) \geq 0,\ b_3(x) \not\equiv 0,\ k \geq 0.$$

$$f_3(x,u) = c_1(x)\sqrt[r]{(1+c_2(x)u^{2k})^q}, \qquad c_i(x) \geq 0,\ c_i(x) \in C(\overline{\Omega}),\ c_1(x) \not\equiv 0,\ k \in \mathbb{N},\ 0 < 2k \leq r.$$

$$f_4(x,u) = \frac{-e_1(x)u^{q+k}}{4+\cot^{-1}(e_2(x)u)} + e_3(x), \quad e_i(x) \in C(\overline{\Omega}),\ e_i(x) \geq 0,\ e_3(x) \not\equiv 0,\ k \geq 0.$$

This paper is organized into 3 sections. In section 2 we present some notations and preliminary results and moreover, we will recall the properties which shall be required of the Nehari manifold. In section 3 we will prove the existence of positive solutions of problem (1.1) by establishing the existence of local minima for the Euler functional, associated with problem (1.1) on Nehari manifold.

2. Preliminaries and auxiliary results

We shall throughout use the function space $W_0^{1,2}(\Omega)$ with the norm

$$\|u\|_{W_0^{1,2}} = \left(\int_\Omega (|\nabla u|^2 + v(x)|u|^2)dx \right)^{\frac{1}{2}},$$

which is equivalent to the standard norm, and we use the standard $\mathrm{L}^p(\Omega)$ spaces whose norms are denoted by $\|u\|_p$. We denoted by S_r the best Sobolev constant for the embedding of $W_0^{1,2}(\Omega)$ into $L^r(\Omega)$, so for $1 \leq r < 2^*$ ($2^* = \frac{2N}{N-2}$ if $N > 2$, $2^* = \infty$ if $N \leq 2$) we have

$$\frac{(\|u\|^2_{W_0^{1,2}(\Omega)})^{p+1}}{(\int_\Omega \mid u \mid^{p+1} dx)^2} \geq \frac{1}{S_{p+1}^{2(p+1)}}. \tag{2.1}$$

The Euler functional associated with problem (1.1) is $I_\lambda : W_0^{1,2}(\Omega) \to \mathbb{R}$, such that

$$I_\lambda(u) = \frac{1}{2}M(u) - \frac{1}{p+1}A(u) - \lambda \int_\Omega F(x,|u|)dx, \tag{2.2}$$

where

$$A(u) = \int_\Omega a(x)|u|^{p+1}dx \quad \text{and} \quad M(u) = \int_\Omega (|\nabla u|^2 + v(x)|u|^2)dx = \|u\|^2_{W_0^{1,2}} \tag{2.3}$$

and $F(x,u)$ is introduced in (1.2). The critical points of the functional I_λ are in fact weak solutions of problem (1.1).

Definition 2.1 *We say that $u \in W_0^{1,2}(\Omega)$ is a weak solution of problem (1.1) if for any $\varphi \in W_0^{1,2}(\Omega)$*

$$\int_\Omega \big(\nabla u.\nabla\varphi + v(x)u\varphi\big)dx = \int_\Omega a(x)|u|^{p-1}u\varphi dx + \lambda \int_\Omega \big(f(x,u)\varphi\big)dx.$$

If I_λ is bounded below and has a minimizer on $W_0^{1,2}(\Omega)$, then this minimizer is a critical point of I_λ, so it is a solution of the corresponding elliptic problem. However, the energy functional I_λ is not bounded below on the whole space $W_0^{1,2}(\Omega)$, but is bounded on an appropriate subset of $W_0^{1,2}(\Omega)$ and a minimizer on this set gives rise to a solution of problem (1.1). In order to obtain the existence results, we introduce the Nehari manifold

$$\mathcal{N}_\lambda(\Omega) = \{u \in W_0^{1,2}(\Omega) \setminus \{0\} : \ \langle I_\lambda'(u), u\rangle = 0\},$$

where $\langle , \rangle$ denotes the usual duality between $W_0^{1,2}(\Omega)$ and $W_0^{1,2}(\Omega)^{-1}$, here $(W_0^{1,2}(\Omega))^{-1}$ is the dual space of the Sobolev space $W_0^{1,2}(\Omega)$. Note that $N_\lambda(\Omega)$ contains every nonzero solution of problem (1.1). Therefore, $u \in \mathcal{N}_\lambda(\Omega)$ if and only if

$$M(u) - A(u) - \lambda \int_\Omega f(x,|u|)|u|dx = 0; \tag{2.4}$$

so we have the following theorem.

Theorem 2.1 *There exists $0 < \lambda_1$ such that for $\lambda < \lambda_1$, I_λ is coercive and bounded below on $\mathcal{N}_\lambda(\Omega)$.*

Proof It follows from (2.1), (2.2), (2.3), (2.4) and (f4)

$$\begin{aligned} I_\lambda(u) &= (\frac{1}{2} - \frac{1}{p+1})M(u) - \lambda \int_\Omega (F(x,|u|) - \frac{1}{p+1} f(x,|u|)|u|)dx \\ &\geq \frac{p-1}{2(p+1)} M(u) - \lambda C_3 \int_\Omega \big(1 + |u|^2\big)dx \\ &\geq \frac{p-1}{2(p+1)} M(u) - \lambda C_3 \big(|\Omega| + S_2^2 M(u)\big). \end{aligned}$$

Thus I_λ is coercive and bounded below on $\mathcal{N}_\lambda(\Omega)$ for $0 < \lambda < \lambda_1 = \frac{p-1}{2(p+1)C_3 S_2^2}$. □

Define

$$\psi_\lambda(u) = \langle I_\lambda'(u), u\rangle. \tag{2.5}$$

Then by (2.2) and (2.5) we have

$$\begin{aligned} I_\lambda(tu) &= \frac{t^2}{2} M(u) - \frac{t^{p+1}}{p+1} A(u) - \lambda \int_\Omega F(x,|u|)dx, \\ \psi_\lambda(tu) &= \langle I_\lambda'(tu), tu\rangle = t^2 M(u) - t^{p+1} A(u) - \lambda \int_\Omega f(x,t|u|)|tu|dx, \\ \langle \psi_\lambda'(tu), tu\rangle &= 2t^2 M(u) - (p+1)t^{p+1} A(u) - \lambda \int_\Omega f_u(x,t|u|)(tu)^2 dx - \lambda \int_\Omega f(x,t|u|)|tu|dx. \end{aligned} \tag{2.6}$$

It is easy to see that $\psi_\lambda(tu) = 0$ if and only if $tu \in \mathcal{N}_\lambda(\Omega)$ and in particular $u \in \mathcal{N}_\lambda(\Omega)$ if and only if $\psi_\lambda(u) = 0$. Thus, it is natural to split $\mathcal{N}_\lambda$ into three parts corresponding to local minima, local maxima and points of inflection and so we define

$$\begin{aligned} \mathcal{N}_\lambda^+ &= \{u \in \mathcal{N}_\lambda(\Omega) : \ \langle \psi_\lambda'(u), u\rangle > 0\}, \\ \mathcal{N}_\lambda^- &= \{u \in \mathcal{N}_\lambda(\Omega) : \ \langle \psi_\lambda'(u), u\rangle < 0\}, \\ \mathcal{N}_\lambda^0 &= \{u \in \mathcal{N}_\lambda(\Omega) : \ \langle \psi_\lambda'(u), u\rangle = 0\}. \end{aligned} \tag{2.7}$$

The following lemma shows that minimizers for $I_\lambda(u)$ on $N_\lambda(\Omega)$ are usually critical points for I_λ, as proved by Brown and Zhang in [4].

Lemma 2.1 *Let u_0 be a local minimizer for $I_\lambda(u)$ on $\mathcal{N}_\lambda(\Omega)$ such that $u_0 \notin \mathcal{N}_\lambda^0(\Omega)$, then u_0 is a critical points of I_λ.*

Proof We let u_0 be a local minimizer for I_λ on $\mathcal{N}_\lambda$. By definition of $\mathcal{N}_\lambda$, u_0 is a minimizer for $I_\lambda(u)$ that subjects to $\langle I_\lambda^{'}(u_0), u_0\rangle = 0$. Hence, by the theory of Lagrange multipliers and (2.5), there exists $\mu \in \mathbb{R}$ such that

$$I_\lambda^{'}(u_0) = \mu\psi_\lambda^{'}(u_0) \quad \textit{in } (W_0^{1,2}(\Omega))^{-1},$$

thus,

$$\langle I_\lambda^{'}(u), u\rangle_{W_0^{1,2}(\Omega)} = \mu\langle \psi_\lambda^{'}(u), u\rangle_{W_0^{1,2}(\Omega)},$$

but $u_0 \notin \mathcal{N}_\lambda^0$ and so $\langle \psi_\lambda^{'}(u_0), u_0\rangle_{W_0^{1,2}(\Omega)} \neq 0$. Hence $\mu = 0$, which concludes the proof. □

Lemma 2.2 *There exists $\lambda_2 > 0$ such that for $0 < \lambda < \lambda_2$, we have $\mathcal{N}_\lambda^0 = \emptyset$.*

Proof Suppose otherwise. For $u \in \mathcal{N}_\lambda^0$ we have

$$\psi_\lambda(u) = M(u) - A(u) - \lambda \int_\Omega f(x, |u|)|u|dx = 0, \tag{2.8}$$

and also

$$\langle \psi_\lambda'(u), u\rangle = 2M(u) - (p+1)A(u) - \lambda \int_\Omega f_u(x, |u|)u^2dx - \lambda \int_\Omega f(x, |u|)|u|dx = 0. \tag{2.9}$$

Using (2.1), (2.3), (2.8), (2.9) and (f2) we have

$$\begin{aligned} M(u) &= pA(u) + \lambda \int_\Omega f_u(x, |u|)u^2dx \\ &\leq L\|u\|^{p+1}_{W_0^{1,2}(\Omega)} + \lambda L'\|u\|^2_{W_0^{1,2}}, \end{aligned}$$

where $L = p\|a\|_\infty S_{p+1}^{p+1}$ and $L' = \|f_u(x, |u|)\|_{L^\infty(\Omega\times\mathbb{R})}S_2^2$. Therefore

$$(1 - \lambda L')\|u\|^2_{W_0^{1,2}(\Omega)} \leq L\|u\|^{p+1}_{W_0^{1,2}(\Omega)},$$

which concludes

$$M(u) \geq \Big(\frac{1 - L'\lambda}{L}\Big)^{\frac{2}{p-1}}. \tag{2.10}$$

On the other hand from (2.1), (2.8), (2.9) and (f3), we obtain

$$\begin{aligned} (p-1)M(u) &= \lambda\Big(\int_\Omega pf(x, |u|) - f_u(x, |u|)|u|\Big)|u|dx \\ &\leq 2C_2\lambda \int_\Omega (1 + |u|^2)dx \leq 2C_2\lambda(|\Omega| + S_2^2\|u\|_{W_0^{1,2}}), \end{aligned}$$

which concludes

$$M(u) \le \frac{2C_2\lambda|\Omega|}{p-1-2C_2\lambda S_2^2}. \tag{2.11}$$

Now by (2,10) and (2,11) we must have

$$\Big(\frac{1-L'\lambda}{L}\Big)^{\frac{2}{p-1}} \le \Big(\frac{2C_2\lambda|\Omega|}{p-1-2C_2\lambda S_2^2}\Big),$$

which is a contradiction for λ sufficiently small. So there exists $\lambda_2 > 0$ such that for $0 < \lambda < \lambda_2$, $\mathcal{N}_\lambda^0 = \emptyset$. □

Definition 2.2 *A sequence $u_n \subset W_0^{1,2}(\Omega)$ is called a Palais-Smale sequence if $\{I_\lambda(u_n)\}$ is bounded and $I'_\lambda(u_n) \to 0$ as $n \to \infty$. It is said that the functional I_λ satisfies the Palais-Smale condition (or $(PS)_c -$ condition) if each Palais-Smale sequence ($(PS)_c -$ sequence) has a convergent subsequence.*

Now we will prove the boundedness of a Palais-Smale sequence.

Lemma 2.3 *If $\{u_n\}$is a $(PS)_c$-sequence for I_λ, then $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$ for $0 < \lambda < \lambda_1$.*

Proof By using (2.1), (2.2), (2.6) and (f4) we get

$$\begin{aligned} I_\lambda(u_n) - \frac{1}{p+1}\langle I_\lambda^{'}(u_n), u_n\rangle &= \frac{p-1}{2p+2}\|u\|^2_{W_0^{1,2}} - \lambda\int_\Omega \Big(F(x,|u_n|) - \frac{1}{p+1}f(x,|u_n|)|u_n|\Big)dx \\ &\ge \frac{p-1}{2p+2}\|u\|^2_{W_0^{1,2}} - \lambda C_3\int_\Omega (1+|u_n|^2)dx \\ &\ge \frac{p-1-2(p+1)\lambda C_3 S_2^2}{2p+2}\|u\|^2_{W_0^{1,2}} - \lambda C_3|\Omega|, \end{aligned}$$

so for $\lambda < \lambda_1 = \frac{p-1}{2(p+1)C_3S_2^2}$, $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$.

Now we describe the nature of the derivative of the $I_\lambda(tu)$ for all possible signs of $A(u)$ and $\int_{\partial\Omega}\frac{\partial}{\partial u}f(x,t|u|)u^2dx$ using assumption (f2), (2.5) and (2.6). We will find it useful to consider the functions

$$h_u(t) = \frac{1}{2}t^2M(u) - \frac{1}{p+1}t^{p+1}A(u) \qquad \text{and} \qquad k_u(t) = \int F(x,t|u|)dx, \tag{2.12}$$

in which $I_\lambda(tu) = h_u(t) - \lambda k_u(t)$ and $\langle I'_\lambda(tu.tu)\rangle = \psi_\lambda(tu) = 0$ if and only if $h_u^{'}(t) = \lambda k_u^{'}(t)$. Then, for fixed u and $t > 0$ by using (2.6) we illustrate the nature of the $\phi'_u(t) = h_u^{'}(t) - \lambda k_u^{'}(t)$ graphically in Figure 1(a)–(d) with possibility signs of $A(u)$ and $\int_{\partial\Omega}\frac{\partial}{\partial u}f(x,t|u|)u^2dx$.

If $\int_\Omega f_u(x,t|u|)u^2dx > 0$ and $A(u) \le 0$ clearly $h_u^{'}$ and $\lambda k_u^{'}$ have graphs as in Figure 1(a). The plot 1(b) shows the graphs of $h_u^{'}$ and $\lambda k_u^{'}$ by letting $\int_\Omega f_u(x,t|u|)u^2dx \le 0$ and $A(u) \le 0$. Clearly in these cases there is exactly one point t_1 such that $t_1u \in N_\lambda(\Omega)$ and since $\psi_\lambda(tu) < 0$ for $0 < t < t_u$ and $\psi_\lambda(tu) > 0$ for $t > t_1$ so $t_1u \in N_\lambda^+(\Omega)$.

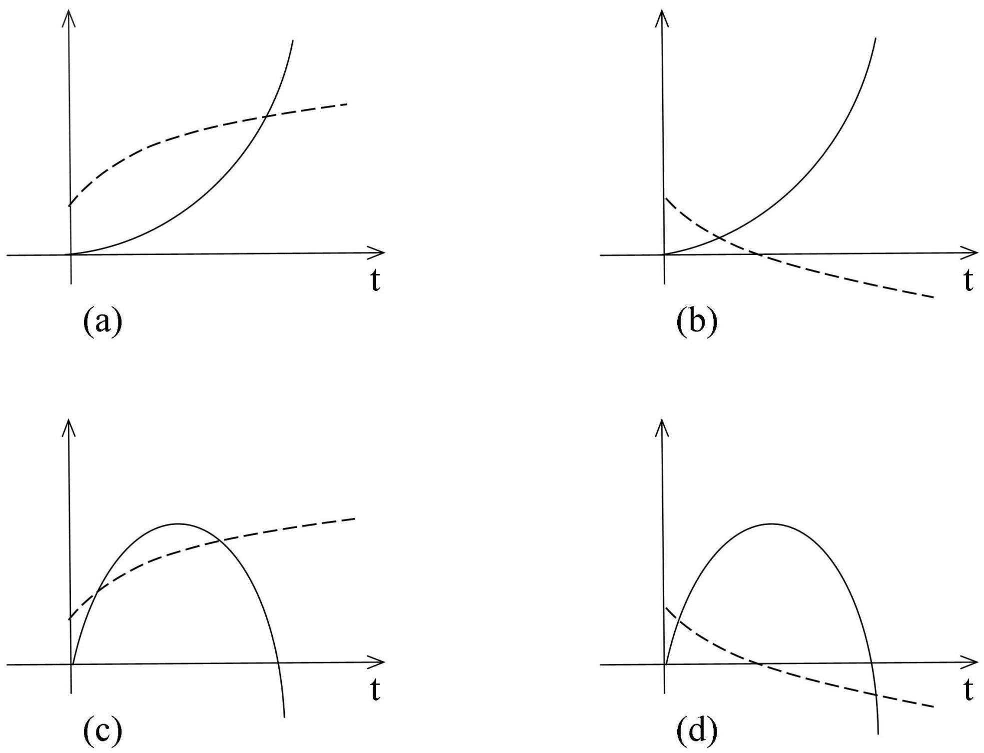

Figure 1. Possible forms for h' (—) and $\lambda k'$ (– – –)

Suppose now $\int_\Omega f_u(x,t|u|)u^2dx > 0$ and $A(u) > 0$ then $h_u^{'}$ and $\lambda k_u^{'}$ have graph as shown in Figure 1(c). The plot 1(d) represents the graphs of $h_u^{'}$ and $\lambda k_u^{'}$ when $\int_\Omega f_u(x,t|u|)u^2dx \le 0$ and $A(u) > 0$. In these cases, if $\lambda > 0$ is sufficiently large, then $\psi_\lambda(tu) = 0$ has no solutions and hence no multiple of u lies in $N_\lambda(\Omega)$. On the other hand, if $\lambda > 0$ is sufficiently small, there are exactly two solutions $t_1 < t_2$ of $\psi_\lambda(tu) = 0$ where $I_\lambda(tu)$ is decreasing in $(0,t_1)$, increasing in (t_1,t_2) and decreasing in (t_2,∞), so t_1 is a local minimum and t_2 is a local maximum of $I_\lambda(tu)$, hence $t_1u \in N_\lambda^+(\Omega)$ and $t_2u \in N_\lambda^-(\Omega)$. □

The following result ensures that when λ is sufficiently small then the graph of $I_\lambda(u)$ has positive value for all non zero $u \in W_0^{1,2}(\Omega)$.

Lemma 2.4 *There exists $\lambda_3 > 0$ such that when $\lambda < \lambda_3$, then $I_\lambda(tu)$ takes on positive values for all non-zero $u \in W_0^{1,2}$.*

Proof If $A(u) \le 0$, then by (2.5), $I_\lambda(tu) > 0$, for t sufficiently large. Suppose there exists $u \in W_0^{1,2}(\Omega)$ such that $A(u) > 0$. Elementary calculus shows that $h_u(t)$ takes on a maximum at

$$t_{\max} = \left(\frac{\|u\|^2_{W_0^{1,2}(\Omega)}}{A(u)}\right)^{\frac{1}{p-1}}, \tag{2.13}$$

where $h_u(t)$ was introduced in (2.12). Hence by (2.1), (2.12) and (2.13)

$$
\begin{aligned}
h_u(t_{\max}) &= \frac{p-1}{2(p+1)}\left(\frac{(\|u\|^2_{W_0^{1,2}(\Omega)})^{p+1}}{\left(\int_\Omega a(x)|u|^{p+1}\right)^2}\right)^{\frac{1}{p-1}} \\
&\geq \frac{p-1}{2(p+1)}\left(\frac{1}{\|a^+\|^2_\infty S_{p+1}^{2(p+1)}}\right)^{\frac{1}{p-1}} := \delta_1,
\end{aligned} \tag{2.14}
$$

and δ_1 is independent of u.

Also by (2.1), (2.13) and (2.14) for $1 \leq r < 2^*$

$$
\begin{aligned}
(t_{\max})^r \int_\Omega |u|^r dx &\leq S_r^r \left(\frac{\|u\|^2_{W_0^{1,2}(\Omega)}}{A(u)}\right)^{\frac{r}{p-1}} (\|u\|^2_{W_0^{1,2}(\Omega)})^{\frac{r}{2}} \\
&= S_r^r \left\{\frac{(\|u\|^2_{W_0^{1,2}(\Omega)})^{p+1}}{(A(u))^2}\right\}^{\frac{r}{2(p-1)}} \\
&= S_r^r \big(\frac{2(p+1)}{p-1}\big)^{\frac{r}{2}} (h_u(t_{\max}))^{\frac{r}{2}} = c(h_u(t_{\max}))^{\frac{r}{2}}.
\end{aligned} \tag{2.15}
$$

Hence from (f1), (f4) and (2.15) we have

$$
\begin{aligned}
\int_\Omega F(x, t_{\max}|u|)dx &\leq \frac{1}{p+1}\int_\Omega C_4(2 + |t_{\max}u|^2)dx + \int_\Omega C_1(|t_{\max}u| + |t_{\max}u|^{q+1})dx \\
&\leq B_0 + B_1 h_u(t_{\max}) + B_2(h_u(t_{\max})^{\frac{1}{2}} + B_3(h_u(t_{\max}))^{\frac{q+1}{2}}.
\end{aligned} \tag{2.16}
$$

Then, using (2.5), (2.14) and (2.16) we obtain

$$
\begin{aligned}
I_\lambda(t_{\max}u) &= h_u(t_{\max}) - \lambda \int_\Omega F(x, t_{\max}|u|)dx \\
&\geq h_u(t_{\max})\left(1 - \lambda\left[B_0(h_u(t_{\max}))^{-1} + B_1 + B_2(h_u(t_{\max}))^{\frac{-1}{2}} + B_3(h_u(t_{\max}))^{\frac{q-1}{2}}\right]\right) \\
&\geq \delta_1\left(1 - \lambda(B_0\delta_1^{-1} + B_1 + B_2\delta_1^{\frac{-1}{2}} + B_3\delta_1^{\frac{q-1}{2}})\right),
\end{aligned}
$$

so $I_\lambda(t_{\max}u) > 0$ for all nonzero u, provided that $\lambda < \lambda_3 = \big(2(B_0\delta_1^{-1} + B_1 + B_2\delta_1^{\frac{-1}{2}} + B_3\delta_1^{\frac{q-1}{2}})\big)^{-1}$, and this completes the proof. □

Corollary 2.1 *If $\lambda < \lambda_3$, then $I_\lambda(u) > 0$ for all $u \in \mathcal{N}_\lambda^-$.*

Proof If $u \in \mathcal{N}_\lambda^-$, due to (f2), $I_\lambda(tu)$ has a global maximum at $t = 1$, so

$$I_\lambda(u) \geq I_\lambda(t_{\max}, u) > 0.$$

□

Lemma 2.5 *There exists* $\lambda_4 > 0$ *such that, when* $\lambda < \lambda_4$, $\psi(tu) = \langle I'_\lambda(tu), tu\rangle$ *takes on positive values for all non-zero* $u \in W_0^{1,2}(\Omega)$.

Proof As the proof of the Lemma 2.4, if $A(u) \leq 0$, then by using (2.6), $\psi_\lambda(tu) > 0$ for t sufficiently large. Suppose there exists $u \in W_0^{1,2}(\Omega)$ such that $A(u) > 0$. Let

$$\bar{h}_u(t) = th'_u(t) = t^2 M(u) - t^{p+1} A(u).$$

By elementary calculus we can show that $\bar{h}_u(t)$ achieves its maximum at

$$\bar{t}_{\max} = \left(\frac{2\|u\|^2_{W_0^{1,2}(\Omega)}}{(p+1)A(u)}\right)^{\frac{1}{p-1}}. \tag{2.17}$$

Therefore, by (2.12) and (2.17) we obtain

$$\begin{aligned}\bar{h}_u(\bar{t}_{\max}) &= (\frac{p-1}{p+1})(\frac{2}{p+1})^{(\frac{2}{p-1})}\left\{\frac{(\|u\|^2_{W_0^{1,2}(\Omega)})^{p+1}}{(\int_\Omega a(x)|u|^{p+1}dx)^2}\right\}^{\frac{1}{p-1}} \\ &\geq (\frac{p-1}{p+1})(\frac{2}{p+1})^{\frac{2}{p-1}}\left(\frac{1}{\|a^+\|^2_\infty S_{p+1}^{2(p+1)}}\right)^{\frac{1}{p-1}} := \delta_2 > 0,\end{aligned} \tag{2.18}$$

where δ_2 is independent of u. Similar to (2.15), for $1 \leq r < 2^*$, we have

$$(\bar{t}_{\max})^r \int_\Omega |u|^r dx = \bar{c}(\bar{h}_u(\bar{t}_{\max}))^{\frac{r}{2}}, \tag{2.19}$$

then using (2.19) and (f1) we conclude that

$$\begin{aligned}\int_\Omega f(x, \bar{t}_{\max}|u|)|\bar{t}_{\max}u|dx &\leq C_1 \int_\Omega \left(|\bar{t}_{\max}u| + |\bar{t}_{\max}u|^{q+1}\right)dx \\ &\leq b_0\big(\bar{h}_u(\bar{t}_{\max})\big)^{\frac{1}{2}} + b_1\big(\bar{h}_u(\bar{t}_{\max})\big)^{\frac{q+1}{2}},\end{aligned} \tag{2.20}$$

where b_0 and b_1 are independent of u. So, from (2.18), (2.20) and (2.6) we get

$$\begin{aligned}\psi_\lambda(\bar{t}_{\max}u) &= \bar{h}_u(\bar{t}_{\max}) - \lambda \int_\Omega f(x, \bar{t}_{\max}|u|)\bar{t}_{\max}|u|dx \\ &\geq \big(\bar{h}_u(\bar{t}_{\max})\big)^{\frac{1+q}{2}}\left(\big(\bar{h}_u(\bar{t}_{\max})\big)^{\frac{1-q}{2}} - \lambda(b_0(\bar{h}_u(\bar{t}_{\max}))^{\frac{-q}{2}} + b_1)\right) \\ &\geq \delta_2^{\frac{1+q}{2}}\left(\delta_2^{\frac{1-q}{2}} - \lambda(b_0\delta_2^{\frac{-q}{2}} + b_1)\right).\end{aligned}$$

Clearly $\psi_\lambda(\bar{t}_{\max}u) > 0$, for all nonzero u, provided that $\lambda < \lambda_4$ where $\lambda_4 = \delta_2^{\frac{1-q}{2}}/2(b_0\delta_2^{\frac{-q}{2}} + b_1)$, this completes the proof. □

Corollary 2.2 *If* $A(u) \leq 0$ *for* $u \in W_0^{1,2}(\Omega)\setminus\{0\}$, *then there exists* t_1 *such that* $t_1 u \in N_\lambda^+$ *and* $I_\lambda(t_1 u) < 0$.

Proof From the definition of $I_\lambda(tu)$ and (2.6) for a fixed u, we know $\psi_\lambda(0) < 0$ and $\lim_{t\to\infty}\psi_\lambda(tu) = +\infty$, so by the intermediate value theorem, there exists $t_1 > 0$ such that $\psi_\lambda(t_1u) = 0$. Now since $\psi_\lambda(tu) < 0$ for $0 < t < t_1$ and $\psi_\lambda(tu) > 0$ for $t_1 < t$, then (2.7) follows that $t_1u \in N_\lambda^+$ and $I_\lambda(t_1u) < I_\lambda(0) = 0$. □

Corollary 2.3 *If $A(u) > 0$ for $u \in W_0^{1,2}(\Omega) \setminus \{0\}$, and $\lambda < \lambda_1$, then there exist $t_1 < t_2$ such that $t_1u \in N_\lambda^+$, $t_2u \in N_\lambda^-$ and $I_\lambda(t_1u) < 0$.*

Proof As in the proof of the above corollary, we obtain $\psi_\lambda(0) < 0$, $\lim_{t\to\infty}\psi_\lambda(tu) = -\infty$ and by using Lemma 2.5 we get $\psi_\lambda(Tu) > 0$ for a suitable T, so the intermediate value Theorem concludes that there exist t_1, t_2 such that $0 < t_1 < T < t_2$, $\psi_\lambda(t_1u) = \psi_\lambda(t_2u) = 0$, $t_1u \in N_\lambda^+$, $t_2u \in N_\lambda^-$ and $I_\lambda(t_1, u) < I_\lambda(0) = 0$. □

3. Existence of solutions

In this section, using the properties of $I_\lambda(tu)$, we study the existence of positive solutions of problem (1.1). For simplicity let $\lambda^* = \min\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$.

Remark 3.1 *By using (f1) we get $|f(x,u)| \le C_1(1+|u|^q) \le 2C_1(1+|u|^r)$ for $0 < q < 1 < r < \frac{N+2}{N-2}$. Hence from the compactness of the embedding $W_0^{1,2}(\Omega) \hookrightarrow \mathrm{L}^r(\Omega)$ for $1 \le r < \frac{N+2}{N-2}$ (the Rellich-Kondrachov theorem [3]) and the fact that the operator $u \longmapsto f(x,u)$ is continuous, we conclude that the functional $J(u) = \int_\Omega F(x,u)dx$ is weakly continuous , i.e. if $u_n \rightharpoonup u$, then $J(u_n) \to J(u)$ and the operator $J'(u) = \int_\Omega f(x,u)udx$ is weak to strong continuous, i.e. if $u_n \rightharpoonup u$, then $J'(u_n) \to J'(u)$.*

Theorem 3.1 *For $\lambda < \lambda^*$, there exists a minimizer of I_λ on $\mathcal{N}_\lambda^+(\Omega)$.*

Proof As in Theorem 2.1 I_λ is bounded below on $\mathcal{N}_\lambda(\Omega)$ and so on $\mathcal{N}_\lambda^+(\Omega)$. Let $\{u_n\}$ be a minimizing sequence for I_λ on $\mathcal{N}_\lambda^+(\Omega)$, i.e.

$$\lim_{n\to\infty} I_\lambda(u_n) = \inf_{u\in\mathcal{N}_\lambda^+} I_\lambda(u) = c,$$

and by Ekeland's variational principle [10] we may assume

$$\langle I_\lambda^{'}(u_n), u_n\rangle \to 0.$$

Then by Lemma 2.3 $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$ and by Rellich theorem [3] without loss of generality, we may assume that $u_n \rightharpoonup u_1$ in $W_0^{1,2}(\Omega)$ and $u_n \to u_1$ in $L^r(\Omega)$ for $1 \le r < 2^*$ and $u_n(x) \to u_1(x)$, a.e.

By Corollaries 2.2 and 2.3 for $u_1 \in W_0^{1,2}(\Omega) \setminus \{0\}$, there exists t_1 such that $t_1u_1 \in N_\lambda^+$ and so $\psi_\lambda(t_1u_1) = 0$. Now we show that $u_n \to u_1$ in $W_0^{1,2}(\Omega)$. Suppose this is false, then

$$M(u_1) < \liminf_{n\to\infty} M(u_n). \tag{3.1}$$

Also we have

$$\psi_\lambda(tu_n) = t^2M(u_n) - t^{p+1}A(u_n) - \lambda\int_\Omega f(x, t|u_n|)|tu_n|dx \tag{3.2}$$

and

$$\psi_\lambda(tu_1) = t^2 M(u_1) - t^{p+1} A(u_1) - \lambda \int_\Omega f(x, t|u_1|)|tu_1| dx. \tag{3.3}$$

So, from (3.1), (3.2), (3.3) and Remark 3.1, $\psi_\lambda(t_1 u_n) > \psi_\lambda(t_1 u_1) = 0$ for n sufficiently large. Since $\{u_n\} \subseteq N_\lambda^+(\Omega)$, by considering possible maps it is easy to see that $\psi_\lambda(tu_n) < 0$ for $0 < t < 1$ and $\psi_\lambda(t_1 u_n) = 0$ for all n. Hence we must have $t_1 > 1$, but $t_1 u_1 \in N_\lambda^+$ and so

$$I_\lambda(t_1 u_1) < I_\lambda(u_1) < \lim_{n\to\infty} I_\lambda(u_n) = \inf_{u \in \mathcal{N}_\lambda^+} I_\lambda(u_n),$$

which is a contradiction. Therefore $u_n \to u_1$ in $W_0^{1,2}(\Omega)$ and so

$$I_\lambda(u_1) = \lim_{n\to\infty} I_\lambda(u_n) = \inf_{u \in \mathcal{N}_\lambda^+} I_\lambda(u).$$

Thus u_1 is a minimizer for I_λ on $\mathcal{N}_\lambda^+(\Omega)$.

Next, we establish the existence of a local minimum for I_λ on $\mathcal{N}_\lambda^-$. □

Theorem 3.2 *If $\lambda < \lambda^*$, there exists a minimizer of I_λ on $\mathcal{N}_\lambda^-(\Omega)$.*

Proof By Corollary 2.1 we have $I_\lambda(u) > 0$ for all $u \in \mathcal{N}_\lambda^-$, i.e.

$$\inf_{u \in \mathcal{N}_\lambda^-} I_\lambda(u) \geq 0.$$

Hence there exists a minimizing sequence $\{u_n\} \subseteq \mathcal{N}_\lambda^-(\Omega)$ such that

$$\lim_{n\to\infty} I_\lambda(u_n) = \inf_{u \in \mathcal{N}_\lambda^-} I_\lambda(u) \geq 0. \tag{3.4}$$

Similarly, as in the proof of the previous theorem, we find that $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$, $u_n \rightharpoonup u_2$ in $W_0^{1,2}(\Omega)$ and $u_n \to u_2$ in $L^r, 1 < r < 2^*$. Also we have

$$I_\lambda(u_n) = \frac{1}{2} M(u_n) - \frac{1}{p+1} A(u_n) - \lambda \int_\Omega F(x, |u_n|) dx. \tag{3.5}$$

We claim that $A(u_n) > 0$. Suppose this is false. Thus $-pA(u_n) \geq 0$, since $u_n \in \mathcal{N}_\lambda^-$, so by (f2), (2.1), (2.6) and (2.7) we have

$$M(u_n) < \lambda \int_\Omega f_u(x, t|u_n|) u_n^2 dx \leq \lambda \|f_u(x, |u_n|)\|_{L^\infty(\Omega\times\mathbb{R})} S_2^2 M(u_n),$$

which gives a contradiction for λ sufficiently small, hence $A(u_n) > 0$. Letting $n \to \infty$, we see that $A(u_2) > 0$.

So by Corollary 2.3, there exists $t_2 > 0$ such that $t_2 u_2 \in \mathcal{N}_\lambda^-(\Omega)$. We claim that $u_n \to u_2$ in $W_0^{1,2}(\Omega)$, suppose that this is false, so

$$M(u_2) < \liminf_{n\to\infty} M(u_n). \tag{3.6}$$

But $u_n \in \mathcal{N}_\lambda^-$ and so $I_\lambda(u_n) \geq I_\lambda(tu_n)$ for all $t \geq 0$, therefore, by using (2.5), (3.4), (3.5), (3.7) and Remark 3.1, we get

$$\begin{aligned} I_\lambda(t_2 u_2) &= \frac{1}{2}t_2^2 M(u_2) - \frac{1}{p+1}t_2^{p+1}A(u_2) - \lambda\int_\Omega F(x, t_2|u_2|)dx \\ &< \lim_{n\to\infty}(\frac{1}{2}t_2^2 M(u_n) - \frac{1}{p+1}t_2^{p+1}A(u_n) - \lambda\int_\Omega F(x, t_2|u_n|)dx \\ &= \lim_{n\to\infty} I_\lambda(t_2 u_n) \leq \lim_{n\to\infty} I_\lambda(u_n) = \inf_{u\in\mathcal{N}_\lambda^-} I_\lambda(u), \end{aligned}$$

which is a contradiction. Therefore $u_n \to u_2$ in $W_0^{1,2}(\Omega)$ and so the proof is complete. □

Corollary 3.1 *Problem (1.1) has at least two positive solutions for* $0 < \lambda < \lambda^*$.

Proof By Theorems 3.1 and 3.2 there exist $u_1 \in N_\lambda^+(\Omega)$ and $u_2 \in N_\lambda^-(\Omega)$ such that $I_\lambda(u_1) = \inf_{u\in N_\lambda^+} I_\lambda(u)$ and $I_\lambda(u_2) = \inf_{u\in N_\lambda^-} I_\lambda(u)$. By Lemma 2.1, and (2,2), u_1 and u_2 are critical points of I_λ on $W_0^{1,2}$ and hence are weak solutions of problem (1.1). On the other hand $I_\lambda(u) = I_\lambda(|u|)$, so we may assume u_1 and u_2 are positive solutions. It remains to show that the solutions found in Theorems 3.1 and 3.2 are distinct. Since $N_\lambda^+ \cap N_\lambda^- = \emptyset$, this implies that u_1 and u_2 are distinct and the proof is complete. □

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